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Arc-based Traffic Assignment: Equilibrium Characterization and Learning

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***Abstract*— Arc-based traffic assignment models (TAMs) are a popular framework for modeling traffic network congestion generated by self-interested travelers who sequentially select arcs based on their perceived latency on the network. However, existing arc-based TAMs either assign travelers to cyclic paths, or do not extend to networks with bidirectional arcs (edges) between nodes. To overcome these difficulties, we propose a new modeling framework for stochastic arc-based TAMs. Given a traffic network with bidirectional arcs, we replicate its arcs and nodes to construct a directed acyclic graph (DAG), which we call the *Condensed DAG* (CoDAG) representation. Self-interested travelers sequentially select arcs on the CoDAG representation to reach their destination. We show that the associated equilibrium flow, which we call the *Condensed DAG equilibrium*, exists, is unique, and can be characterized as a strictly convex optimization problem. Moreover, we propose a discrete-time dynamical system that captures a natural adaptation rule employed by self-interested travelers to learn about the emergent congestion on the network. We show that the arc flows generated by this adaptation rule converges to a neighborhood of Condensed DAG equilibrium. To our knowledge, our work is the first to study learning and adaptation in an arc-based TAM. Finally, we present numerical results that corroborate our theoretical results.**

I. INTRODUCTION

Traffic assignment models (TAMs) [1–7] play a central role in congestion modeling for transportation networks, by informing crucial decisions about infrastructure investment, capacity management, and tolling for congestion regulation. The central dogma behind this modeling approach is that self-interested travelers select routes with minimal *perceived* latency (i.e., the Wardrop or user equilibrium), which can be modeled as deterministic [1, 2] or stochastic [3–7]. Empirical studies confirm that stochastic TAMs achieve greater success at interpreting congestion levels, compared to their deterministic counterparts [8].

There exist two dominant modeling paradigms in TAM: the route-based model [1, 5, 7, 9]—where each traveler makes a single choice between set of available routes from origin to destination—and the arc (or edge) based model [3, 10–13]—where the traveler sequentially makes routing decision at each node on the network, based on their perception of arc latencies. There are two major drawbacks of route-based models on real-world networks: route correlation and route enumeration. Specifically, the utility generated from different routes is correlated due to overlapping arcs on different routes. Moreover, exhaustive route enumeration is prohibitive in terms of computational cost, memory storage, and information acquisition, since the number of routes in a traffic network can be exponential in the number of arcs.

To avoid explicit route enumeration, Akamatsu [6] proposed the first arc-based stochastic TAM, which was further generalized by Baillon and Cominetti [3]. More recently, Fosgerau et al. and Mai et al. [4, 12] presented similar arc-based models based on dynamic discrete choice analysis, which are mathematically similar to the models proposed by Akamatsu [6] and Baillon and Cominetti [3]. However, these models suggest that travelers take cyclic routes with positive probability. To overcome this fundamental modeling challenge, Oyama et al. [14, 15] recently proposed various methods to explicitly avoid routing on cyclic routes. Unfortunately, these methods either do not apply beyond acyclic graphs [15] or restrict the set of feasible routes, at the expense of modeling accuracy [14], or restrictive assumptions on cost structure [3]. Sequential arc selection models in network routing have also been studied by Calderone et al. [16, 17] where each arc selection is accompanied by stochastic transitions to the next arc, and a deterministic transition cost. This stands in contrast to the stochastic TAM literature, where transitions from arc to arc are assumed deterministic and the travel cost (latency) is assumed stochastic.

In this work, we propose an arc-based stochastic TAM that explicitly avoids cycles by considering routing on a directed acyclic graph derived from the original network, henceforth referred to as the *Condensed Directed Acyclic Graph* (CoDAG). The CoDAG representation duplicates an appropriate subset of nodes and arcs in the original network, to explicitly avoids cycles while preserving all feasible routes. Travelers sequentially select arcs on the CoDAG network at every intermediate node, based on perceived arc latencies. This route choice behavior is akin to the models prescribed by Akamatsu [6] and Baillon and Cominetti [3], but with routing occurring over the CoDAG associated with original network. We show that the corresponding equilibrium congestion pattern—which we term the *Condensed DAG equilibrium* (CoDAG equilibrium)—can be characterized as the unique minimizer of a strictly convex optimization problem.

Moreover, we propose a discrete-time dynamical system that captures a natural adaptation rule used by self-interested travelers who progressively learn towards equilibrium arc selections. In the game theory literature, an equilibrium notion is only considered useful if there exists an adaptive

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learning scheme that allows self interested players to converge to it [18]. Despite research progress on both theoretical and algorithmic aspects of stochastic arc-based TAMs, to the best of our knowledge, there has been no research on adaptive learning schemes that ensure convergence to such equilibria. Recently, adaptive learning schemes that converge to equilibria in route-based TAMs have been extensively studied [19–24], by considering self-interested travelers who repeatedly select routes by observing route latencies in past rounds of interaction. In this work, we extend this line of research to arc-based TAMs by proposing a discrete-time dynamics, in which in every round, travelers update arc selections at every node on the CoDAG network based on previous interactions. We prove that the emergent aggregate arc selection probabilities at every node (and the resulting congestion levels on each arc) globally asymptotically converge to a neighborhood of the CoDAG equilibrium.

To establish convergence, we appeal to the theory of stochastic approximation [25], which requires two conditions: (i) The vector field of the discrete-time dynamical system is Lipschitz, and (ii) The trajectories of an associated continuous-time dynamical system asymptotically converge to the CoDAG equilibrium. To prove (i), we establish recursive Lipschitz bounds for vector fields associated with every node. For (ii), we first construct a Lyapunov function using a strictly convex optimization objective associated with the CoDAG representation. We then show that the value of this Lyapunov function decreases along the trajectories of the continuous-time dynamical system. Our contributions are:

1) We introduce a new arc-based traffic equilibrium concept—the *Condensed DAG equilibrium*—which overcomes some limitations of existing traffic equilibrium notions. Furthermore, we show that the Condensed DAG equilibrium is characterized by a solution to a strictly convex optimization problem.
2) We present, to the best of our knowledge, the first adaptive learning scheme in the context of stochastic arc-based TAM. Furthermore, we establish formal convergence guarantees for this learning scheme.
3) We validate our theorems on a simulated traffic network.

The paper proceeds as follows. Section II introduces the setup considered in this paper, and defines the Condensed DAG representation. Section III defines the Condensed DAG equilibrium, and characterize it as a solution to a strictly convex optimization problem. Section IV presents discrete-time dynamics that converges to the Condensed DAG equilibrium and also provides a proof sketch. In Section V, we numerically study the convergence of the discrete-time dynamics on a simulated traffic network. Finally, Section VI presents concluding remarks and future work directions.

Notation: For each positive integer $n \in \mathbb{N}$, we denote $[n] := \{1, \cdots, n\}$. For each $i \in [n]$ in an Euclidean space $\mathbb{R}^n$, we denote by e_i the i-th standard unit vector.

II. CONDENSED DAG REPRESENTATION

A. Setup

Consider a traffic network represented by a directed graph $G_O = (I_O, A_O)$, possibly with bidirectional arcs, where I_O and A_O denote nodes and arcs, respectively. An example is depicted in Figure 1 (top left). Let the *origin nodes* and *destination nodes* be two disjoint subsets of nodes in G_O. Each traveler enters the network through an origin node to travel to a destination node, by sequentially selecting arcs at every intermediate node. This gives rise to congestion on each arc, which in turn decides the travel times. Specifically, each arc $\tilde{a} \in A_O$ is associated with a strictly increasing *latency function* $s_{\tilde{a}} : [0, \infty) \to [0, \infty)$, which gives for each arc the travel time as a function of traffic flow. To simplify our exposition, we assume that there is only one origin-destination tuple (o, d), although the results presented in this paper naturally extend to settings where the traffic network has multiple origin-destination pairs. We denote by g_o the demand of (infinitesimal) travelers who travel from the origin o to the destination d.

Remark 1: Arc selections made by travelers at different nodes are independent of one another. Therefore, if the underlying network has bidirectional edges, then sequential arc selection by a traveler can result in a cyclic route. For example, sequential arc selection in the original network shown on the top left in Figure 1 may lead a traveler to loop between i_2^O and i_3^O before reaching destination. To overcome this, we introduce a directed acyclic graph (DAG) representation of the original graph G_O in the following subsections, called the *condensed DAG*. Sequential arc selections made on this network encodes the travel history by design and therefore avoids cyclic routes.

B. Preliminaries on DAG: Depth and Height

Before introducing condensed DAG representation, we first present the notions of *height* and *depth* of a DAG. These concepts are crucial for the construction and analysis of condensed DAGs in the following sections. For the exposition in this subsection, let G be a DAG with a single origin-destination pair (o, d). Furthermore, let $\mathbf{R}$ be the set of all acyclic routes in G which start at the origin node o and end at the destination node d.

*Definition 1 (**Depth**):* For each $r \in \mathbf{R}$ and $a \in r$, let $\ell_{a,r}$ denote the location of arc a in route r, i.e., a is the $\ell_{a,r}$-th arc in the route r, and with a slight abuse of notation, define: $\ell_a := \max_{r \in \mathbf{R}: a \in r} \ell_{a,r}$, We say that a is an ℓ_a*-th depth arc* in the Condensed DAG G. Moreover, we define the *depth of a node* $i \in I \backslash \{o\}$ by:

$$\bar{\ell}_i := \max_{a \in A_i^-} \ell_a$$

with $\bar{\ell}_o = 0$.

*Definition 2 (**Height**):* For each $r \in \mathbf{R}$ and $a \in r$, let $m_{a,r}$ denote the location of arc a in route r, i.e., a is the $(|r| - m_{a,r})$-th arc in route r, and with a slight abuse of notation, define: $m_a := \max_{r \in \mathbf{R}: a \in r} m_{a,r}$. We say that a is an m_a*-th height arc* in the Condensed DAG G . Moreover, we define the *height of a node* $i \in I \backslash \{d\}$ by:

$$\bar{m}_i := \max_{a \in A_i^+} m_a$$

with $\bar{m}_d = 0$.

C. Construction of Condensed DAG

For ease of description, we illustrate the construction through an example in Figure 1. We also present a pseudo-code to generate the condensed DAG representation.

A straightforward way to associate G_O with a DAG would be to brute-force enumerate all acyclic (simple) routes and construct a tree network by replicating arcs and nodes by the number of routes passing through them (see Figure 1, bottom). However, the resulting tree network may contain a significantly larger number of arcs and nodes compared with the original network. To ameliorate this, we present the *condensed DAG* representation (Figure 1, top right). The condensed DAG is formed by merging superfluous nodes and arcs in the tree network, while ensuring that the graph remains acyclic, and preserving the set of acyclic routes from the original network.

TABLE I: Arc correspondences between the graphs in Figure 1: The original network (top left), fully expanded tree (bottom), and the CoDAG (top right).

Original	Tree DAG	CoDAG
a_1^O	$a_1^T, a_2^T, a_3^T, a_4^T, a_5^T$	a_1^C
a_2^O	$a_6^T, a_7^T, a_8^T, a_9^T, a_{10}^T$	a_2^C
a_3^O	a_{12}^T, a_{13}^T	a_4^C
a_4^O	$a_{18}^T, a_{19}^T, a_{20}^T$	a_7^C
a_5^O	$a_{14}^T, a_{15}^T, a_{23}^T, a_{24}^T$	a_5^C, a_9^C
a_6^O	$a_{16}^T, a_{17}^T, a_{21}^T, a_{22}^T$	a_6^C, a_8^C
a_7^O	a_{11}^T, a_{25}^T	a_3^C, a_{10}^C
a_8^O	$a_{26}^T, a_{28}^T, a_{30}^T, a_{32}^T$	a_{11}^C
a_9^O	$a_{27}^T, a_{29}^T, a_{31}^T, a_{33}^T$	a_{12}^C

One can design a condensed DAG representation as follows:

(S1) Convert the original network G_O to a tree structure $G_T = (I_T, A_T)$, in which every branch emanating from the origin represents a route. Each node and arc is replicated by the number of acyclic routes that contains it. For every node i in G_T, compute the depth $\bar{\ell}_i$ and height $\bar{m}_i$ (see Definition 1-2).

(S2) Generate a partition P_T of I_T such that:
  (i) For each $X \in P_T$, all nodes in X replicate the node in I_O that shares the same height or depth in G_T.
  (ii) For any $X, Y \in P_T$, there exists no $i, i' \in X, j, j' \in Y$, such that $\bar{m}_j > \bar{m}_i$ and $\bar{m}_{j'} < \bar{m}_{i'}$

(S3) For each set element X of P_T, merge all nodes in X into a single node. Then, merge arcs which have the same start and end nodes, and are replicas of the same edge in the original network G_O.

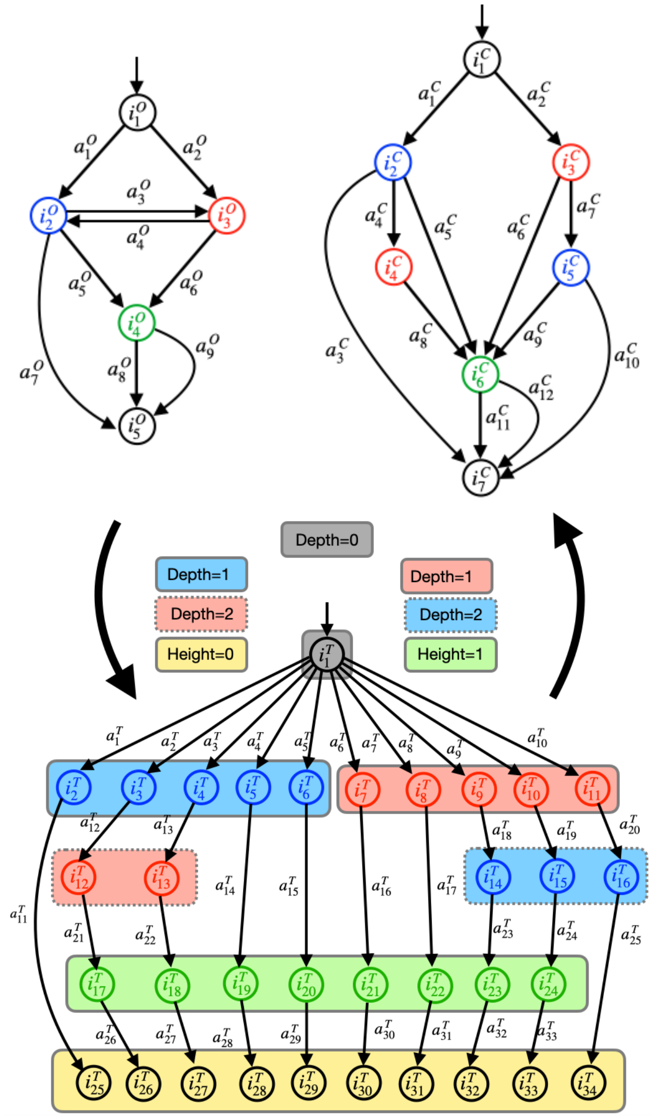

Fig. 1: Example of a single-origin single-destination original network G_O (top left, with superscript O), and its corresponding tree network (bottom, with superscript T) and condensed DAG G (top right, with superscript C). The blocks in G_T represent a partition P_T (see (S2)). The depth and height of nodes in every partition are denoted above G_T. Arc correspondences between the three networks are given by Table I, while node correspondences are indicated by color.

We refer to any graph generated via (S1)-(S3) as a *condensed DAG* (CoDAG) representation $G = (I, A)$ of the original network, where I and A are the set of nodes and arcs, respectively. By construction, the CoDAG representation explicitly avoids cyclic routes, and preserves all the acyclic routes from the original network. This is because the tree network constructed in (S1) preserves all acyclic routes from original network. Furthermore, the merging conditions stated in (S3) prohibit both the removal and the addition of routes.

Remark 2: A given traffic network with bidirectional arcs may yield several distinct CoDAG representations, any of which would be amenable to our analysis in subsequent sections. The development of an algorithmic procedure to compute a CoDAG with the least number of arcs or nodes is beyond the scope of this work.

Remark 3: The Condensed DAG representation G can be significantly smaller in size, compared to the tree network. There exist original networks whose corresponding tree representation G_T is exponentially larger than its corresponding CoDAG G. For example, consider a network with nodes $i_1, \cdots, i_n$, with two directed arcs connecting i_k to i_{k+1}, for each $k \in [n-1]$. Here, the corresponding tree network would have $2^n - 2$ arcs, while the CoDAG representation only has $2(n-1)$ arcs.

Remark 4: The arc-based TAM literature also considers modified representations of traffic networks with bidirectional arcs. For example, Oyama, Hara et al. [15, 26] consider the Network Generalized Extreme Value (NGEV) representations, which are similar to our CoDAG representation, but applies only to acyclic networks [15]. Thus, NGEV models cannot capture realistic traffic networks where almost all arcs are bidirectional. Meanwhile, Oyama, Hato et al. [14] consider the Choice Based Prism (CBP) representation, which prunes the available set of feasible routes to ameliorate computational inefficiency. While CBP explicitly avoids cyclic routes, it also removes some acyclic routes during the pruning process. In contrast, the CoDAG representation avoids this issue.

To conclude this section, we introduce some notation used throughout the rest of the paper. Recall that CoDAGs are formed by replicating the arcs in G_O. To describe this correspondence between arcs, we define $[\cdot] : A \to A_O$ to be a map from each CoDAG arc $a \in A$ to the corresponding arc $[a] \in A_O$. For each arc $a \in A$, let i_a and j_a denote the start and terminal nodes, and for each node $i \in I$, let $A_i^-, A_i^+ \subset A$ denote the set of incoming and outgoing arcs.

III. EQUILIBRIUM CHARACTERIZATION

In this section, we introduce the *condensed DAG (CoDAG) equilibrium* (Definition 3), which is based on the CoDAG representation of the original traffic network. Specifically, we show that the CoDAG equilibrium exists, is unique, and solves a strictly convex optimization problem (Theorem 1).

A. Condensed DAG Equilibrium

Below, we assume that every traveler knows G_O and has access to the same CoDAG representation of G_O. To avoid cyclic routes, we model travelers as performing sequential arc selection over the CoDAG representation $G = (I, A)$. The aggregate effect of the travelers' arc selections gives rise to the congestion on the network. Concretely, for each $a \in A$, let the *flow* or *congestion level* on arc a be denoted by w_a, and let the total flow on the corresponding arc in the original network be denoted, with a slight abuse of notation, by $w_{[a]} := \sum_{a' \in [a]} w_{a'}$. (Note that unlike existing TAMs, the latency of arcs in G can be coupled through the map $w_{[\cdot]}$, since multiple copies of the same arc in G_O may exist in G.) Then, the perceived latency of travelers on each arc $a \in A$ is described by:

$$\tilde{s}_{[a]}(w_{[a]}) := s_{[a]}(w_{[a]}) + \nu_a,$$

where ν_a is a zero-mean random variable. At each non-destination node $i \in I \backslash \{d\}$, travelers select among outgoing nodes $a \in A_i^+$ by comparing their perceived latencies-to-go $\tilde{z}_a : \mathbb{R}^{|A|} \to \mathbb{R}$, given recursively by:

$$\begin{aligned} \tilde{z}_a(w) &:= \tilde{s}_{[a]}(w_{[a]}) + \min_{a' \in A_{j_a}^+} \tilde{z}_{a'}(w), && j_a \neq d, \\ \tilde{z}_a(w) &:= \tilde{s}_{[a]}(w_{[a]}), && j_a = d. \end{aligned} \tag{1}$$

Consequently, the fraction of travelers who arrive at $i \in I \backslash \{d\}$ and choose arc $a \in A_i^+$ is given by:

$$P_{ij_a} := \mathbb{P}(\tilde{z}_a \leq \tilde{z}_{a'}, \forall a' \in A_i^+). \tag{2}$$

An explicit formula for the probabilities $\{P_{ij_a} : a \in A_i^+\}$ in terms of the statistics of $\tilde{z}_a$, is provided by the discrete-choice theory [27]. In particular, define $z_a(w) := \mathbb{E}[\tilde{z}_a(w)]$ and $\epsilon_a := \tilde{z}_a(w) - z_a(w)$, and define the latency-to-go at each node by:

$$\varphi_i(\{z_{a'}(w) : a' \in A_i^+\}) = \mathbb{E}\left[\min_{a' \in A_i^+} \tilde{z}_{a'}(w)\right]. \tag{3}$$

Then, from discrete-choice theory [27]:

$$P_{ij_a} = \frac{\partial \varphi_i(z)}{\partial z_a}, \quad i \in I \backslash \{d\}, a \in A_i^+, \tag{4}$$

where, with a slight abuse of notation, we write $\varphi_i(z)$ for $\varphi_i(\{z_{a'} : a' \in A_i^+\})$. To obtain a closed form expression of φ, this work considers the *logit Markovian model* [3, 6], which assumes that the zero-mean noise ϵ is Gumbel-distributed with scale $\beta > 0$. Intuitively, $\beta > 0$ is an entropy parameter that models the degree to which the average traveler's perception of network latency is suboptimal. In this case, the corresponding latency-to-go at each node i in G is:

$$\varphi_i(z) = -\frac{1}{\beta} \ln\left(\sum_{a' \in A_i^+} e^{-\beta z_{a'}}\right). \tag{5}$$

Using (1) and (5), the expected minimum latency-to-go $z_a : \mathbb{R}^{|A|} \to \mathbb{R}$, associated with traveling on each arc $a \in A$, is given by:

$$z_a(w) = s_{[a]}\left(\sum_{\bar{a} \in [a]} w_{\bar{a}}\right) - \frac{1}{\beta} \ln\left(\sum_{a' \in A_{j_a}^+} e^{-\beta z_{a'}(w)}\right). \tag{6}$$

Note that (6) is well-posed, as z_a can be recursively computed along arcs of increasing height (Definition 2) from the destination back to the origin. For more details, please see Appendix B [28].

Against the preceding backdrop, we formally define the central equilibrium solution concept studied in this paper: the Condensed DAG Equilibrium (CoDAG Equilibrium).

*Definition 3 (**Condensed DAG Equilibrium**):* Given $\beta > 0$, a vector of arc-flow $\bar{w}^\beta \in \mathbb{R}^{|A|}$ is called a *Condensed DAG equilibrium* if, for each $i \in I \backslash \{d\}$, $a \in A_i^+$:

$$\bar{w}_a^\beta = \left(g_i + \sum_{a' \in A_i^+} \bar{w}_{a'}^\beta\right) \cdot \frac{\exp(-\beta z_a(\bar{w}^\beta))}{\sum_{a' \in A_{i_a}^+} \exp(-\beta z_{a'}(\bar{w}^\beta))}, \tag{7}$$

where $g_i = g_o$ if $i = o$, $g_i = 0$ otherwise, and $w \in \mathcal{W}$, with:

$$\mathcal{W} := \Bigg\{ \bar{w}^\beta \in \mathbb{R}^{|A|} : \sum_{a \in A_i^+} \bar{w}_a^\beta = \sum_{a \in A_i^-} \bar{w}_a^\beta, \forall\, i \neq o, d, \quad (8)$$

$$\sum_{a \in A_o^+} \bar{w}_a^\beta = g_o,\ \bar{w}_a^\beta \geq 0, \forall\, a \in A \Bigg\}.$$

For any CoDAG equilibrium $\bar{w}^\beta$, the fraction of travelers at any node $i \in I\backslash\{d\}$ who selects an arc $a \in A_i^+$ is:

$$\bar{\xi}_a^\beta := \frac{\bar{w}_a^\beta}{\sum_{a' \in A_i^+} \bar{w}_{a'}^\beta}.$$

Remark 5: Essentially, at the CoDAG equilibrium, the traveler population at each intermediate node $i \in I\backslash\{d\}$ (with total flow $g_i + \sum_{a' \in A} w_{a'}$) select from outgoing arcs by comparing their costs-to-go using the softmax function. While the CoDAG equilibrium and Markovian Traffic Equilibrium (MTE) share some similarities (see [3]), there also exist two main fundamental differences. First, by design, the CoDAG equilibrium does not yield cyclic routes with strictly positive probability (as is the case with the MTE). Second, unlike the MTE, congestion levels on arcs (which may be replicas of the same arc in G_O) in the CoDAG representation are coupled to each other. Therefore, MTE analysis does not extend straightforwardly to the CoDAG equilibrium.

B. Existence and Uniqueness of the CoDAG equilibrium

In this subsection, we show the existence and uniqueness of the CoDAG equilibrium, by characterizing it as the unique minimizer of a strictly convex optimization problem over a compact set. First, for each $[a] \in A_O$, define:

$$f_{[a]}(w) := \int_0^{w_{[a]}} s_{[a]}(u)\, du, \tag{9}$$

and for each $i \in I\backslash\{d\}$, set:

$$\chi_i(w_{A_i^+}) := \sum_{a \in A_i^+} w_a \ln w_a - \left(\sum_{a \in A_i^+} w_a \right) \ln \left(\sum_{a \in A_i^+} w_a \right). \tag{10}$$

Finally, define $F : \mathcal{W} \to \mathbb{R}$ by:

$$F(w) = \sum_{[a] \in A_O} f_{[a]}(w) + \frac{1}{\beta} \sum_{i \neq d} \chi_i(w_{A_i^+}), \tag{11}$$

where $w_{A_i^+} \in \mathbb{R}^{|A_i^+|}$ denotes the components of w corresponding to arcs in A_i^+.

Theorem 1: The CoDAG equilibrium $\bar{w}^\beta \in \mathcal{W}$ exists, is unique, and is the unique minimizer of F over $\mathcal{W}$.

To prove Theorem 1, we first show that $F(\cdot)$ is strictly convex over $\mathcal{W}$ (Lemma 1), so F has a unique minimizer in $\mathcal{W}$. It then suffices to show that the CoDAG equilibrium definition (Definition 3) matches the Karush-Kuhn-Tucker (KKT) conditions for the optimization problem (11).

Lemma 1: The map $F : \mathcal{W} \to \mathbb{R}$ is strictly convex.

Proof: (**Proof Sketch**) It suffices to show that $f_{[a]}$ and χ_i are convex for each $[a] \in A_O$, $i \in I\backslash\{d\}$. Each $f_{[a]}$ is convex, since it is the composition of a convex function ($w \mapsto \sum_{a \in A_O} \int_0^{w_a} s_a(u) du$) with a linear function ($w_{[a]} := \sum_{a' \in [a]} w_{a'}$). Furthermore, we establish that for any $i \in I\backslash\{d\}$, $y_i \in \mathbb{R}^{|A_i^+|}$:

$$y_i^\top \nabla_w^2 \chi_i(w) y_i \geq 0,$$

where the equality holds if and only if y_i and $w_{A_i^+}$ are scalar multiples of one another. Strict convexity then follows by a contradiction argument showing that there exists at least one node $i \in I\backslash\{d\}$ such that $y_i^\top \nabla_w^2 \chi_i(w) y_i > 0$. ■

IV. LEARNING DYNAMICS

In this section, we propose a discrete-time dynamical system (PBR) which captures travelers' preferences for minimizing total travel time, as well as their perception uncertainties, while simultaneously learning about the emergent congestion on the network.

We leverage the constant step-size stochastic approximation theory to prove that these discrete-time dynamics converge to a neighborhood of the CoDAG equilibrium (Theorem 2). To this end, we first prove that the continuous-time counterpart to (PBR) globally asymptotically converges to the CoDAG equilibrium (Lemma 2). We then conclude the proof by verifying technical assumptions required to invoke results in stochastic approximation theory [25] (Lemma 3).

A. Discrete-time Dynamics

In this subsection, we present a discrete-time dynamical equation that captures the evolution of flows on the network as a result of learning and adaptation by self-interested travelers. More formally, at each discrete time step $n \geq 0$, g_o units of travelers arrive at the origin node o. At time step n, every traveler who reaches node $i \in I\backslash\{d\}$ selects some arc $a \in A_i^+$. For any $i \in I\backslash\{d\}, a \in A_i^+$, let $\xi_a[n]$ be the *aggregate arc selection probability*: the fraction of travelers at node i choosing arc a at time n. As a result of the arc selections made by every traveler, a flow of $W[n]$ is induced on the arcs as given below. For every $a \in A$:

$$W_a[n] = \left(g_{i_a} + \sum_{a' \in A_{i_a}^-} W_{a'}[n] \right) \cdot \xi_a[n], \tag{12}$$

where, as given in Definition 3, $g_{i_a} = g_o$ if $i_a = o$, and $g_{i_a} = 0$ otherwise.

At the end of each time step, every traveler reaches the destination and observes a noisy estimate of the latency-to-go, independent across travelers, on every arc in the network (including ones they did not visit in that time step). Note that the latency-to-go for any arc is dependent on the congestion $W[n]$, which in turn depends on aggregate decisions taken by travelers (please refer to (12)). Based on the observed latencies, at time $n+1$, at every non-destination node $i \in I\backslash\{d\}$, a $\eta_i[n+1] \cdot K_i$ fraction of travelers at node i switches to an arc with the minimum observed latency-to-go. Meanwhile, a $1 - \eta_i[n+1] \cdot K_i$ fraction of travelers selects the same arc they selected at time step n. We assume that $\{\eta_i[n+1] \in \mathbb{R} : i \in I, n \geq 0\}$ are independent bounded random variables in $[\underline{\mu}, \overline{\mu}]$, independent of travelers' perception stochasticities, with $0 < \underline{\mu} < \mu < \overline{\mu} < 1/\max\{K_i : i \in I\backslash\{d\}\}$ and $\mathbb{E}[\eta_{i_a}[n+1]] = \mu$ for each node index $i \in I$ and discrete time index $n \geq 0$. Meanwhile, the constants K_i represent node-dependent update rates. To summarize, the dynamic evolution of arc selections by infinitesimal travelers is captured by the following evolution of $\xi[n]$. For every $i \in I\backslash\{d\}, a \in A_i^+$:

$$\xi_a[n+1] = \xi_a[n] + \eta_{i_a}[n+1] \cdot K_{i_a} \left(-\xi_a[n] + P_{ij_a}\right),$$

where P_{ij_a} is defined in (2). Using (4) and (5), the previous equation can be rewritten as:

$$\begin{aligned} &\xi_a[n+1] \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(PBR)}\\ =&\xi_a[n] + \eta_{i_a}[n+1] \cdot K_{i_a} \\ &\cdot \left(-\xi_a[n] + \frac{\exp(-\beta\big[z_a(W[n])\big])}{\sum_{a' \in A_{i_a}^+} \exp(-\beta\big[z_{a'}(W[n])\big])} \right), \end{aligned}$$

The dynamics (PBR) bears close resemblance to perturbed best response dynamics in routing games [23], so we shall refer to (PBR) as *perturbed best response* dynamics.

We assume $\xi_a[0] > 0$ for each $a \in A$, i.e., each arc has some strictly positive initial traffic flow. This is reasonable, since the stochasticity in travelers' perception of network congestion ensures that each arc has a nonzero probability of being selected.

B. Convergence Results

Our main theorem establishes that the discrete-time dynamics (PBR) asymptotically converges to a neighborhood of the CoDAG equilibrium $\bar{w}^\beta$.

Theorem 2: Under the discrete-time flow dynamics (PBR), for each $\delta > 0$:

$$\limsup_{n\to\infty} \mathbb{E}\big[\|\xi[n] - \bar{\xi}^\beta\|_2^2\big] \leq O(\mu),$$
$$\limsup_{n\to\infty} \mathbb{P}\big(\|\xi[n] - \bar{\xi}^\beta\|_2^2 \geq \delta\big) \leq O\left(\frac{\mu}{\delta}\right).$$

To prove Theorem 2, we leverage the theory of constant step-size stochastic approximation [25]. This requires proving that the continuous-time dynamics corresponding to the discrete-time update (PBR), presented below, converges to the CoDAG equilibrium. For each arc $a \in A$:

$$\dot{\xi}_a(t) = -K_i \left(\xi_a(t) + \frac{\exp(-\beta \cdot z_a(w(t)))}{\sum_{a' \in A_{i_a}^+} \exp(-\beta \cdot z_{a'}(w(t)))} \right), \tag{13}$$

where $w(t)$ is the resulting arc flow associated with the arc selection probability $\xi(t)$, similar to (12):

$$w_a(t) = \xi_a(t) \cdot \left(g_{i_a} + \sum_{a' \in A_{i_a}^-} w_{a'}(t) \right). \tag{14}$$

*Lemma 2 (**Informal**):* Suppose $w(0) \in \mathcal{W}$, i.e., the initial flow satisfies flow continuity. Under the continuous-time flow dynamics (13) and (14), if $K_i \ll K_{i'}$ whenever $\ell_i < \ell_{i'}$, the traffic flow $w(t)$ globally asymptotically converges to the CoDAG equilibrium $\bar{w}^\beta$.

Proof: (**Proof Sketch**) Recall that $\bar{w}^\beta$ is the unique minimizer of the map $F : \mathcal{W} \to \mathbb{R}$, defined by (11). We show that F is a Lyapunov function for the continuous-time flow dynamics (20) induced by the arc selection dynamics (13). To this end, we first unwind the dynamics (13) and (14) to obtain the recursive relation:

$$\begin{aligned} \dot{w}_a(t) = &-K_{i_a} \left(1 - \frac{1}{K_{i_a}} \cdot \frac{\sum_{a' \in A_{i_a}^-} \dot{w}_{a'}(t)}{\sum_{\hat{a} \in A_{i_a}^+} w_{\hat{a}}(t)} \right) w_a(t) \\ &+ K_{i_a} \cdot \sum_{a' \in A_{i_a}^-} w_{a'}(t) \cdot \frac{\exp(-\beta z_a(w(t)))}{\sum_{a' \in A_{i_a}^+} \exp(-\beta z_{a'}(w(t)))}. \end{aligned}$$

Then, we establish that along any trajectory starting on $\mathcal{W}$ and following the dynamics given by (13), we have for each $t \geq 0$:

$$\dot{F}(t) = \dot{w}(t)^\top \nabla_w F(w(t)) \leq 0.$$

The proof then follows from LaSalle's Theorem (see [29, Proposition 5.22]). For a precise characterization and detailed proof of Lemma 2, please see Appendix C [28]. ■

Remark 6: On a technical level, the statement and proof technique of Theorem 2 share similarities with methods used to establish the convergence of best-response dynamics in potential games [23]. However, there exist crucial distinctions between the two approaches which render our problem more difficult. First, since the map F defined by (11) is not a potential function, the mathematical machinery of potential games cannot be directly applied. Moreover, the continuous-time flow dynamics (13) and (14) allow couplings between arbitrary arcs in the CoDAG. For more details, please see Appendix C [28].

Remark 7: The assumption that $K_i \ll K_{i'}$ whenever the depth of node $i \in I\backslash\{d\}$ is less than the depth of node $i' \in I\backslash\{d\}$ conforms to the intuition that travelers farther away from the destination node face more complex route selection decisions based on more information regarding traffic flow throughout the rest of the network, and thus perform slower updates.

Having established the global asymptotic convergence of the continuous-time dynamics (13) and (14) to the CoDAG equilibrium $\bar{w}^\beta$, it remains to verify the remaining technical conditions necessary to prove Theorem 2 via stochastic approximation theory. To this end, we rewrite the discrete ξ-dynamics (PBR) as a Markov process with a martingale difference term:

$$\xi_a[n+1] = \xi_a[n] + \mu\big(\rho_a(\xi[n]) + M_a[n+1]\big),$$

where $\rho_a : \mathbb{R}^{|A|} \times \mathbb{R}^{|A_O|} \to \mathbb{R}^{|A|}$ is given by:

$$\rho_a(\xi) := K_{i_a} \left(-\xi_a + \frac{\exp(-\beta \cdot z_a(w))}{\sum_{a' \in A_{i_a}^+} \exp(-\beta \cdot z_{a'}(w))} \right), \tag{15}$$

with $w \in \mathbb{R}^{|A|}$ defined arc-wise by $w_a = (g_{i_a} + \sum_{\hat{a} \in A_{i_a}^-} w_{a'}) \cdot \xi_a$, and:

$$M_a[n+1] := \left(\frac{1}{\mu} \eta_{i_a}[n+1] - 1 \right) \cdot \rho_a(\xi[n]). \tag{16}$$

TABLE II: Parameters for simulation.

Notation	Default value
k_0	0, 1, 0, 1, 1, 0, 1, 1, 1 (ordered by edge index)
k_1	2, 1, 1, 1, 1, 1, 2, 2, 2 (ordered by edge index)
g_1	1
β	10
$\eta_{i_a}[n]$	Uniform(0, 0.1), $\forall a \in A, i \in I\backslash\{d\}$

Here, $W_a[n] = \big(g_{i_a} + \sum_{a' \in A^-_{i_a}} W_{a'}[n]\big)$, as given by (12).

The following lemma bounds the magnitude of the discrete-time flow $W[n] \in \mathbb{R}^{|A|}$ and the martingale difference terms $M[n] \in \mathbb{R}^{|A|}$.

Lemma 3: Given initial flows $W[0]$ and arc selection probabilities $\xi[0]$:

1) For each $a \in A$: $\{M_a[n+1] : n \geq 0\}$ is a martingale difference sequence with respect to the filtration $\mathcal{F}_n := \sigma\big(\cup_{a \in A}\,(W_a[1], \xi[1], \cdots, W_a[n], \xi[n])\big)$.
2) There exist $C_w, C_m > 0$ such that, for each $a \in A$, $n \geq 0$, we have $W_a[n] \in [C_w, g_o]$ and $|M_a[n]| \leq C_m$.
3) For each $a \in A$, the map ρ_a, given by (15), is Lipschitz continuous over the range of realizable flow and arc selection probability trajectories $\{W[n] : n \geq 0\}$ and $\{\xi[n] : n \geq 0\}$.

Proof: (**Proof Sketch**) The first part of Lemma 3 follows because, with respect to $\mathcal{F}_n$, the only stochasticity in $M_a[n+1]$ originates from the i.i.d. input flows $\eta_{i_a}[n+1]$. The second part follows by invoking the flow continuity equations in (12) to recursively upper bound each $W_a[n]$ and $z_a(W[n])$, in increasing order of depth and height, respectively (flows are propagated from origin to destination, and latency-to-go values are computed in the opposite direction). These bounds are then used to recursively establish upper and lower bounds for each $\xi_a[n]$, and consequently each $W[n]$, in order of increasing depth. Finally, the Lipschitz continuity of each ρ_a can be proved by establishing that ρ_a is continuously differentiable, with bounded derivatives over the compact domain defined by the bounds on $W[n]$ established in the second part of the lemma. For details, please see the proofs of Lemmas 5 and 6 in Appendix C [28]. ■

V. EXPERIMENT RESULTS

In this section, we conduct numerical experiments to validate the theoretical analysis presented in Section IV. We show in simulation that, under (PBR), the traffic flows converge to a neighborhood of the condensed DAG equilibrium, as claimed by Theorem 2.

Consider the network presented in Figure 1, with affine edge-latency functions $s_{[a]}(w_{[a]}) = k_0 + k_1 w_{[a]}$ for each arc $a \in A$, where $k_0, k_1 > 0$ are simulation parameters provided in Table II. To validate Theorem 2, we evaluate and plot the traffic flow values $W_a[n]$ on each arc $a \in A$ and discrete time $n \geq 0$. Figure 2 presents traffic flow values at the condensed DAG equilibrium (i.e., w^β) for the original network and condensed DAG. While travelers generally prefer routes of lower latency, each route has a nonzero level of traffic flow at equilibrium. The reason is that under the perturbed best response dynamics, users do not allocate all the traffic

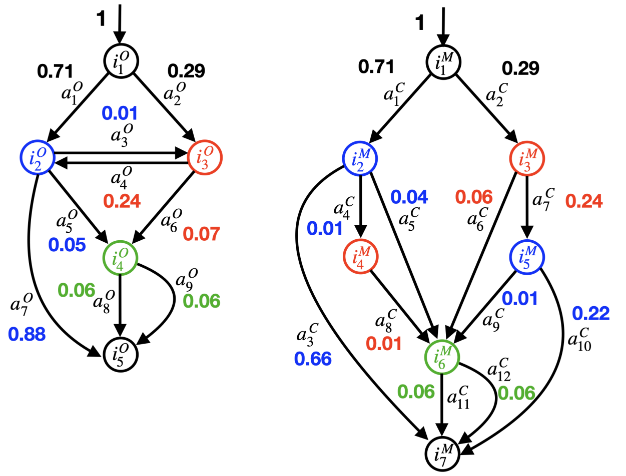

Fig. 2: Steady state traffic flow on each arc for an original network and condensed DAG. Flows on arcs emerging from same node are represented in same color.

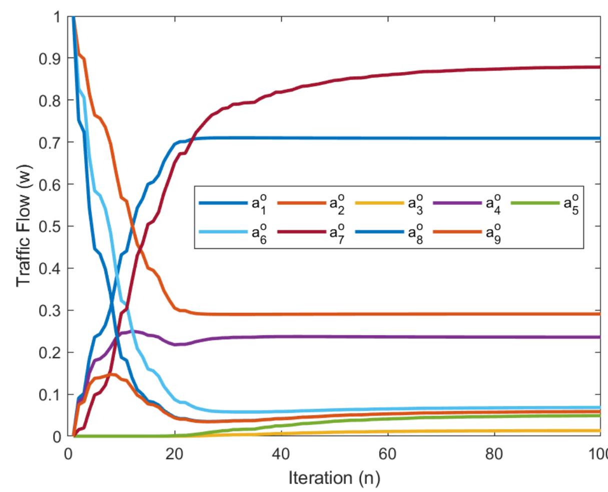

Fig. 3: Traffic flow $W[n]$ for the network in Fig. 2.

flow to the minimum-cost route, but instead distribute their traffic allocation more evenly. Meanwhile, Figure 3 illustrates that w converges to the condensed DAG equilibrium in approximately 100 iterations with some initial fluctuations. The fluctuations are due to the magnitude of the average step-size μ. If μ is small, the discrete-time update is close to the continuous-time dynamics, and the resulting evolution of the traffic flow follows a smoother trend. Note that in practice, flow convergence to the CoDAG equilibrium occurs even when the effects of the constants $\{K_i : i \in I\}$ are ignored, i.e., when each K_i is set to unity.

VI. CONCLUSION AND FUTURE WORK

We present a new equilibrium concept for stochastic arc-based TAMs in which travelers are guaranteed to be routed on acyclic routes. Specifically, we construct a condensed DAG representation of the original network, by replicating arcs and nodes to avoid cyclic routes, while preserving the set of feasible routes from the original network. We characterize the proposed equilibrium as the optimal solution of a strictly convex optimization problem. Furthermore, we propose adaptive learning dynamics for arc-based TAM that characterizes the evolution of flow generated by the simultaneous learning and adaptation of self-interested travelers. Additionally, we prove that the learning dynamics converges to the corresponding equilibrium flow allocation.

Interesting avenues of future research include: (i) Developing an equilibrium notion and corresponding convergent learning dynamics, for the case in which travelers can only access latency-to-go values on the routes they choose; and (ii) Developing dynamic tolling mechanisms to properly allocate equilibrium flows to induce socially optimal loads.

References

[1] Roberto Cominetti, Francisco Facchinei, and Jean B Lasserre. *Modern Optimization Modeling Techniques*. Springer Science & Business Media, 2012.

[2] John Glen Wardrop. "Some Theoretical Aspects of Road Traffic Research." In: *Proceedings of the institution of civil engineers* 1.3 (1952), pp. 325–362.

[3] Jean-Bernard Baillon and Roberto Cominetti. "Markovian Traffic Equilibrium". In: *Mathematical Programming* (Feb. 2008).

[4] Mogens Fosgerau, Emma Frejinger, and Anders Karlstrom. "A Link-based Network Route Choice Model with Unrestricted Choice Set". In: *Transportation Research Part B: Methodological* 56 (2013), pp. 70–80.

[5] Carlos F Daganzo and Yosef Sheffi. "On Stochastic Models of Traffic Assignment". In: *Transportation science* 11.3 (1977), pp. 253–274.

[6] Takashi Akamatsu. "Decomposition of Path Choice Entropy in General Transport Networks". In: *Transportation Science* 31.4 (Nov. 1997), pp. 349–362.

[7] Robert B. Dial. "A Probabilistic Multipath Traffic Assignment Model which Obviates Path Enumeration". In: *Transportation Research* 5.2 (1971), pp. 83–111. ISSN: 0041-1647.

[8] Yosef Sheffi and Warren Powell. "A Comparison of Stochastic and Deterministic Traffic Assignment over Congested Networks". In: *Transportation Research Part B: Methodological* 15.1 (1981), pp. 53–64.

[9] Tetsuo Yai, Seiji Iwakura, and Shigeru Morichi. "Multinomial Probit with Structured Covariance for Route Choice Behavior". In: *Transportation Research Part B: Methodological* 31.3 (1997), pp. 195–207.

[10] Maëlle Zimmermann and Emma Frejinger. "A Tutorial on Recursive Models for Analyzing and Predicting Path Choice Behavior". In: *EURO Journal on Transportation and Logistics* 9.2 (2020), p. 100004.

[11] Yuki Oyama and Eiji Hato. "A Discounted Recursive Logit Model for Dynamic Gridlock Network Analysis". In: *Transportation Research Part C: Emerging Technologies* 85 (2017), pp. 509–527.

[12] Tien Mai, Mogens Fosgerau, and Emma Frejinger. "A Nested Recursive Logit Model for Route Choice Analysis". In: *Transportation Research Part B: Methodological* 75 (2015), pp. 100–112.

[13] Tien Mai. "A Method of Integrating Correlation Structures for a Generalized Recursive Route Choice Model". In: *Transportation Research Part B: Methodological* 93 (2016), pp. 146–161.

[14] Yuki Oyama and Eiji Hato. "Prism-based Path Set Restriction for Solving Markovian Traffic Assignment Problem". In: *Transportation Research Part B: Methodological* 122 (2019), pp. 528–546.

[15] Yuki Oyama, Yusuke Hara, and Takashi Akamatsu. "Markovian Traffic Equilibrium Assignment Based on Network Generalized Extreme Value Model". In: *Transportation Research Part B: Methodological* 155 (2022), pp. 135–159.

[16] Dan Calderone and S Shankar Sastry. "Markov Decision Process Routing Games". In: *Proceedings of the 8th International Conference on Cyber-Physical Systems*. 2017, pp. 273–279.

[17] Dan Calderone and Shankar Sastry. "Infinite-horizon Average-cost Markov Decision Process Routing Games". In: *2017 IEEE 20th International Conference on Intelligent Transportation Systems (ITSC)*. IEEE. 2017, pp. 1–6.

[18] Drew Fudenberg and David K Levine. *The Theory of Learning in Games*. Vol. 2. MIT press, 1998.

[19] Walid Krichene, Benjamin Drighes, and Alexandre Bayen. "On the Convergence of no-Regret Learning in Selfish Routing". In: *International Conference on Machine Learning*. PMLR. 2014, pp. 163–171.

[20] Syrine Krichene, Walid Krichene, Roy Dong, and Alexandre Bayen. "Convergence of Heterogeneous Distributed Learning in Stochastic Routing Games". In: *2015 53rd Annual Allerton Conference on Communication, Control, and Computing (Allerton)*. IEEE. 2015, pp. 480–487.

[21] Robert Kleinberg, Georgios Piliouras, and Éva Tardos. "Multiplicative Updates Outperform Generic no-Regret Learning in Congestion Games". In: *Proceedings of the forty-first annual ACM symposium on Theory of computing*. 2009, pp. 533–542.

[22] Chinmay Maheshwari, Kshitij Kulkarni, Manxi Wu, and S. Shankar Sastry. "Dynamic Tolling for Inducing Socially Optimal Traffic Loads". In: *2022 American Control Conference (ACC)*. 2022, pp. 4601–4607.

[23] William H. Sandholm. *Population Games And Evolutionary Dynamics*. Economic Learning and Social Evolution, 2010.

[24] Emily Meigs, Francesca Parise, and Asuman Ozdaglar. "Learning Dynamics in Stochastic Routing Games". In: *2017 55th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*. IEEE. 2017, pp. 259–266.

[25] Vivek Borkar. *Stochastic Approximation: A Dynamical Systems Viewpoint*. Cambridge University Press, 2008.

[26] Andrea Papola and Vittorio Marzano. "A Network Generalized Extreme Value Model for Route Choice Allowing Implicit Route Enumeration". In: *Computer-Aided Civil and Infrastructure Engineering* 28.8 (2013), pp. 560–580.

[27] M. E. Ben-Akiva. *Discrete Choice Analysis: Theory and Application to Travel Demand*. Cambridge: MIT Press, 1985.

[28] Chih-Yuan Chiu, Chinmay Maheshwari, Pan-Yang Su, and Shankar Sastry. "Arc-based Traffic Assignment: Equilibrium Characterization and Learning". In: *arXiv* (2023).

[29] Shankar Sastry. *Nonlinear Systems: Analysis, Stability, and Control*. Springer, 1999.

Appendix

Please use the following link to access an ArXiv version with the appendix [28] (`https://arxiv.org/pdf/2304.04705.pdf`).