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Transition-dependent Robust MPC for Stochastic Switched Systems

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Abstract— **This study is concerned with the robust MPC for discrete-time stochastic switched systems subject to constraints on states and control inputs. Aiming at achieving optimal control synthesis under the requirement of bumpless transfer control (BTC), the min-max MPC formulation is extended to the transition-dependent paradigm, and a weighted performance index is optimized in the receding horizon, such that the abrupt variation in feedback gains can be mitigated. Meanwhile, a class of more general stochastic switching signals is considered, where the sojourn time may follow any distribution, and the recursive feasibility and mean-square stability are theoretically guaranteed. Compared with existing studies on switched MPC or BTC, this work avoids the assumption of the Markov property on mode switching and reduces conservatism by exploiting the statistical information of sojourn time. An illustrative example is provided to show the potential of the obtained results.**

I. Introduction

Over the decades, model predictive control (MPC) has attracted considerable attention from both academia and industry due to its superiority in control performance optimization subject to various constraints. In concern of model uncertainties or disturbances, great efforts are devoted to robust MPC, and the representative results can be referred to tube-based MPC [1] and min-max MPC [2], both of which rely on the state-feedback control law for robustness guarantee. In recent years, the robust MPC is extended to the switched systems, where the MPC law is naturally designed in the mode-dependent paradigm for less conservatism, i.e., the cost functions [3], terminal constraints [4] or feedback gains [5] are explicitly switched with the system mode. It is worth noting that the mode-dependent control law, which is indispensable for switched robust MPC, is essentially a piecewise continuous function in time domain, and the discontinuity at switching instants may lead to abrupt changes in control input and undesired transient behaviors, which is regarded as the *bump phenomenon* [6] in switched systems.

The issue of control bump suppression gives rise to the research topic of bumpless transfer control (BTC), the basic objective of which is to smoothen the gain variation within control synthesis. A simple approach proposed in [7] constrains the norm difference between the mode-dependent gains and a common gain, such that the gain variation is ensured below a threshold. This approach is improved in [8] by introducing transition progress between the stabilizing gains, which provides more flexible control synthesis and reduce the conservatism in stabilization. Further, [6] reveals that the BTC controllers should depend on both of the current and last modes, such that non-conservative *transition-dependent* gains can be designed to smoothen the transition progress between arbitrary two given stabilizing gains. Recently, the BTC results are further extended to various control issues and different systems [9], but the issue of optimal BTC for constrained switched systems remains to be addressed, let alone BTC synthesis with receding horizon optimization.

On the other hand, the issue of robust MPC with general stochastic switching deserves further investigation. In [3], the conditions on the terminal cost are given to ensure mean-square stability for Markov switched system, based on which an explicit MPC law is obtained. In [10], a min-max MPC is formulated based on scenario generation and solved via an offline-to-online algorithm. Also, the complementary merits of tube-based MPC and min-max MPC are discussed under Markovian switching in [11]. In such studies, the recursive feasibility and stability are based on a common assumption that the mode switching is governed by a Markov chain, where the sojourn time inherently follows a geometric distribution. In practice, the mode switching may not conform to the Markov property, which restricts the application of the aforementioned results. Some attempts on overcoming this limitation can be seen in nondeterministic switched MPC. [5] propose a min-max MPC approach under average dwell-time constraints, but provides no guarantee of recursive feasibility. In [12] and [13], a set-theoretic analysis is developed to determine the modal dwell time and feasible region, while the issue of MPC synthesis is not addressed. Moreover, such studies ignore the statistical information about mode switching, leading to conservatism in feasibility and stability. Therefore, the issue of stochastic switched MPC with sojourn time conformed to general distribution is still largely open, which motivates us to fill this gap.

In this paper, the bumpless transfer robust MPC for a class of discrete-time stochastic switched systems subject to hard constraints on state and control input is investigated. The main contributions lie in that: (*i*) The switched robust MPC is extended to be with a class of more general stochastic switching signal, where the sojourn time may follow arbitrary distributions. (*ii*) A min-max MPC is formulated with the transition-dependent control law and weighted objective optimization, such that the control bump can be suppressed without much detriment to performance. The recursive feasibility and stability are also addressed, and an illustrative example is given to validate the proposed approach.

Notations: In this paper, $\mathbb{R}$ and $\mathbb{N}$ denote the sets of

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real numbers and nonnegative integers, respectively. $\mathbb{E}_k[\cdot]$ stands for the mathematical expectation of a random variable conditioned on the information accessible at time k. $(x)_l$ denotes the lth component of $x \in \mathbb{R}^n$, $(P)_{ll}$ denotes the element of the lth row and lth column of a matrix P. In addition, $diag\{A_1, A_2, ..., A_N\}$ stands for a block-diagonal matrix constituted by $\{A_1, A_2, ..., A_N\}$. Symbol $*$ is used as an ellipsis for the terms that are introduced by symmetry.

II. Problem Formulation and Preliminaries

Consider a class of discrete-time switched linear systems

$$x_{k+1} = A^{\theta}_{r_k} x_k + B^{\theta}_{r_k} u_k \tag{1}$$

with hard constraints on the state $x_k \in \mathbb{R}^{n_x}$ and control input $u_k \in \mathbb{R}^{n_u}$:

$$Ex_k + Fu_k \leq d \tag{2}$$

where r_k is a piece-wise constant switching signal taking value in a finite integer set $\mathbb{M} = \{1, 2, ..., M\}$, which governs the switching among M subsystems. The switching sequence $k_0, k_1, k_2, ..., k_s, ...$ is considered to be unknown a priori, but the switching instant k_s with $s \in \mathbb{Z}_+$ can be instantly accessed by the controller. When $k \in [k_s, k_{s+1})$, the r_k-th subsystem is said to be active with sojourn time $k_{s+1} - k_s$.

In this work, the running interval $[k_s, k_{s+1})$ for the active subsystem is considered to be with a deterministic stage $[k_s, k_d)$ and a stochastic stage $[k_d, k_{s+1})$, and the running time of the two stages is denoted by $h_{dt} = k_d - k_s$ and $h_{ss} = k_{s+1} - k_d$, respectively. The duration of deterministic stages is fixed and dependent on the mode, i.e., $h_{dt} = \tau_m, m \in \mathbb{M}$ with $\tau_m \geq 0$ are given constants. For the stochastic stage, the duration follows some mode-dependent distribution X_m that can be obtained from statistical information. To conclude, the considered switching signal can be represented by $r_k = m$ for $k \in [k_s, k_{s+1}), k_{s+1} - k_s = \tau_m + \bar{\tau}_m$, with the random variable $\bar{\tau}_m \sim \mathcal{T}_m$.

At each time instant k, the values of $A^{\theta}_{r_k}, B^{\theta}_{r_k}$ are unknown but belong to a polytopic uncertainty class Ω_{r_k}, i.e.,

$$\left[A^{\theta}_{r_k} B^{\theta}_{r_k}\right] \in \Omega_{r_k} \triangleq \text{Co}\left\{\left[A^{v}_{r_k} B^{v}_{r_k}\right], v \in \mathbb{D} \triangleq \{1, \ldots, D\}\right\} \tag{3}$$

where $A^{v}_{r_k}, B^{v}_{r_k}$ are known constant matrices. In this paper, the performance of predicted control sequence $\{u_{k+i|k}, i \in \mathbb{N}\}$ is evaluated by the expectation of the linear quadratic (LQ) cost in the infinite prediction horizon, with the worst-case LQ performance for instant k represented by

$$\tilde{J}_{LQ}(k) = \max_{\substack{[A^v_m, B^v_m] \in \Omega_m, \\ m = r_{k+i|k}, i \in \mathbb{N}}} \mathbb{E}\{\sum_{i=0}^{\infty} \left[x^T_{k+i|k} \mathcal{Q} x_{k+i|k} + u^T_{k+i|k} \mathcal{R} u_{k+i|k}\right]\} \tag{4}$$

where $\mathcal{Q}$, $\mathcal{R}$ are the given positive weight matrices. To ensure the stability and obtain optimal performance under the polytopic uncertainty and stochastic switching, the MPC is designed to minimize the worst-case LQ performance. Also, for the sake of the recursive feasibility, all the predicted states and control inputs should satisfy the hard constraint (2). Thereby, the stochastic switched robust MPC can be formulated as solving the following min-max problem in a receding horizon manner:

$$\min_{u_{k+i|k}, i \in \mathbb{N}} \tilde{J}_{LQ}(k) \quad s.t.(1), (2) \tag{5}$$

In consideration of computational tractability, a common strategy is adopting a state-feedback control law to generate the predicted control sequence in the infinite horizon, such that the problem (5) can be parameterized with decision variables of finite dimensions. In this paper, the transition-dependent linear control law is formulated as

$$u_{k+i|k} = K(k+i|k) x_{k+i|k} \tag{6}$$

where $K(k+i|k) = K_{m'm,t}$, $m = r_{k+i|k}$, $m' = r_{k_s-1|k}$, $t_i = k+i-k_s$, $m = r_{k+i|k}$, $k+i \in [k_s, k_{s+1})$. Note that $K_{r'r,t}$ are the transition-dependent gains, which depend on the current mode $r \in \mathbb{M}$, the last mode $r' \in \mathbb{M}$ and the time spent in the current mode $t \in \mathbb{N}$. The gains are considered to be time-varying only within each transition interval $[k_s, k_s + T_m)$ with $T_m < \tau_m$, i.e., $K_{m'm,t} = K_{m'm,T_m}$ for $t \in [k_s + T_m, k_{s+1})$. The length of the transition duration is designed as mode-dependent, denoted by T_r, and the transition interval is denoted by $\mathbb{T}_r \triangleq [0, T_r]$. Such a strategy can provide extra design degree to smoothen the gain transition at switching instants. For the purpose of BTC, an additional objective is introduced into the min-max formulation, evaluated by the following worst-case control bump cost

$$\tilde{J}_{BT}(k) = \max_{\substack{[A^v_r, B^v_r] \in \Omega_r, v \in \mathbb{D} \\ r = r_{k+i|k}, i \in \mathbb{N}}} \mathbb{E}\left[\|\Delta u_{k+i|k}\|^2\right] \tag{7}$$

Finally, the definition of the mean square stability (MSS) are recalled for later derivations:

Definition 1: The switched system (1) is said to be mean-square stable (MSS) within the feasible region $\mathbb{X}_0$, if the following condition holds

$$lim_{k \to \infty} \mathbb{E}_0[\|x_k\|^2] = 0 \tag{8}$$

for any initial state $x_0 \in \mathbb{X}_0$ and initial mode $r_0 \in \mathbb{M}$.

III. Main Results

In this section, an approximated optimal solution to the origin min-max problem (5) is obtained by minimizing the upper bound of the cost with additional constraints for the invariance of predicted states and the boundness of LQ cost. Then, the transition-dependent MPC is achieved by solving the constructed semidefinite programming (SDP) problem by receding horizon, where the recursive feasibility and mean-square stability under stochastic switching are proved.

A. Transition-dependent Robust MPC

Firstly, the upper bound of the LQ performance of the stochastic switched system (1) in the infinite horizon is given as follows.

Lemma 1: Consider the infinite horizon MPC problem for the stochastic switched system (1) subject to hard constraints (2) and polytopic uncertainties (3). If there exists scalars

$\{0 < \alpha_r \leq 1, \mu_r \geq 1\}_{r\in\mathbb{M}}$ and positive definite matrices $\{P_r > 0\}_{r\in\mathbb{M}}$ with the corresponding quadratic functions $V_m(k+i|k) = x^T_{k+i|k} P_m x_{k+i|k}$ satisfying that $\forall (r_{k_s}) \in \mathbb{M}$,

$$\begin{aligned} &V_m(k+i+1|k) - \alpha_m V_m(k+i|k) \\ &\leq -\left[x^T_{k+i|k}\mathcal{Q}x_{k+i|k} + u^T_{k+i|k}\mathcal{R}u_{k+i|k}\right] \end{aligned} \quad (9)$$

$\forall\,(r_{k_s} = m', r_{k_s-1} = m) \in \mathbb{M}\times\mathbb{M}$, $m' \neq m$,

$$V_{m'}(k_s|k) - \mu_m V_m(k_s|k) \leq 0 \quad (10)$$

and the stochastic switching signal satisfying

$$\mu_r \alpha_r^{\tau_r} \mathbb{E}_{\bar{\tau}_r\sim\mathcal{T}_r}\left[\alpha_r^{\bar{\tau}_r}\right] < 1, \forall r \in \mathbb{M}. \quad (11)$$

then the predicted state $x_{k+i|k}$ satisfies

$$lim_{i\to\infty}\mathbb{E}_k\left[\|x_{k+i|k}\|^2\right] = 0 \quad (12)$$

and there holds

$$\tilde{J}_{LQ}(k) \leq \mu_{r_k} V_{r_k}(k) \quad (13)$$

where $t_k = k - k_s$ is the current running time, $k \in [k_s, k_{s+1})$.

Proof: According to (9) and $\alpha_m^{-1} > 1$, one has

$$\begin{aligned} &V_m(k+i|k) - \alpha_m^{-n} V_m(k+i+n|k) \\ &\geq \sum_{j=i}^{i+n-1}\left(x^T_{k+j|k}\mathcal{Q}x_{k+j|k} + u^T_{k+j|k}\mathcal{R}u_{k+j|k}\right) \end{aligned} \quad (14)$$

For each running interval $[k_s, k_{s+1})$, taking expected value on both sides of (14) and substituting (10), it yields

$$\begin{aligned} &V_m(k_s|k) - \mathbb{E}_{k_s|k}\left[\alpha_m^{-(k_{s+1}-k_s)}\mu_m^{-1}V_{m'}(k_{s+1}|k)\right] \\ &\geq \mathbb{E}_{k_s|k}\left[\sum_{j=k_s}^{k_{s+1}-1}\left(x^T_{j|k}\mathcal{Q}x_{j|k} + u^T_{j|k}\mathcal{R}u_{j|k}\right)\right] \end{aligned} \quad (15)$$

Summing (15) up from $k_s = k_1$ to k_∞, according to (11) and the Jensen's inequality, one has

$$\begin{aligned} &V_{m_1}(k_1|k) - \mathbb{E}_{k_1|k}\left[V_{m_\infty}(k_\infty|k)\right] \\ &\geq \mathbb{E}_{k_1|k}\left[\sum_{j=k_1}^{k_\infty-1}\left(x^T_{j|k}\mathcal{Q}x_{j|k} + u^T_{j|k}\mathcal{R}u_{j|k}\right)\right] \end{aligned} \quad (16)$$

where $m_i = r_{k_i}$. Similarly, in the last interval $[k_\infty, \infty)$ and the first interval $[k, k_1)$, (9) and (10) ensure that

$$\begin{aligned} &V_{m_\infty}(k_\infty|k) - \mathbb{E}_{k_\infty|k}\left[V_{m_\infty}(\infty|k)\right] \\ &\geq \mathbb{E}_{k_\infty|k}\left[\sum_{j=k_\infty}^{\infty}\left(x^T_{j|k}\mathcal{Q}x_{j|k} + u^T_{j|k}\mathcal{R}u_{j|k}\right)\right] \end{aligned} \quad (17a)$$

$$\begin{aligned} &\mu_{m_0}V_{m_0}(k|k) - \mathbb{E}_k[V_{m_1}(k_1|k)] \\ &\geq \mathbb{E}_k\left[\sum_{j=k}^{k_1-1}\left(x^T_{j|k}\mathcal{Q}x_{j|k} + u^T_{j|k}\mathcal{R}u_{j|k}\right)\right] \end{aligned} \quad (17b)$$

Summing (16) (17a) and (17b) up, one has

$$\begin{aligned} &\mu_{m_0}V_{m_0}(k) - \mathbb{E}_k[V_{m_\infty}(\infty|k)] \\ &\geq \mathbb{E}_k\left[\sum_{j=k}^{\infty}\left(x^T_{j|k}\mathcal{Q}x_{j|k} + u^T_{j|k}\mathcal{R}u_{j|k}\right)\right] \end{aligned} \quad (18)$$

Therefore, $\mathbb{E}_k\left[\sum_{j=k}^{\infty}\left(x^T_{j|k}\mathcal{Q}x_{j|k} + u^T_{j|k}\mathcal{R}u_{j|k}\right)\right]$ is upper-bounded. Moreover, since $x^T_{j|k}\mathcal{Q}x_{j|k} > 0$ for $x_{j|k} \neq 0$, one can obtain (12) and $lim_{j\to\infty}\mathbb{E}_k[V_m(j|k)] = 0$, which further implies (13) due to (18). ■

Remark 1: In Lemma 1, the scenario tree is generated with nodes on switching instants instead of every time instant (commonly adopted for Markov switching, cf. [10], [14]), and then the stochastic convergence constraint is imposed on nodes to ensure $\mathbb{E}_k[V(k_s|k)]$ decreasing with a variation larger than LQ costs. The obtained result is more general, because the sojourn-time distribution can be deduced from transition probability matrices, but not the opposite.

Remark 2: With the quadratic function given in Lemma 1, it is straightforward that the mode-dependent ellipsoidal sets $\mathcal{E}_{m,t} = \{x|x^T P_m x \leq \alpha_m^t, t = k - k_s\}$ are positive invariant for subsystems within each running interval, where $V_m(k_s + t|k) \leq \alpha_m^t V_m(k_s|k) < V_m(k_s|k)$. Further, given the duration of the deterministic stage τ_m, (9) and (10) ensure $V_m(k_{s+i}|k) < \alpha_m^{\tau_m+\bar{\tau}_m}\mu_m V_m(k_s|k)$. Thus, with the initial condition $V_m(k) < \alpha_m^{t_k}$, Lemma 1 ensures that all the ellipsoidal sets $\mathcal{E}_r = \{x|x^T P_m x \leq 1\}, \forall r \in \mathbb{M}$ is positive invariant, based on which the constraints can be imposed with a guarantee of recursive feasibility.

Based on Lemma 1, the control synthesis under the requirements of LQ performance, bumpless transfer performance and hard constraints are summarized as follows.

Theorem 1: Consider switched system (1) with feedback gains $K_{r'r,t}$ subject to hard constraints (2). If there exist positive definite matrices $\{Q_r, U, W\}_{r\in\mathbb{M}}$, arbitrary matrices $\{S_{r'r,t}\}_{r,r'\in\mathbb{M}, r\neq r', t\in\mathbb{T}_r}$ and positive scalars $\{\gamma, \delta\}$, such that $\forall r', r, r'' \in \mathbb{M}$, $r \neq r'$, $r'' \neq r$, $t \in \mathbb{T}_r$, $v \in \mathbb{D}$

$$\begin{bmatrix} \alpha_{r_k}^{t_k} & * \\ x_k & Q_{r_k} \end{bmatrix} \geq 0 \quad (19a)$$

$$\begin{bmatrix} \Psi^v_{r'r,t} & * \\ \Gamma_{rr',t} & \gamma I \end{bmatrix} \geq 0, \quad (19b)$$

$$Q_r - \mu_r Q_{r''} \leq 0 \quad (19c)$$

$$\begin{bmatrix} \Theta_{r'r,t} & * \\ \Xi^v_{r'r,r'r,tt'} & \Lambda_r \end{bmatrix} \geq 0, \quad (19d)$$

$$\begin{bmatrix} \Theta_{r'r,T_r} & * \\ \Xi^v_{r'r,rr'',T_r 0} & \Lambda_{r''} \end{bmatrix} \geq 0, \quad (19e)$$

$$trace(U) \leq \delta, \quad (19f)$$

$$\begin{bmatrix} W & EQ_r + FS_{r'r,t} \\ * & Q_r \end{bmatrix} \geq 0, (W)_{ll} \leq (d)_l^2, \quad (19g)$$

where $\Psi^v_{r'r,t} = [\alpha_r Q_r, *; A^v_r Q_r + B^v_r S_{r'r,t}, Q_r]$, $\Gamma_{r'r,t} = [\mathcal{Q}^{1/2}Q_r, 0; \mathcal{R}^{1/2}S_{r'r,t}, 0]$, $\Theta_{r'r,t} = [Q_r, *; S_{r'r,t}, U]$, $\Xi^v_{ab,cd,ij} = \text{diag}\{A^v_b Q_b + B^v_b S_{ab,i}, -S^T_{cd,j}\}$, $\Lambda_r = [I, I; I, Q_r]$, $t' = \min(t+1, T_r)$, then the worst-case LQ performance can be bounded by $\tilde{J}_{LQ}(k) \leq \rho_k\gamma$ with $\rho_k = \alpha_{r_k}^{t_k}\mu_{r_k}$, and the worst-case BT performance can be bounded by $\tilde{J}_{BT}(k) \leq \delta$. The corresponding gains of predictive control are given by

$$K_{r'r,t} = S_{r'r,t}Q_r^{-1} \quad (20)$$

Proof: By applying congruent transformation to (19b) with $diag\{Q_r^{-1}, I, ..., I\}$ and Schur complement, one has

$$\begin{aligned}&\alpha_r Q_r^{-1} - (A_r^v + B_r^v K_{r'r,t})^T Q_r^{-1} (A_r^v + B_r^v K_{r'r,t}) \\ &\quad - \gamma^{-1}[\mathcal{Q} + K_{r'r,t}^T \mathcal{Q} K_{r'r,t}] \geq 0\end{aligned} \tag{21}$$

By introducing the matrices $P_r = \gamma Q_r^{-1}$ and the control law $u_k = K_{r'r,t} x_k$, it can be obtained that

$$\begin{aligned}&\alpha_{r'} x_k^T P_r x_k - x_{k+1}^T P_r x_{k+1} \\ &\quad - [x_k^T \mathcal{Q} x_k + u_k^T \mathcal{R} u_k] \geq 0\end{aligned} \tag{22}$$

Consider the following quadratic functions

$$V_{m_i}(k+i|k) = x_{k+i|k}^T P_{m_i} x_{k+i|k}, i \geq 0 \tag{23}$$

with $m_i = r_{k+i|k}$. It is straightforward that (22) ensures (9), and (19c) ensures (10). According to Lemma 1, the LQ performance in infinite horizon can be up-bounded with the quadratic function (23). Moreover, by Schur complement, (19a) ensures that $x_k^T Q_{r_k}^{-1} x_k \leq \alpha_{r_k}^{t_k}$ and $x_k^T P_{r_k} x_k \leq \alpha_{r_k}^{t_k} \gamma$, so one has $\tilde{J}_{LQ}(k) \leq \mu_{r_k} V_{r_k}(k|k) \leq \rho_k \gamma$.

By Schur complement, (19d) ensures that

$$\begin{bmatrix} Q_r & * & * \\ S_{r'r,t} & U & * \\ A_r^v Q_r + B_r^v S_{r'r,t} & (S_{r'r,t'} Q_r^{-1})^T & I \end{bmatrix} \geq 0, \tag{24}$$

By applying congruent transformation to (19b) with $diag\{Q_r^{-1}, I, I\}$ and substituting $K_{r'r,t'} = S_{r'r,t'} Q_r^{-1}$, (24) ensures that

$$\begin{bmatrix} Q_r & * \\ S_{r'r,t} - K_{r'r,t'}(A_r^v Q_r + B_r^v S_{r'r,t}) & U \end{bmatrix} \geq 0, \tag{25}$$

By apply Schur complement and substituting $K_{r'r,t} = S_{r'r,t} Q_r^{-1}$, one has

$$\Phi_{r'r,t} Q_r \Phi_{r'r,t}^T \leq U \tag{26}$$

where $\Phi_{r'r,t} = K_{r'r,t'}(A_r^v + B_r^v K_{r'r,t}) - K_{r'r,t}$. Similarly, (19e) ensures that

$$\Phi_{rr'',T_r} Q_r \Phi_{rr'',T_r}^T \leq U \tag{27}$$

where $\Phi_{rr'',T_r} = K_{rr'',0}(A_r^v + B_r^v K_{r'r,T_r}) - K_{r,T_r}$.

Then, in virtual of the invariant sets $\{x | x^T Q_r^{-1} x \leq 1\}$ and the Cauchy-Schwarz inequality, one has

$$\begin{aligned}&\max_{\substack{[A_m^v, B_m^v] \in \Omega_r, v \in \mathbb{D} \\ m = r_{k+i|k}, i \in \mathbb{N}}} |(\Delta u_{k+i|k})_l|^2 \\ &\leq \max \|(\Phi_{k+i|k}^v Q_m^{1/2})_l\|^2 \\ &= \max (\Phi_{k+i|k}^v Q_m (\Phi_{k+i|k}^v)^T)_{ll}\end{aligned} \tag{28}$$

where $\Phi_{k+i|k}^v = K_{m'_+ m_+, t_{i+1}}(A_m^v + B_m^v K_{m'm,t_r}) - K_{m'm,t_i}$, $m = r_{k+i|k}$, $m_+ = r_{k+i+1|k}$, $m' = r_{k_s - 1}$, $m'_+ = r_{k_{s_+} - 1}$, $t_i = k+i-k_s$, $t_{i+1} = k+i+1-k_{s_+}$, with $k+i \in [k_s, k_{s+1})$ and $k+i+1 \in [k_{s_+}, k_{s_+ + 1})$.

Indeed, there are only two possible cases, where $s_+ = s+1$ if $k+i+1$ is a switching instant in prediction horizon, and $s_+ = s$ otherwise. Note that (27) and (26) ensure

$$\Phi_{k+i|k}^v Q_m (\Phi_{k+i|k}^v)^T \leq U \tag{29}$$

for such two cases, respectively. With (19f) (28) and (29), it can be obtained that

$$\begin{aligned}\tilde{J}_{BT}(k) &= \max_{\substack{[A_m^v, B_m^v] \in \Omega_r, v \in \mathbb{D} \\ m = r_{k+i|k}, i \in \mathbb{N}}} \sum_l |(\Delta u_{k+i|k})_l|^2 \\ &\leq \max \sum_l (\Phi_{k+i|k}^v Q_m (\Phi_{k+i|k}^v)^T)_{ll} \\ &\leq trace(U) \leq \delta\end{aligned} \tag{30}$$

In a similar way, one can get from (19g) that

$$\begin{aligned}|(Ex_{k+i|k} + Fu_{k+i|k})_l|^2 &= |(\mathcal{D} Q_m^{1/2} Q_m^{-1/2} x)_l|^2 \\ &\leq (\mathcal{D} Q_m \mathcal{D}^T)_{ll} \leq (W)_{ll} \leq (d)_l^2\end{aligned} \tag{31}$$

where $\mathcal{D} = E + F K_{m'm,t_i}$. Thus, the requirements on the bumpless transfer performance and hard constraints are satisfied for the closed-loop system with gains $K_{r'r,t} = S_{r'r,t} Q_r^{-1}$. This completes the proof. ∎

Remark 3: In practice, the gains beyond the transition interval $[k_s, k_s + T_m)$ are set to be only dependent on the mode, i.e., $K_{r'r,T_r} = K_{r,T_r}$ for $r' \in \mathbb{M}, r' \neq r$, according to the fact that the optimal linear feedback law of the subsystem is unique. In specific, the matrix variables in optimization (32) satisfy $S_{r'r,T_r} = S_{r,T_r}$, so that the variables and constraints are reduced, where Theorem 1 still holds. Different from previous studies [13], the stabilizing gains for subsystems are not manually given but designed within the receding horizon optimization together with transitional gains.

B. Recursive Feasibility and Stability

Based on Theorem 1, the prediction gains $K(k+i|k)$ can be determined by solving the following optimization problem with weighted objective for minimizing LQ cost and bumpless transfer cost

$$\min_{\gamma, \delta, S_{r'r,t}, Q_r, U, W} \lambda_g \rho_k \gamma + \lambda_d \delta \quad \text{s.t. } (19) \tag{32}$$

where λ_g, λ_d are the positive weights. Then, the bumpless transfer MPC controller can be implemented by solving the SDP problem (32) and applying the first prediction control input $u_k = S_{r'_k r_k, t_k} Q_{r_k}^{-1} x_k$ by receding horizon. The recursive feasibility and stability based on the proposed MPC approach are summarized as the following theorem.

Theorem 2: Consider the stochastic switched uncertain system described by (1)-(3). If there exists a feasible solution for the optimization problem (32) at any switching instant k_s, then there will also exist a feasible solution at any time instant $k \geq k_s$, and the MPC controller based on (32) will stabilize the system in the mean-square sense.

Proof: (Recursive feasibility) Suppose that $\Upsilon^*(k) \triangleq \{\gamma^*(k), \delta^*(k), Q_r^*(k), S_{r'r,t}^*(k), U^*(k), W^*(k)\}$ is the optimal solution of the problem (32) at time instant k. In (32), only the constraint (19a) is dependent on the current state x_k and r_k, while the others are unchanged in the receding horizon optimization. Thus, once (19a) is satisfied for x_{k+1} and r_{k+1}, $\Upsilon^*(k)$ will be a feasible solution for the optimization problem (32) at $k+1$. The feasibility of $\Upsilon^*(k)$ indicates (19a) is satisfied at k and thus $V_m(k) < \alpha_m^{t_k}$. According to Remark 2, $\mathcal{E}_{r,t}$ is invariant within the running interval, and $\mathcal{E}_r$ is invariant at switching instant. Therefore, one has $x_{k+1}^T P_m x_{k+1} \leq \alpha_m x_k^T P_m x_k \leq \alpha_m^{t_{k+1}}$ and $x_{k+1}^T P_{m'} x_{k+1} \leq 1 \leq \alpha_m^{t_{k+1}}$ with $t_{k+1} = k+1-k_{s+1} = 0$, which implies that (19a) is satisfied for both running interval $k+1 \in [k_s, k_{s+1})$ and switching instants $k+1 = k_{s+1}$. Hence, $\Upsilon^*(k)$ is a feasible solution for instant $k+1$, and there will exist a feasible solution at any time instant $k+i \geq k$.

(Closed-loop stability) Since recursive feasibility of the optimization problem (32) is guaranteed based on the above analysis, $\Upsilon^*(k)$ is always a feasible solution at $k+1$. According to the optimality, one has

$$\begin{aligned}
&\lambda_g \rho_{k_{s+1}} \gamma^*(k_{s+1}) + \lambda_d \delta^*(k_{s+1}) \\
&\leq \lambda_g \rho_{k_{s+1}-1} \gamma^*(k_{s+1}-1) + \lambda_d \delta^*(k_{s+1}-1) \\
&\leq \lambda_g \rho_{k_s} \gamma^*(k_s) + \lambda_d \delta^*(k_s)
\end{aligned}$$

Further, according to Lemma 1 and Theorem 1, one has

$$\begin{aligned}
&\lambda_g \rho_{k_s} \gamma^*(k_s) + \lambda_d \delta^*(k_s) \\
&\quad \geq \mathbb{E}_{k_s} \left[\lambda_g \rho_{k_{s+1}} \gamma^*(k_{s+1}) + \lambda_d \delta^*(k_{s+1}) \right] \\
&\quad + \lambda_g \mathbb{E}_{k_s} \left[\sum_{k=k_s}^{k_{s+1}-1} \left(x_{k|k}^T \mathcal{Q} x_{k|k} + u_{k|k}^T \mathcal{R} u_{k|k} \right) \right]
\end{aligned}$$

Summing up both sides in a similar way to (18), the expectation of the infinite horizon LQ cost is bounded by

$$\begin{aligned}
&\mathbb{E}_0 \left[\sum_{k=0}^{\infty} \left(x_k^T \mathcal{Q} x_k + u_k^T \mathcal{R} u_k \right) \right] \\
&\leq \rho_0 \gamma^*(0) + \lambda_g^{-1} \lambda_d \delta^*(0) \\
&- \mathbb{E}_0 \left[\rho_\infty \gamma^*(\infty) + \lambda_g^{-1} \lambda_d \delta^*(\infty) \right]
\end{aligned}$$

In line with Lemma 1, it is straightforward that the actual system state satisfies $lim_{k\to\infty} \mathbb{E}_0 \left[\|x_k\|^2 \right] = 0$, so the closed-loop system with the MPC controller based on (32) is mean-square stable. ■

IV. Illustrative Example

A. System Model and Experimental Settings

In this section, a DC motor device system from [15] is utilized to validate the obtained results, where the mode switching is caused by abrupt failures on the power delivered to the shaft. The system is considered with three operation modes, including the normal ($r_k = 1$), low ($r_k = 2$), and medium ($r_k = 3$) power modes. The system state denoted by $x_k = [x_k(1), x_k(2), x_k(3)]^T$ includes the angular rate, the electrical current, and the integrator term. Then, the DC motor device with power failures can be modeled as the discrete-time switched system (1) with

$$A_i^v = \begin{bmatrix} a_i^v(11) & a_i^v(12) & 0 \\ a_i^v(21) & a_i^v(22) & 0 \\ a_i^v(31) & 0 & a_i^v(33) \end{bmatrix}, B_i^v = \begin{bmatrix} b_i^v(1) \\ b_i^v(2) \\ 0 \end{bmatrix}$$

where $v \in \{1, 2\}$ and the model parameters of the DC motor can be referred to [15]. The switching signal is with $\tau_m = 4$, $\bar{\tau}_m \in \mathbb{Z}_{[4,12]}$, $\mathcal{T}_m = [0.1, 0.05, 0.05, ..., 0.05, 0.55]$, so we set $\alpha_r = 0.97$ and $\mu_r = 1.5$ to ensure (11). The transition duration is set as $T_r = 2$. The state and control input are

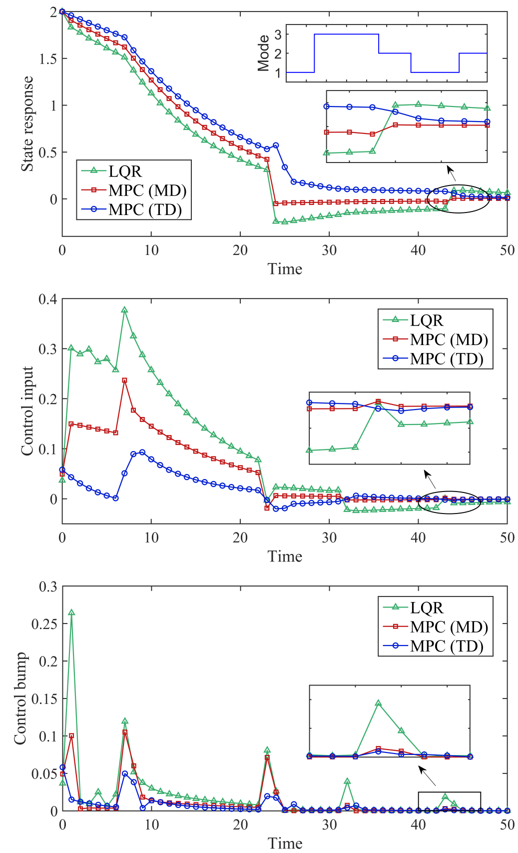

Fig. 1. Comparison of the state response, control input and control bump.

considered to be constrained by $|(x_k)_l| \leq 5$, $|(u_k)_l| \leq 1$, and the LQ cost is set with the weight matrices $\mathcal{Q} = I_3$ and $R = 1$. The initial values of the state and the mode are given by $x_0 = [2, 0.5, 0.5]^T$ and $r_0 = 1$. Based on above settings, the transition-dependent MPC is implemented by solving the SDP problem (32) in a receding horizon manner.

B. Control Performance Comparison

The state response, control input, and control bump of the closed-loop system with LQR, mode-dependent (MD) MPC, and transition-dependent (TD) MPC are shown in Fig. 1. The LQR controllers are designed for the subsystems separately, and the MD/TD MPC is designed for the switched system as detailed in Section IV-A. It can be seen from Fig. 1 that although LQR controllers achieve satisfactory stabilization performance within each subsystem, the system state may increase significantly at switching instants and thus exhibit the slowest convergence rate under stochastic switching ($x_{50}(1)$ of LQR is more than 6 times of the one of MPC). By considering the stochastic switching in the prediction horizon according to Lemma 1, the MPC controllers minimize the accumulative cost of all running intervals within different modes, thus exhibiting better performance. Nevertheless, limited by the mode-dependent paradigm, the MPC controller performs abrupt gain variation at switching instants and thus suffers from the control bumps. By comparison, the proposed transition-dependent MPC based on (32) inserts

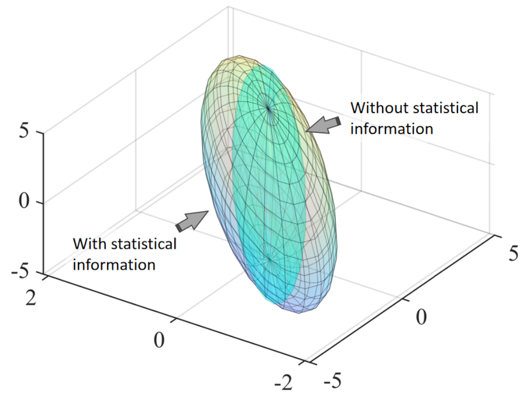

Fig. 2. Comparison of the feasible region of switched robust MPC with/without exploiting statistical information.

transitional stages at switching instants and simultaneously optimizes the LQ cost and bump cost, so that the control input can be changed gradually at the switching instants. This result validates the effectiveness of the proposed method in alleviating the bump phenomenon for switched systems.

C. Conservatism Comparison

It is expected that the obtained result can cover not only Markovian switching (see Remark 1) but also nondeterministic switching [12]. In the former case, the distribution of sojourn time follows the geometric distribution, i.e., $\bar{\tau}_r \sim \mathcal{T}_r$ with probability mass function $p_{\bar{\tau}_r}(\tau) = \Pi_{rr}^{\tau-1}(1-\Pi_{rr})$, where Π_{rr} is the diagonal element of the transition probabilities matrix Π_{rr}, standing for the probability of remaining the mode r. Since $\mathcal{T}_r$ can be obtained, the proposed MPC synthesis approach can be applied.

In the latter case, the switching signal is considered to be arbitrary and none statistical information can be obtained. Note that the only requirement of Lemma 1 on switching signals is the condition (11), which still holds if only conservative estimation of the sojourn time is accessible. For instance, if we only have a lower bound (may be conservative) of the sojourn time, i.e., $\hat{\tau}_m \leq \tau_m \leq h_{dt} + h_{ss}$, then the parameter can be set to satisfy $\mu_r \alpha_r^{\hat{\tau}_r} < 1$ and the condition (11) is consequently guaranteed due to $\alpha_r^{\tau_r - \hat{\tau}_r} \mathbb{E}_{\bar{\tau}_r \sim \mathcal{T}_r}[\alpha_r^{\bar{\tau}_r}] < 1$. Thereby, Lemma 1 and the subsequent results developed in Section III also hold. Note that $\hat{\tau}_m$ can be regarded as the mode-dependent dwell time (MDT), so the obtained results in stability analysis and MPC synthesis can cover the one for MDT switching. Moreover, compared with such results, our method reduces the conservatism in switched MPC synthesis by exploiting more statistical information, which may relax constraints and enlarge the invariant sets. Using the maximal ellipsoid approximation in [16], the feasible regions of switched MPC with/without statistical information are computed and shown in Fig. 2. The proposed MPC controller exploits the distribution of sojourn time and thus exhibits a significantly larger feasible region, which verifies the above discussion.

V. CONCLUSIONS

This study investigates the issue of bumpless transfer MPC synthesis for discrete-time stochastic switched systems, where the switching signal may follow a more general stochastic model other than the Markov chain. In concern of the bump phenomenon caused by the mode-dependent control law, the robust MPC is extended to a transition-dependent paradigm, where the feedback gains are obtained by online optimization of a weighted objective with LQ cost and control bump cost. It is illustrated that the obtained results can cover the previous studies on MPC with Markov switching or MDT switching and reduce conservatism. Also, the proposed MPC strategy can effectively alleviate the bump phenomenon without much detriment to performance.

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