2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Computing the L_1-Induced Norm of Sampled-data Systems

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***Abstract*— This paper is concerned with developing a method for computing the L_1-induced norm of sampled-data systems. We first derive an operator-based form of the L_1-induced norm in the lifted representation of sampled-data systems. The corresponding operators are further considered on the top of the fast-lifted treatment, in which the sampling interval $[0, h)$ is divided into M subintervals with an equal width. This treatment allows us to develop a piecewise constant approximation of the input and output signals of sampled-data systems, by which an upper bound and a lower bound on the L_1-induced norm can be obtained. The gap between these bounds is shown to converge to 0 at the rate of $1/M$ with the fast-lifting parameter M.**

I. Introduction

As evaluating the effects of disturbances on the outputs has been regarded as one of the most important issues in control engineering, there have been a number of studies on computing various system norms. For instance, the L_2-induced norm is employed in [1]–[5] to deal with energy-bounded disturbances while the L_∞-induced norm is used in [6]–[14] for tackling magnitude-bounded disturbances. However, they do not fit into some practical problems such as the fuel efficiency maximization [15], the population management [16], [17], and so on. For these problems, the L_1-induced norm can be taken since it corresponds to the total amount of the output. This induced norm has been also extensively used for various systems such as positive systems [18], switched systems [19], [20], Markov jump systems [21], and so on.

The issue of computing system norms is important also in sampled-data systems taking into account their inter-sample behavior. In connection with this, similarly to the above case of continuous-time systems, there are various studies on computing the L_2-induced norm [22]–[26] and the L_∞-induced norm [27]–[33] for sampled-data systems. However, no computation procedure to the L_1-induced norm of sampled-data systems is discussed, although an operator-based formula for the induced norm is introduced in [34].

Motivated by this fact, we are concerned with developing a method for computing the L_1-induced norm of sampled-data systems. As a preliminary step to establish such a computation procedure, we introduce a tractable form of the L_1-induced norm by applying the lifting technique [35]–[37] to sampled-data systems. More precisely, the L_1-induced norm of sampled-data systems can be regarded as the l_1-induced norm of operator-based discrete-time systems. To derive computable upper and lower bounds on the L_1-induced norm, the tractable form is also re-interpreted in the fast-lifted representation [38] of sampled-data systems, by which the sampling interval $[0, h)$ is divided into M subintervals of with an equal width, where M is the fast-lifting parameter. To put it another way, we develop constant approximation approaches the corresponding input and output operators on such smaller intervals, and this leads to a piecewise constant approximation of the input and output signals of sampled-data systems. Based on the piecewise constant approximation, we can obtain an upper bound and a lower bound on the L_1-induced norm of sampled-data systems whenever M is fixed. Furthermore, the efficacy of taking M large enough for computing the L_1-induced norm is validated by showing that the gap between the upper and lower bounds converges to 0 at the rate of $1/M$.

This paper is organized as follows. An operator-based representation of sampled-data systems is introduced in Section II by taking the lifting technique [35]–[37]. The L_1-induced norm of sampled-data systems is described by a tractable form and its fast-lifted counterpart are provided in Section III. The main results, i.e., a computation method and its convergence analysis with respect to the L_1-induced norm, are derived in Section IV. Finally, the notations used in this paper are shown in Table I.

TABLE I: Notations in the paper

Notations	Definitions
$\mathbb{R}_1^\nu$	Banach space of ν-dimensional real vectors equipped with the vector 1-norm
$\mathbb{N}$	Set of positive integers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
h'	$h := h/M$ with the sampling period h
$\mathcal{K}_\nu$	The Banach space $(L_1[0, h))^\nu$
$\mathcal{K}'_\nu$	The Banach space $(L_1[0, h'))^\nu$
$\lVert\cdot\rVert_1$	The 1-norm of a matrix
$\lVert\cdot\rVert_{L_1}$	The $L_1[0, T)$ norm of a function with $T = h, h'$ or ∞, i.e., $\lVert f\rVert_{L_1} := \int_0^T \lVert f(t)\rVert_1 dt$
$\lVert\cdot\rVert_{Y/X}$	The induced norm of an operator defined as $\lVert\mathcal{T}\rVert_{Y/X} := \sup_{f\neq 0} \lVert\mathcal{T}f\rVert_Y / \lVert f\rVert_X$

II. Operator-Based Representation of Sampled-Data Systems

This section introduces the operator-based representation of sampled-data systems via the lifting approach [35]–[37].

*This study was carried out with the support of ‘R&D Program for Forest Science Technology (RS-2024-00403460)’ provided by Korea Forest Service(Korea Forestry Promotion Institute).

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Let us consider the sampled-data system $\Sigma_{\rm SD}$ as shown in Fig. 1, where P is the continuous-time linear-time-invariant (LTI) plant, Ψ is the discrete-time LTI controller, $\mathcal{S}$ is the ideal sampler, and $\mathcal{H}$ is the zero-order hold (ZOH). The discrete-time devices (i.e., $\mathcal{S}$ and $\mathcal{H}$) are assumed to synchronously with the sampling period h, and suppose that P and Ψ are described respectively by

$$P: \begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x \end{cases} \tag{1}$$

$$\Psi: \begin{cases} \psi_{k+1} = A_\psi \psi_k + B_\psi y_k \\ u_k = C_\psi \psi_k + D_\psi y_k \end{cases} \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^{n_z}$, $y(t) \in \mathbb{R}^{n_y}$, $w(t) \in \mathbb{R}^{n_w}$, $u(t) \in \mathbb{R}^{n_u}$, $\psi_k \in \mathbb{R}^{n_\psi}$, and $u(t) = u_k$ $(kh \le t < (k+1)h)$. Let us denote $y(kh)$ as y_k .

The actions of $\mathcal{S}$ and $\mathcal{H}$ lead to the linear periodically time-varying (LPTV) nature of $\Sigma_{\rm SD}$, and the relevant difficulty can be alleviated by taking the lifting approach [35]–[37]. More precisely, for a given function $f \in (\mathcal{L}_1[0,\infty))^\nu$, its lifting, denoted by $\hat{f}_k \in \mathcal{K}_\nu$ $(k \in \mathbb{N}_0)$, is defined as follows.

$$\hat{f}_k(s) := f(kh+s) \quad (0 \le s < h). \tag{3}$$

On the basis of applying the lifting technique to w and z, the lifted representation of $\Sigma_{\rm SD}$ is described by

$$\begin{cases} \xi_{k+1} = \mathcal{A}\xi_k + \mathcal{B}\hat{w}_k \\ \hat{z}_k = \mathcal{C}\xi_k + \mathcal{D}\hat{w}_k \end{cases} \tag{4}$$

where $\xi_k := \begin{bmatrix} x_k^T & \psi_k^T \end{bmatrix}^T$ $(x_k := x(kh))$ and the matrix $\mathcal{A}$ and the operators $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are given by

$$\mathcal{A} = \begin{bmatrix} A_d + B_{2d} D_\Psi C_{2d} & B_{2d} C_\Psi \\ B_\Psi C_{2d} & A_\Psi \end{bmatrix} : \mathbb{R}^{n+n_\psi} \to \mathbb{R}^{n+n_\psi}$$
$$\mathcal{B} = J_\Sigma \mathbf{B}_1 : \mathcal{K}_{n_w} \to \mathbb{R}^{n+n_\psi}$$
$$\mathcal{C} = \mathbf{M}_1 C_\Sigma : \mathbb{R}^{n+n_\psi} \to \mathcal{K}_{n_z}$$
$$\mathcal{D} = \mathbf{D}_{11} : \mathcal{K}_{n_w} \to \mathcal{K}_{n_z}$$

with

$$A_d := e^{Ah}, \quad B_{2d} := \int_0^h e^{As} B_2 ds, \quad C_{2d} := C_2, \tag{5}$$

$$J_\Sigma := \begin{bmatrix} I \\ 0 \end{bmatrix} \in \mathbb{R}^{(n+n_\psi)\times n}, \quad C_\Sigma := \begin{bmatrix} I & 0 \\ D_\psi C_{2d} & C_\psi \end{bmatrix} \tag{6}$$

$$M_1 := \begin{bmatrix} C_1 & D_{12} \end{bmatrix}, \quad A_2 := \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} \tag{7}$$

$$\mathbf{B}_1 \hat{w}_k = \int_0^h e^{A(h-s)} B_1 \hat{w}_k(s)\, ds \tag{8}$$

$$\left(\mathbf{M}_1 \begin{bmatrix} x_k \\ u_k \end{bmatrix} \right)(s) = M_1 e^{A_2 s} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \tag{9}$$

$$(\mathbf{D}_{11}\hat{w}_k)(s) = \int_0^s C_1 e^{A(s-\tau)} B_1 \hat{w}_k(\tau)\, d\tau + D_{11}\hat{w}_k(s) \tag{10}$$

We also suppose the internal stability of $\Sigma_{\rm SD}$ for the L_1-induced norm to be well-defined and bounded; all the eigenvalues of $\mathcal{A}$ are assumed to be located in the open unit disc.

III. The L_1-Induced Norm of $\Sigma_{\rm SD}$ and Its Fast-Lifted Expression

This section introduces a tractable form of the input/output relation of $\Sigma_{\rm SD}$ by taking the fast-lifting technique [38], tailored to the analysis for the L_1-induced norm $\|\Sigma_{\rm SD}\|_{L_1/L_1}$ defined as follows.

$$\|\Sigma_{\rm SD}\|_{L_1/L_1} := \sup_{\|w\|_{L_1}=1} \|z\|_{L_1} \tag{11}$$

A. Operator-Based Expression of $\|\Sigma_{\rm SD}\|_{L_1/L_1}$ via Toeplitz Structure of Input/Output Relation of $\Sigma_{\rm SD}$

We first note from (4) that the input/output relation of $\Sigma_{\rm SD}$ is described by the following Toeplitz structure.

$$\begin{bmatrix} \hat{z}_0 \\ \hat{z}_1 \\ \hat{z}_2 \\ \hat{z}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathcal{D} & 0 & \cdots & & & \\ \mathcal{CB} & \mathcal{D} & 0 & \cdots & & \\ \mathcal{CAB} & \mathcal{CB} & \mathcal{D} & 0 & \cdots & \\ \mathcal{CA}^2\mathcal{B} & \mathcal{CAB} & \mathcal{CB} & \mathcal{D} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \hat{w}_0 \\ \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \\ \vdots \end{bmatrix} \tag{12}$$

Here, we define $\mathcal{F}^{[k]}$ for $k \in \mathbb{N}_0$ as

$$\mathcal{F}^{[0]} := \begin{bmatrix} \mathcal{D}^T & (\mathcal{CB})^T & (\mathcal{CAB})^T & (\mathcal{CA}^2\mathcal{B})^T & \cdots \end{bmatrix}^T$$
$$\mathcal{F}^{[1]} := \begin{bmatrix} 0 & \mathcal{D}^T & (\mathcal{CB})^T & (\mathcal{CAB})^T & (\mathcal{CA}^2\mathcal{B})^T & \cdots \end{bmatrix}^T$$
$$\mathcal{F}^{[2]} := \begin{bmatrix} 0 & 0 & \mathcal{D}^T & (\mathcal{CB})^T & (\mathcal{CAB})^T & (\mathcal{CA}^2\mathcal{B})^T & \cdots \end{bmatrix}^T$$

Then, it immediately follows from (12) that

$$\begin{aligned} \|z\|_{L_1} &= \left\| \begin{bmatrix} \hat{z}_0^T & \hat{z}_1^T & \hat{z}_2^T & \cdots \end{bmatrix}^T \right\|_{L_1} \\ &= \left\| \mathcal{F}^{[0]}\hat{w}_0 + \mathcal{F}^{[1]}\hat{w}_1 + \mathcal{F}^{[2]}\hat{w}_2 + \cdots \right\|_{L_1} \\ &\le \|\mathcal{F}^{[0]}\hat{w}_0\|_{L_1} + \|\mathcal{F}^{[1]}\hat{w}_1\|_{L_1} + \|\mathcal{F}^{[2]}\hat{w}_2\|_{L_1} + \cdots \\ &\le \|\mathcal{F}^{[0]}\|_{L_1/L_1} \cdot \left(\|\hat{w}_0\|_{L_1} + \|\hat{w}_1\|_{L_1} + \|\hat{w}_2\|_{L_1} + \cdots \right) \end{aligned} \tag{13}$$

where the last inequality holds from $\|\mathcal{F}^{[0]}\|_{L_1/L_1} = \|\mathcal{F}^{[k]}\|_{L_1/L_1}$ $(\forall k \in \mathbb{N})$. In the following, we take the notation $\mathcal{F}$ instead of $\mathcal{F}^{[0]}$ for simplicity. If we note the fact that the equality in (13) establishes when $\hat{w}_k = 0$ $(\forall k \in \mathbb{N})$, then the L_1-induced norm $\|\Sigma_{\rm SD}\|_{L_1/L_1}$ admits the following representation.

$$\|\Sigma_{\rm SD}\|_{L_1/L_1} = \|\mathcal{F}\|_{L_1/L_1} \tag{14}$$

Even though the L_1-induced norm $\|\Sigma_{\rm SD}\|_{L_1/L_1}$ can be described by the tractable form of (14), its direct computation is still a non-trivial task since $\mathcal{F}$ is an infinite-dimensional operator-based matrix. To alleviate such a difficulty, we

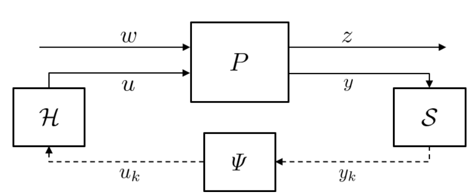

Fig. 1: Sampled-data system $\Sigma_{\rm SD}$.

introduce for an $N \in \mathbb{N}_0$ the two operators $\mathcal{F}_N^-$ and $\mathcal{F}_N^+$ defined as

$$\mathcal{F}_N^- := \begin{bmatrix} \mathcal{D}^T & (\mathcal{C}\mathcal{B})^T & (\mathcal{C}\mathcal{A}\mathcal{B})^T & \cdots & (\mathcal{C}\mathcal{A}^N\mathcal{B})^T \end{bmatrix}^T \quad (15)$$

$$\mathcal{F}_N^+ := \begin{bmatrix} (\mathcal{C}\mathcal{A}^{N+1}\mathcal{B})^T & (\mathcal{C}\mathcal{A}^{N+2}\mathcal{B})^T & \cdots \end{bmatrix}^T \quad (16)$$

In terms of applying the triangular inequality to $\mathcal{F}_N^-$ and $\mathcal{F}_N^+$, we can see that

$$\|\mathcal{F}_N^-\|_{L_1/L_1} \le \|\mathcal{F}\|_{L_1/L_1} \le \|\mathcal{F}_N^-\|_{L_1/L_1} + \|\mathcal{F}_N^+\|_{L_1/L_1} \quad (17)$$

Because $\mathcal{A}^k$ exponentially converges to 0 as k becomes larger from the internal stability assumption on Σ_{SD}, $\|\mathcal{F}_N^+\|_{L_1/L_1}$ also converges to 0 by taking N larger.

In connection with this, we are in a position to compute $\|\mathcal{F}_N^-\|_{L_1/L_1}$ as accurately as possible while $\|\mathcal{F}_N^+\|_{L_1/L_1}$ is obtained in a relatively rough way when the truncation parameter N is large enough.

B. Fast-Lifted Representation of $\mathcal{F}_N^-$

Concerning dealing with $\|\mathcal{F}_N^-\|_{L_1/L_1}$, the fast-lifting technique [38] plays an important role in developing an approximate computation method. For the corresponding parameter $M \in \mathbb{N}$ with $h' := h/M$, the fast-lifting operator, denoted by $\mathbf{L}_M$, maps from $\hat{f}_k \in \mathcal{K}_\nu$ to $\check{f}_k := \begin{bmatrix} (\hat{f}_k^{(1)})^T & \cdots & (\hat{f}_k^{(M)})^T \end{bmatrix}^T \in (\mathcal{K}'_\nu)^M$, where

$$\hat{f}_k^{(i)}(s') := \hat{f}_k((i-1)h' + s') \quad (0 \le s' < h') \quad (18)$$

Because $\mathbf{L}_M$ is norm-preserving, $\|\mathcal{F}_N^-\|_{L_1/L_1}$ admits the representation

$$\|\mathcal{F}_N^-\|_{L_1/L_1} = \left\| \begin{bmatrix} \mathbf{L}_M \mathcal{D} \mathbf{L}_M^{-1} \\ \vdots \\ \mathbf{L}_M \mathcal{C}\mathcal{A}^N\mathcal{B}\mathbf{L}_M^{-1} \end{bmatrix} \right\|_{L_1/L_1} \quad (19)$$

and the fast-lifted operators $\mathbf{L}_M\mathcal{D}\mathbf{L}_M^{-1}$ and $\mathbf{L}_M\mathcal{C}\mathcal{A}^j\mathcal{B}\mathbf{L}_M^{-1}$ in (19) can be described by [38]

$$\mathbf{L}_M\mathcal{D}\mathbf{L}_M^- = \overline{\mathbf{M}'_1}\Delta_M\overline{\mathbf{B}'_1} + \overline{\mathbf{D}'_{11}} \quad (20)$$

$$\mathbf{L}_M\mathcal{C}\mathcal{A}^j\mathcal{B}\mathbf{L}_M^- = \overline{\mathbf{M}'_1}A'_{2dM}C_\Sigma\mathcal{A}^jJ_\Sigma A'_{dM}\overline{\mathbf{B}'_1} \quad (21)$$

where $\mathbf{M}'_1$, $\mathbf{B}'_1$ and $\mathbf{D}'_{11}$ are defined equivalently as $\mathbf{M}_1, \mathbf{B}_1$ and $\mathbf{D}_{11}$, respectively, by changing the interval $[0, h)$ with $[0, h')$, $\overline{(\cdot)}$ represents M copies of $(\cdot)$, i.e., $\mathrm{diag}[(\cdot), \ldots, (\cdot)]$ and the matrices are given by

$$A'_{dM} := [(A'_d)^{M-1} \cdots I], \quad A'_{2dM} := \begin{bmatrix} I \\ \vdots \\ (A'_{2d})^{M-1} \end{bmatrix}$$

$$\Delta_M := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ J & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (A'_{2d})^{M-2}J & \cdots & J & 0 \end{bmatrix} \quad (22)$$

with

$$A'_d := e^{Ah'}, \; A'_{2d} := e^{A_2h'}, \; J := \begin{bmatrix} I \\ 0 \end{bmatrix} \in \mathbb{R}^{(n+n_u)\times n} \quad (23)$$

Based on the above notations, we define $\mathcal{F}_{N,M}^-$ as

$$\mathcal{F}_{N,M}^- = \begin{bmatrix} \overline{\mathbf{M}'_1}\Delta_M\overline{\mathbf{B}'_1} + \overline{\mathbf{D}'_{11}} \\ \overline{\mathbf{M}'_1}\mathcal{A}_{M,0}\overline{\mathbf{B}'_1} \\ \vdots \\ \overline{\mathbf{M}'_1}\mathcal{A}_{M,N}\overline{\mathbf{B}'_1} \end{bmatrix} \quad (24)$$

where

$$\mathcal{A}_{M,j} := A'_{2dM}C_\Sigma\mathcal{A}^jJ_\Sigma A'_{dM} \;\; (j = 0, \ldots, N) \quad (25)$$

Then, it readily follows that

$$\|\mathcal{F}_N^-\|_{L_1/L_1} = \|\mathcal{F}_{N,M}^-\|_{L_1/L_1} \quad (26)$$

The main objective of employing the fast-lifting technique is to derive the operators $\mathbf{B}'_1$, $\mathbf{M}'_1$ and $\mathbf{D}'_{11}$ instead of $\mathbf{B}_1$, $\mathbf{M}_1$ and $\mathbf{D}_{11}$, by which it could be expected that discretizing the former operators lead to smaller error bounds rather than a direct discretization of the latter operators. The subsequent section is devoted to developing such a discretization.

IV. Main Results

This section develops a piecewise constant approximation scheme of $\mathbf{M}_1$, $\mathbf{B}_1$ and $\mathbf{D}_{11}$ on the top of the fast-lifted representation $\mathcal{F}_{N,M}^-$, by which the L_1-induced norm $\|\Sigma_{\mathrm{SD}}\|_{L_1/L_1}$can be obtained within any degree of accuracy.

A. Piecewise Constant Approximation and Convergence Rate Analysis

We consider the averaging $\mathbf{J}'_{\mathrm{A}} : \mathcal{K}'_{n_w} \to \mathcal{K}'_{n_w}$ defined as

$$(\mathbf{J}'_{\mathrm{A}}w)(s') = \frac{1}{h'}\int_0^{h'} w(\tau')\,d\tau' \quad (0 \le s' < h') \quad (27)$$

The rationale to take this operator is to alleviate difficulties of $\mathbf{B}'_1$ on computing $\|\mathcal{F}_{N,M}^-\|_{L_1/L_1}$, and we introduce the operator $\mathbf{B}'_{\mathrm{A}}$ given by

$$\mathbf{B}'_{\mathrm{A}}w := \mathbf{B}'_1\mathbf{J}'_{\mathrm{A}}w = \int_0^{h'} e^{A(h'-s')}B_1 \cdot (\mathbf{J}'_{\mathrm{A}}w)(s')\,ds' \quad (28)$$

On the other hand, we also provide the approximate operators $\mathbf{M}'_{\mathrm{A}}$ and $\mathbf{D}'_{\mathrm{A}}$ for $\mathbf{M}'_1$ and $\mathbf{D}'_1$, respectively, which are defined as

$$\left(\mathbf{M}'_{\mathrm{A}}\begin{bmatrix} x \\ u \end{bmatrix}\right)(s') = M_1\begin{bmatrix} x \\ u \end{bmatrix} \quad (0 \le s' < h') \quad (29)$$

$$(\mathbf{D}'_{\mathrm{A}}w)(s') = D_{11}w(s') \quad (0 \le s' < h') \quad (30)$$

$\mathbf{M}'_{\mathrm{A}}$ corresponds to a constant approximation of the output of $\mathbf{M}'_1$ at $s' = 0$ while $\mathbf{D}'_{\mathrm{A}}$ ignores the convolution integral part of $\mathbf{D}'_{11}$.

With $\mathbf{B}'_{\mathrm{A}}$, $\mathbf{M}'_{\mathrm{A}}$ and $\mathbf{D}'_{\mathrm{A}}$, we introduce the approximate $\mathcal{P}_{N,M}^-$ defined as

$$\mathcal{P}_{N,M}^- = \begin{bmatrix} \overline{\mathbf{M}'_{\mathrm{A}}}\Delta_M\overline{\mathbf{B}'_{\mathrm{A}}} + \overline{\mathbf{D}'_{\mathrm{A}}} \\ \overline{\mathbf{M}'_{\mathrm{A}}}\mathcal{A}_{M,0}\overline{\mathbf{B}'_{\mathrm{A}}} \\ \vdots \\ \overline{\mathbf{M}'_{\mathrm{A}}}\mathcal{A}_{M,N}\overline{\mathbf{B}'_{\mathrm{A}}} \end{bmatrix} \quad (31)$$

Then, we are in a position to compute $\|\mathcal{P}_{N,M}^{-}\|_{L_1/L_1}$ instead of $\|\mathcal{F}_{N,M}^{-}\|_{L_1/L_1} (= \|\mathcal{F}_{N}^{-}\|_{L_1/L_1})$, and the following part is devoted to establishing the corresponding convergence rate. In other words, it is shown that $\|\mathcal{P}_{N,M}^{-}\|_{L_1/L_1}$ converges to $\|\mathcal{F}_{N,M}^{-}\|_{L_1/L_1}$ with the rate of $1/M$ as M becomes larger, and the following three lemmas play important role in deriving such a convergence order.

Lemma 1: The inequality

$$\|\mathbf{B}'_1 - \mathbf{B}'_{\mathrm{A}}\|_{1/L_1} \leq \frac{K_{\mathbf{B}M}}{M} \tag{32}$$

holds, where

$$K_{\mathbf{B}M} := \frac{3}{2} h \|A\|_1 e^{\|A\|_1 h'} \cdot \|A'_d B_1\|_1 \tag{33}$$

and $K_{\mathbf{B}M}$ has a uniform upper bound independent of M given by

$$K_{\mathbf{B}} = \frac{3}{2} h \|A\|_1 \cdot \|B_1\|_1 \cdot e^{\|A\|_1 h} \tag{34}$$

Lemma 2: The inequality

$$\|\mathbf{M}'_1 - \mathbf{M}'_{\mathrm{A}}\|_{L_1/1} \leq \frac{K_{\mathbf{M}M}}{M^2} \tag{35}$$

is satisfied, where

$$K_{\mathbf{M}M} := \frac{h^2}{2} \|M_1 A_2\|_1 \cdot e^{\|A_2\|_1 h'} \tag{36}$$

and $K_{\mathbf{M}M}$ has a uniform upper bound independent of M described by

$$K_{\mathbf{M}} = \frac{h^2}{2} \|M_1 A_2\|_1 \cdot e^{\|A_2\|_1 h} \tag{37}$$

Lemma 3: The inequality

$$\|\mathbf{D}'_{11} - \mathbf{D}'_{\mathrm{A}}\|_{L_1/L_1} \leq \frac{K_{\mathbf{D}M}}{M} \tag{38}$$

is established, where

$$K_{\mathbf{D}M} := M \|C_1\|_1 \cdot \|B_1\|_1 \cdot \frac{e^{\|A\|_1 h'} - 1}{\|A\|_1} \tag{39}$$

and $K_{\mathbf{D}M}$ has a uniform upper bound independent of M given by

$$K_{\mathbf{D}} = h \|B_1\|_1 \cdot \|C_1\|_1 \cdot e^{|A_1| h} \tag{40}$$

Based on Lemmas 1–3, we can obtain the following theorem relevant to the convergence order for the first block-row of $\mathcal{P}_{N,M}^{-}$ in (31).

Proposition 1: The inequality

$$\left\| (\overline{\mathbf{M}'_1} \Delta_M \overline{\mathbf{B}'_1} + \overline{\mathbf{D}'_{11}}) - (\overline{\mathbf{M}'_{\mathrm{A}}} \Delta_M \overline{\mathbf{B}'_{\mathrm{A}}} + \overline{\mathbf{D}'_{\mathrm{A}}}) \right\|_{L_1/L_1} \leq \frac{K_{\mathcal{D}M}}{M} \tag{41}$$

holds, where

$$K_{\mathcal{D}M} := \sum_{k=0}^{M-2} \|(A'_d)^k\|_1 \cdot \Big(\frac{K_{\mathbf{M}M}}{M} \cdot e^{\|A\|_1 h'} \|B_1\|_1 + \frac{K_{\mathbf{B}M}}{M} h \|M_1\|_1 \Big) + K_{\mathbf{D}M} \tag{42}$$

Furthermore, $K_{\mathcal{D}M}$ has a uniform upper bound regardless of M given by

$$K_{\mathcal{D}M}^{U} := K_{\mathbf{D}} + (K_{\mathbf{M}} \cdot \|B_1\|_1 \cdot e^{\|A\|_1 h} + h \cdot K_{\mathbf{B}} \cdot \|M_1\|_1) \cdot e^{\|A_2\|_1 h} \tag{43}$$

With respect to the convergence order relevant to the other block-rows of $\mathcal{P}_{N,M}^{-}$ in (31), combining Lemmas 1 and 2 establish the following result.

Proposition 2: For a $j \in \{0, \ldots, N\}$, the inequality

$$\left\| \overline{\mathbf{M}'_1} \mathcal{A}_{M,j} \overline{\mathbf{B}'_1} - \overline{\mathbf{M}'_{\mathrm{A}}} \mathcal{A}_{M,j} \overline{\mathbf{B}'_{\mathrm{A}}} \right\|_{L_1/L_1} \leq \frac{K_{\mathcal{A}M,j}}{M} \tag{44}$$

holds, with

$$K_{\mathcal{A}M,j} := \frac{\|\mathcal{A}_{M,j}\|_1}{M} \cdot (K_{\mathbf{M}M} \cdot \|B_1\|_1 \cdot e^{\|A\|_1 h'} + h \cdot K_{\mathbf{B}M} \cdot \|M_1\|_1) \tag{45}$$

Furthermore, $K_{\mathcal{A}M,j}$ has a uniform upper bound regardless of M and j as well as N given by

$$K_{\mathcal{A}} := K_{\Sigma} \cdot (K_{\mathbf{M}} \cdot \|B_1\|_1 \cdot e^{\|A\|_1 h} + h \cdot K_{\mathbf{B}} \cdot \|M_1\|_1) \tag{46}$$

where

$$K_{\Sigma} := \max_{j \in \mathbb{N}_0} \|\mathcal{A}^j\|_1 \cdot e^{(\|A\|_1 + \|A_2\|_1) h} \cdot \|C_{\Sigma}\|_1 \tag{47}$$

Remark 1: There should exist the maximum defined as $\max_{j \in \mathbb{N}_0} \|\mathcal{A}^j\|_1$ in (47) because the internal stability assumption on Σ_{SD} ensures that $\lim_{j \to \infty} \|\mathcal{A}^j\| = 0$.

From Propositions 1 and 2, we readily have the following theorem associated with an upper bound and a lower bound on $\|\mathcal{F}_{N}^{-}\|_{L_1/L_1} = \|\mathcal{F}_{N,M}^{-}\|_{L_1/L_1}$.

Theorem 1: The inequality

$$\|\mathcal{P}_{N,M}^{-}\|_{L_1/L_1} - \frac{K_{N,M}}{M} \leq \|\mathcal{F}_{N}^{-}\|_{L_1/L_1} \leq \|\mathcal{P}_{N,M}^{-}\|_{L_1/L_1} + \frac{K_{N,M}}{M} \tag{48}$$

holds, where

$$K_{N,M} := K_{\mathcal{D}M} + \sum_{j=0}^{N} K_{\mathcal{A}M,j} \tag{49}$$

In addition, $K_{N,M}$ has a uniform upper bound regardless of M given by

$$K_N := K_{\mathbf{D}} + (N+1) \cdot K_{\mathcal{A}} \tag{50}$$

This theorem obviously implies that $\|\mathcal{P}_{N,M}^{-}\|_{L_1/L_1}$ converges to $\|\mathcal{F}_{N,M}^{-}\|_{L_1/L_1}$ within the order of $1/M$, and thus computing the former instead of the latter is theoretically valid when M is large enough.

B. Computation of $\|\mathcal{F}\|_{L_1/L_1}$ via Matrix 1-Norm

Even though the convergence rate of $1/M$ is derived in Theorem 1, it is still unclear how to compute $\|\mathcal{P}_{N,M}^{-}\|_{L_1/L_1}$ and also $\|\mathcal{F}\|_{L_1/L_1}$.

In connection with this, we first consider the exact computation of $\|\mathcal{P}_{N,M}^{-}\|_{L_1/L_1}$ through the 1-norm of a matrix. If we note that $\|\mathbf{J}_{\rm A}'w\|_{L_1} \leq \|w\|_{L_1}$ and $\mathbf{J}_{\rm A}'w$ becomes a constant function on $[0,h')$ for an arbitrary $w \in \mathcal{K}_{n_w}'$, then we can see for an arbitrary $\alpha(>0)$ that

$$\{\mathbf{B}_1'\mathbf{J}_{\rm A}'w \mid \|w\|_{L_1} \leq \alpha\} = \{\frac{1}{h'}B_{0d}'w_d \mid \|w_d\|_1 \leq \alpha\} \quad (51)$$

where $B_{0d}' := \int_0^{h'} e^{A(h'-s')}B_1\, ds'$. It readily also follows from the definition of $\mathbf{M}_{\rm A}'$ in (29) that

$$\|\mathbf{M}_{\rm A}'p\|_{L_1} = \|h'M_1p\|_1, \quad \forall p \in \mathbb{R}_1^{n+n_u} \quad (52)$$

From the definition of the $L_1[0,h')$-induced norm and the lower-triangular structure of $\mathit{\Delta}_M$, on the other hand, the operator $\mathbf{D}_{\rm A}'$ (defined as (30)) can be regarded by the feed through matrix D_{11} in computing $\|\mathcal{P}_{N,M}^{-}\|_{L_1/L_1}$.

To summarize, the operators $\mathbf{B}_{\rm A}'$, $\mathbf{M}_{\rm A}'$ and $\mathbf{D}_{\rm A}'$ in $\mathcal{P}_{N,M}^{-}$ can be replaced with $\dfrac{1}{h'}B_{0d}'$, $h'M_1$ and D_{11}', respectively, without changing $\|\mathcal{P}_{N,M}^{-}\|_{L_1/L_1}$. In other words, we can obtain the following theorem.

Theorem 2: Define the matrix $P_{N,M}^{-}$ as

$$P_{N,M}^{-} := \begin{bmatrix} \overline{M_1}\mathit{\Delta}_M\overline{B_{0d}'} + \overline{D_{11}} \\ \overline{M_1}\mathcal{A}_{M,0}\overline{B_{0d}'} \\ \vdots \\ \overline{M_1}\mathcal{A}_{M,N}\overline{B_{0d}'} \end{bmatrix} \quad (53)$$

Then, $\|P_{N,M}^{-}\|_1 = \|\mathcal{P}_{N,M}^{-}\|_{L_1/L_1}$.

Theorem 2 obviously implies that the operator induced norm $\|\mathcal{P}_{N,M}^{-}\|_{L_1/L_1}$ can be exactly computed by the matrix 1-norm $\|P_{N,M}^{-}\|_1$. Because we can compute an upper bound and a lower bound on $\|\mathcal{F}_N^{-}\|_{L_1/L_1}$ by using the arguments in Theorems 1 and 2, the remaining step is to deal with $\left\|\mathcal{F}_N^{+}\right\|_{L_1/L_1}$. To do this, let us first define $\mathcal{A}_{N,L}$ as

$$\mathcal{A}_{N,L} := [(\mathcal{A}^{N+1})^T \ (\mathcal{A}^{N+2})^T \ \cdots \ (\mathcal{A}^{N+L})^T]^T \quad (54)$$

with a sufficiently large $L \in \mathbb{N}_0$ such that $\|\mathcal{A}^L\|_1 < 1$. By taking such a $\mathcal{A}_{N,L}$, we can lead to the following lemma associated with an upper bound on $\|\mathcal{F}_N^{+}\|_{L_1/L_1}$.

Lemma 4: The inequality

$$\left\|\mathcal{F}_N^{+}\right\|_{L_1/L_1} \leq K_{N,L} \quad (55)$$

holds, where

$$K_{N,L} := \|C_\Sigma\|_1\|M_1\|_1\frac{e^{\|A_2\|_1h}-1}{\|A_2\|_1}\frac{\|\mathcal{A}_{N,L}\|_1}{1-\|\mathcal{A}^L\|_1}e^{\|A\|_1h}\|B_1\|_1 \quad (56)$$

Furthermore, $K_{N,L}$ converges to 0 regardless of L by taking N larger.

Finally, substituting Theorems 1, 2 and Lemma 4 into (17) derives the following result relevant to an upper bound and a lower bound on the L_1-induced norm $\|\mathcal{F}\|_{L_1/L_1}$.

Theorem 3: For a sufficiently large $L \in \mathbb{N}_0$ such that $\left\|\mathcal{A}^L\right\|_1 < 1$, the following inequality holds.

$$\|P_{N,M}^{-}\|_1 - \frac{K_{N,M}}{M} \leq \|\mathcal{F}\|_{L_1/L_1} \leq \|P_{N,M}^{-}\|_1 + \frac{K_{N,M}}{M} + K_{N,L} \quad (57)$$

Furthermore, $K_{N,M}$ has a uniform upper bound K_N defined as (50), and $K_{N,M}/M$ converges to 0 by taking M larger, while $K_{N,L}$ tends to 0 regardless of L by taking N larger.

V. Conclusions

A method for computing the L_1-induced norm of sampled-data systems was developed in this paper. To this end, we first derived a tractable form of the L_1-induced norm in the lifting-based representation [35]–[37] of sampled-data systems. We next applied the fats-lifting technique [38] to the tractable form, by which the input/output relation of sampled-data systems can be considered on the interval $[0,h/M)$, where h is the sampling period and M is the fast-lifting parameter. This fast-lifted representation of sampled-data systems allowed us to develop a piecewise constant approximation leading to an upper bound and a lower bound on the L_1-indueced norm. Furthermore, the gap between these bounds was shown to converge to 0 at the rate of $1/M$.

Finally, it would be worthwhile to note that an (approximate) equivalent discretization of sampled-data systems might be required for an optimal controller synthesis to minimize the L_1-induced norm of sampled-data systems, and the developed piecewise constant approximation could be expected to establish such a discretization. However, this is a non-trivial task and is left for an interesting future study.

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