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Inverse Kalman Filtering for Systems with Correlated Noises

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Abstract— **This paper focuses on two inverse problems of the Kalman filter in which the process and measurement noises are correlated. The unknown covariance matrix in a stochastic system is reconstructed from observations of its posterior beliefs. For the standard inverse Kalman filtering problem, a novel duality-based formulation is proposed, where a well-defined inverse optimal control (IOC) problem is solved instead. Identifiability of the underlying model is proved, and a least squares estimator is designed that is statistically consistent. The time-invariant case using the steady-state Kalman gain is further studied. Since this inverse problem is ill-posed, a canonical class of covariance matrices is constructed, which can be uniquely identified from the dataset with asymptotic convergence. Finally, the performances of the proposed methods are illustrated by numerical examples.**

I. Introduction

In complex tasks, autonomous agents are required to adaptively interact with the environment and understand the world around them, where an effective estimator becomes a vital component in autonomous decision-making [1], [2]. In forward filtering problems, given a stochastic process observed in noise, an optimal estimator is designed for the unknown signal, usually in the sense of minimum mean-squared error (or its Bayes risk), maximum likelihood, minimum conditional KL divergence, and so on. On the contrary, the goal of inverse filtering is to reconstruct the unknown parameters or likelihoods in a filtering model from the observations of its posterior updates. For different application purposes, quantities of interest may include the transition kernel of the state, raw measurements, sensors' capabilities, or posterior distributions of the filter [3], [4], [5].

Among variants of the inverse filtering problem, in this paper we aim to recover the statistical characteristics of a stochastic system from its posterior updates. Such problems arise in the pressing need to interact with a filter-based agent or remote sensor fusion, where the transition kernel is unavailable due to security or communication reasons. Most existing results on inverse filtering focus on Bayesian filters of Markov models, such as [6] and a series of follow-up works. Cognitive sensing systems are studied in [7], where remote calibration of a cognitive sensor is achieved by analyzing its sensing strategy in reaction to the probing signals. Similar topics also arise in counter-adversarial scenarios [4]. However, there are only limited results on continuous state space models. An inverse filter is designed in [5] to estimate the filter's updates of a linear Gaussian model. For the specific issue of reconstructing transition kernels, the single-output system is studied in [8]. In our early work [9], the unknown parameters and signals in an observed filter are reconstructed by solving several inverse problems on multiple-output systems that are driven by independent stochastic processes.

In this paper, we further study general multiple-output systems in continuous state spaces, where the process and measurement noises may be correlated. Such phenomenon is observed in numerous applications such as multisensor systems [10] and motion estimation of moving targets with onboard sensors [2]. For example, in an aircraft's navigation system, random gusts of wind will influence both the aircraft dynamics and wind speed measured by an anemometer, thus leading to cross-correlation in process and observation noises. However, by allowing a cross-term in the covariance matrix, additional degrees of freedom are introduced in the inverse problem, which leads to further difficulties in analyzing its well-posedness and reconstructing the unknown parameters. Moreover, in many applications such as nano-satellites and portable devices, instead of updating the Kalman gains online, a predetermined constant gain is usually preferred due to the limited onboard computational power [11], [12]. This is no exception for the case with correlated noises. As for the corresponding inverse problem, different from the standard optimal filter, its well-posedness can no longer be guaranteed. How to parameterize the solution space and design a well-defined estimator remains challenging. Among various choices of the constant gain, in this paper we study the case in which the steady-state Kalman gain is used.

For the specific inverse filtering problem, we aim to reconstruct the unknown covariance in a filtering-based agent from noisy observations of its posterior beliefs. Regarding the inverse problem, there exist a number of fundamental issues that are widely acknowledged in the literature. In this paper, several of such challenges, namely, the ill-posedness and lack of guaranteed performances for data-driven algorithms, are addressed for both the optimal filter and the Steady-State Kalman Filter (SSKF) with correlated noises. The main contributions of this paper include:

1) A novel duality-based formulation is proposed to address the inverse Kalman filtering problem. An equivalent well-posed inverse optimal control problem is solved by designing a sequence of empirical estimators with asymptotic convergence.
2) Ill-posedness of the inverse SSKF is shown. The structure of the solution space is studied and an analytic

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expression is derived for each equivalence set.

3) A well-defined canonical class is designed for the SSKF model, which can be uniquely identified by a statistically consistent estimator.

Notations: Denote e_i as the i-th standard basis vector of $\mathbb{R}^n$. Let $\mathrm{Tr}(\cdot)$ be the trace of a square matrix and $\otimes$ denote the Kronecker product. Denote $\mathrm{col}\{x_1, ..., x_n\} = [x_1^\mathsf{T}, ..., x_n^\mathsf{T}]^\mathsf{T}$ as a stacked column vector. For a block matrix A, denote $[A]_i$ as the i-th block column of A. Let $\mathbb{S}_+^n$ and $\mathbb{S}_{++}^n$ denote the set of $n \times n$ positive semi-definite matrices and positive definite matrices, respectively.

II. Kalman Filtering and Its Inverse Problems

A. Kalman Filtering and Duality Principle

Consider a discrete-time linear time-invariant system

$$\begin{aligned} x_{k+1} &= Ax_k + w_k, \\ y_k &= Cx_k + v_k, \end{aligned} \tag{1}$$

where $\mathrm{Cov}[x_0] = P_0 \succeq 0$ is given and we can assume $\mathbb{E}[x_0] = 0$ without loss of generality [9]. At each time instant, the process noise w_k and measurement noise v_k are zero-mean random vectors with a correlated covariance

$$\Sigma := \mathrm{Cov}([w_k^\mathrm{T}, v_k^\mathrm{T}]^\mathrm{T}) = \begin{bmatrix} Q & S \\ S^\mathrm{T} & R \end{bmatrix} \succeq 0, \quad \forall k \geq 0, \tag{2}$$

where $Q \in \mathbb{S}_+^n$, $R \in \mathbb{S}_{++}^m$, $S \in \mathbb{R}^{n \times m}$ is allowed to be non-zero, and the distributions of the random variables are unknown. The random vectors x_0, w_t, and v_s are assumed to be mutually independent for any $t \neq s$.

In this paper, we have the following standing assumption on the system matrices.

Assumption 1. *Assume that A is nonsingular, C has full row rank and (C, A) is observable.*

Given measurements $Y_k = \mathrm{col}\{y_0, y_1, \cdots, y_k\}$, the Kalman filter provides an optimal linear estimate of x_{k+1} by

$$\hat{x}_{k+1} = A\hat{x}_k + K_k(y_k - C\hat{x}_k), \quad \hat{x}_0 = 0, \tag{3}$$

where $K_k = (AP_kC^\mathrm{T} + S)\Gamma_k^{-1}$, $\Gamma_k = CP_kC^\mathrm{T} + R$ and P_k is derived by the difference Riccati equation (DRE)

$$P_{k+1} = AP_kA^\mathrm{T} - (AP_kC^\mathrm{T} + S)\Gamma_k^{-1}(CP_kA^\mathrm{T} + S^\mathrm{T}) + Q. \tag{4}$$

The duality theorem [13, Ch.7] provides an alternative representation of the Kalman iterates by

$$a^\mathrm{T}\hat{x}_t = -\sum_{k=0}^{t-1} y_k^\mathrm{T} u_{k+1}^*, \tag{5}$$

where $\{u_k^*\}_{k=1}^t$ is the optimal control to the LQR in (6) and a is an arbitrary boundary condition of the dual system.

$$\begin{aligned} \min_u \quad & z_0^\mathrm{T} P_0 z_0 + \sum_{k=1}^{t} (z_k^\mathrm{T} Q z_k + u_k^\mathrm{T} R u_k + 2 z_k^\mathrm{T} S u_k) \\ s.t. \quad & z_k = A^\mathrm{T} z_{k+1} + C^\mathrm{T} u_{k+1}, \quad z_t = a. \end{aligned} \tag{6}$$

One can show that the estimator obtained by (5) is equivalent to the Kalman iterates [13]. Such duality in optimal control and filtering will be further developed in the next part to address the corresponding inverse problems.

B. Inverse Kalman Filtering

In this paper, we aim to reconstruct the unknown covariance of an observed filtering system. Given the measurement sequence of a known sensor, we consider general scenarios where only noisy observations of the posterior estimates

$$\tilde{x}_k = \hat{x}_k + \nu_k,$$

are available, for example, due to quantization errors and measurement errors. The zero-mean random vector ν_k is assumed to be independent with x_0, $\{w_t\}_{t\geq 0}$ and $\{v_t\}_{t\geq 0}$, and has a bounded covariance.

Problem 1. *Consider the stochastic system (1) with known $\{A, C, R\}$ and its Kalman filter on time interval $[0, t]$. Given the measurement sequence $\{y_k\}_{k=0}^{t-1}$ and noisy observations on the posterior estimates $\{\tilde{x}_k\}_{k=1}^t$ (without knowing $\{P_k\}_{k=1}^t$), the goal is to recover the unknown parameters Q and S in the covariance matrix.*

Solving the above problem from the Kalman iterations (3) and (4) directly is not easy. One key novelty of our work lies in a duality-based formulation, where an equivalent inverse optimal control problem is to be solved.

Problem 2. *Consider the LQR in (6) with known system matrices (A, C) and penalty matrix R. Given an implicit observation equation (5) on the optimal solution where $\{\tilde{x}_k\}_{k=1}^t$ and $\{y_k\}_{k=0}^{t-1}$ are available, the goal is to reconstruct the unknown penalty matrices Q and S in the cost function.*

Different from the classic IOC problem where exact values of the optimal solution $\{z_k^*, u_k^*\}_{k=1}^t$ are available, the key challenge of Problem 2 lies in the implicit and nonlinear observation equation (5). Its well-posedness and solvability will be studied in the next section.

In this paper, we further study the inverse problem of SSKF with a non-zero cross-covariance. Since the SSKF is a sub-optimal filter, the duality principle no longer holds. In that case, the posterior estimates are updated by

$$\hat{x}_{k+1} = A\hat{x}_k + K(y_k - C\hat{x}_k), \quad \hat{x}_0 = 0, \tag{7}$$

where $K = (APC^\mathrm{T} + S)\Gamma^{-1}$, $\Gamma = CPC^\mathrm{T} + R$ and P is the unique stabilizing solution to the algebraic Riccati equation (ARE)

$$P = APA^\mathrm{T} - (APC^\mathrm{T} + S)\Gamma^{-1}(CPA^\mathrm{T} + S^\mathrm{T}) + Q. \tag{8}$$

Problem 3. *Suppose the system parameters $\{A, C, R\}$ are known. Given a SSKF on time interval $[0, t]$, the goal is to reconstruct the unknown covariance matrices Q and S from noisy observations on the posterior estimates $\{\tilde{x}_k\}_{k=1}^t$ and the measurement sequence $\{y_k\}_{k=0}^{t-1}$.*

III. Duality-Based Inverse Kalman Filtering

Following the duality-based formulation, the problem of inverse Kalman filtering is addressed by solving the inverse LQR in Problem 2. The identifiability and solvability of the filtering model are studied. An empirical least squares estimator is proposed that is shown to be statistically consistent.

A. Identifiablty of the Inverse Filtering Model

The fundamental issue of well-posedness is first studied, namely, whether there exists a unique parameter that characterizes the model underlying the observed dataset. By formulating the inverse problem as a system identification problem, its well-posedness is closely related to the identifiability of the underlying model structure. For each measurement Y_{k-1} driven by the random process $\eta_{k-1} = \mathrm{col}\{x_0, w_0, \cdots .w_{k-2}, v_0, \cdots, v_{k-1}\}$, the posterior updates $\hat{x}_k$ can be considered as the output of the filtering model

$$\hat{x}_k = \mathcal{M}_k(Q,S)Y_{k-1}, \tag{9}$$

where $\mathcal{M}_k(Q,S)$ is defined by (3) and (4). The model structure $\mathcal{M}_k(Q,S)$ is said to be *strictly globally identifiable* [14] if it is uniquely characterized by the unknown parameter (Q,S) for any $k \geq 0$.

Definition 1. *For any $k = 1, \cdots, t$, the model structure $\mathcal{M}_k(Q,S) : \mathbb{R}^{km} \to \mathbb{R}^n$ is a $1 \times k$ block matrix, where the i-th block column takes the form*

$$[\mathcal{M}_k(Q,S)]_i = \big(\prod_{j=i}^{k-1}(A - K_jC)\big)K_{i-1}, \tag{10}$$

for $i = 1, \cdots, k-1$ and $[\mathcal{M}_k(Q,S)]_k = K_{k-1}$.

We look into more details on the Kalman gains.

Theorem 1. *Under Assumption 1, suppose $t \geq n+1$. Consider two covariance matrices $\Sigma_1, \Sigma_2 \in \mathbb{J}$ with given R. Let $\{K_k^1\}_{k=0}^{t-1}$ and $\{K_k^2\}_{k=0}^{t-1}$ be the corresponding sequence of Kalman gains, respectively. Then $K_k^1 = K_k^2$ for $k = 0, \cdots, t-1$ if and only if $\Sigma_1 = \Sigma_2$.*

Proof. Sufficiency is straightforward and we show the necessity as follows. Denote

$$\Sigma_1 = \begin{bmatrix} Q_1 & S_1 \\ S_1^{\mathrm{T}} & R \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} Q_2 & S_2 \\ S_2^{\mathrm{T}} & R \end{bmatrix},$$

and let $\{P_k^i\}_{k=0}^{t-1}$ with $i = 1,2$ solve the corresponding DRE. Rewriting $K_k^i(CP_k^iC^{\mathrm{T}} + R) = (AP_k^iC^{\mathrm{T}} + S_i)$ gives

$$K_k^iR = (A - K_k^iC)P_k^iC^{\mathrm{T}} + S_i, \quad i = 1,2. \tag{11}$$

Evaluating (11) at $k = 0$ together with $P_0^i = P_0$ implies $S_1 = S_2$. Based on Sylvester's determinant theorem, one can show that the closed-loop matrix $A - K_k^iC$ is nonsingular if A is nonsingular. Hence, $K_k^1 = K_k^2$ implies $P_k^1C^{\mathrm{T}} = P_k^2C^{\mathrm{T}}$ for any $k = 0, \cdots, t-1$. Substituting it into the DRE yields

$$\begin{cases} \Delta P_kC^{\mathrm{T}} = 0, \\ A\Delta P_kA^{\mathrm{T}} - \Delta P_{k+1} + \Delta Q = 0, \end{cases} \tag{12}$$

where $\Delta P_k = P_k^1 - P_k^2$ and $\Delta Q = Q_1 - Q_2$. Using $\Delta P_0 = 0$ and applying recursions to (12) for $k = 0, \cdots, n$ then gives

$$\Delta Q \begin{bmatrix} C^{\mathrm{T}} & A^{\mathrm{T}}C^{\mathrm{T}} & \cdots & (A^{n-1})^{\mathrm{T}}C^{\mathrm{T}} \end{bmatrix} = 0. \tag{13}$$

Thus $\Delta Q = 0$ if (C,A) is observable, namely, $\Sigma_1 = \Sigma_2$. □

Proposition 1. *Under Assumption 1, suppose $t \geq n+1$. The model structure in (10) is strictly globally identifiable, namely, for any $(Q,S) \in \mathbb{S}_+^n \times \mathbb{R}^{n\times m}$,*

$$\mathcal{M}_k(Q,S) = \mathcal{M}_k(\bar{Q},\bar{S}), \quad k = 0, \cdots, t-1,$$

implies $(Q,S) = (\bar{Q},\bar{S})$.

Proof. Based on the structure of $\mathcal{M}_k(Q,S)$ in (10), the claim follows directly from Theorem 1. Details are omitted. □

B. Design of Consistent Estimators

With a strictly globally identifiable model structure, it is then possible to develop data-driven algorithms to uniquely reconstruct the unknown parameter, where a condition of persistent excitation is required on the random process η_k.

Assumption 2. *Denote $Y_k|\eta_k$ as the random variable driven by η_k according to the dynamics (1). For any $k \geq 1$ and $\xi \in \mathbb{R}^{(k+1)m}$, there exists some $\beta(\xi) \neq 0$ such that $\mathbb{P}(Y_k|\eta_k \in B_\epsilon(\beta\xi)) > 0, \forall \epsilon > 0$, where $B_\epsilon(\beta\xi)$ is the open ϵ-ball centered at $\beta\xi$.*

For brevity, in the sequel we identify each parameter pair (Q,S) with Σ as in (2), where R is given. Let $\{z_k\}_{k=0}^{t-1}$ be the optimal state sequence to (6), then there exists $\{\lambda_k\}_{k=0}^{t}$ such that the Pontryagin's Maximum Principle (PMP) is satisfied [15], which can be rewritten in the compact form

$$\underbrace{\begin{bmatrix} E & -F & & \\ & \ddots & \ddots & \\ & & E & -F \\ \tilde{F} & & & \tilde{E} \end{bmatrix}}_{\mathscr{F}(\Sigma)} \underbrace{\begin{bmatrix} \psi_0 \\ \vdots \\ \psi_{t-1} \end{bmatrix}}_{\Psi} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ A^{\mathrm{T}} - C^{\mathrm{T}}R^{-1}S^{\mathrm{T}} \\ 0 \end{bmatrix}}_{\mathscr{H}(\Sigma)} z_t, \tag{14}$$

where $\psi_k = [z_k^{\mathrm{T}}, \lambda_k^{\mathrm{T}}]^{\mathrm{T}}$ and

$$E = \begin{bmatrix} I & C^{\mathrm{T}}R^{-1}C \\ 0 & A - SR^{-1}C \end{bmatrix}, \; F = \begin{bmatrix} A^{\mathrm{T}} - C^{\mathrm{T}}R^{-1}S^{\mathrm{T}} & 0 \\ SR^{-1}S^{\mathrm{T}} - Q & I \end{bmatrix},$$

$$\tilde{E} = \begin{bmatrix} I & C^{\mathrm{T}}R^{-1}C \\ 0 & 0 \end{bmatrix}, \; \tilde{F} = \begin{bmatrix} 0 & 0 \\ P_0 & -I \end{bmatrix}.$$

Based on Bellman's principle of optimality [16], the duality-based formulation (5) can also be evaluated at any time instant in the interval $[0,t]$. Substituting the PMP further gives

$$z_k{}^{\mathrm{T}}\hat{x}_k = -\sum_{i=0}^{k-1} y_i^{\mathrm{T}}u_{i+1}^* = \sum_{i=0}^{k-1}(z_{i+1}^{\mathrm{T}}SR^{-1}y_i + \lambda_i^{\mathrm{T}}C^{\mathrm{T}}R^{-1}y_i),$$

for $k = 1, \ldots, t$. We consider n optimal trajectories corresponding to linearly independent terminal states $z_t^i = e_i$, where the superscript "i" is used to distinguish related quantities for the LQR with terminal constraint z_t^i. Taking into account all the time instants for the LQRs then yields

$$\begin{aligned} Z(\Sigma,k)\hat{x}_k = \begin{bmatrix} Z(\Sigma,1) & \cdots & Z(\Sigma,k) \end{bmatrix} (I_k \otimes (SR^{-1}))Y_{k-1} \\ + \Lambda(\Sigma,k)\bar{C}Y_{k-1}, \end{aligned} \tag{15}$$

where $k = 1, 2, \cdots, t$, $\bar{C} = I_k \otimes (C^{\mathrm{T}} R^{-1})$ and

$$Z(\Sigma, k) = \begin{bmatrix} (z_k^1)^{\mathrm{T}} \\ \vdots \\ (z_k^n)^{\mathrm{T}} \end{bmatrix}, \ \Lambda(\Sigma, k) = \begin{bmatrix} \lambda_0^1 & \cdots & \lambda_0^n \\ \vdots & & \vdots \\ \lambda_{k-1}^1 & \cdots & \lambda_{k-1}^n \end{bmatrix}^{\mathrm{T}}.$$

By minimizing the residual of (15), a least squares estimator of the covariance matrix is designed for Problem 2 via

$$\begin{aligned} \min_{\Sigma, \{\Psi^i\}_{i=1}^n} \mathcal{L}(\Sigma) &= \mathbb{E}_{\eta,\nu}[\frac{1}{t}\sum_{k=1}^{t} \|r(\Sigma, k; Y|\eta, \nu)\|^2] \\ s.t.\ \mathscr{F}(\Sigma) &\begin{bmatrix} \Psi^1 & \cdots & \Psi^n \end{bmatrix} = \mathscr{H}(\Sigma) \\ Q \in \mathbb{S}_+^n,& \ \|Q\| \le \alpha_1, \ \|S\| \le \alpha_2 \end{aligned} \tag{16}$$

where we assume that the true value of (Q, S) is contained in a compact set (with known α_1 and α_2) to avoid ill-conditioning, and the residual function is defined as

$$\begin{aligned} &r(\Sigma, k; Y|\eta, \nu) = \tilde{x}_k - Z(\Sigma, k)^{-1}\Lambda(\Sigma, k)\bar{C}Y_{k-1} \\ &-Z(\Sigma, k)^{-1}\begin{bmatrix} Z(\Sigma, 1) & \cdots & Z(\Sigma, k) \end{bmatrix}(I_k \otimes (SR^{-1}))Y_{k-1}. \end{aligned}$$

As the PMP provides a necessary and sufficient condition for the optimal solution to an LQR, $\mathscr{F}(\Sigma)$ must be nonsingular for any Σ. For brevity, we denote the cost in (16) only as a function of Σ, where $\{\Psi^i\}_{i=1}^n$ can be uniquely determined by the first constraint for each candidate Σ.

Without prior knowledge on the distributions of η and ν, in practice $\mathcal{L}(\Sigma)$ can only be computed by Monte Carlo approximations. An empirical estimator is further designed, whose asymptotic convergence is guaranteed.

Theorem 2. *Let $\bar{\Sigma}$ (with $\bar{Q}$ and $\bar{S}$) be the true covariance matrix based on which the observations $\{\tilde{x}_{1:t}^i, Y_{t-1}^i\}_{i=1}^N$ are generated. Suppose $t \ge n+1$ and Σ_N^* is the local minimizer to the following least squares estimation problem,*

$$\begin{aligned} \Sigma_N^* = \arg\min_{\Sigma} \ \frac{1}{tN}\sum_{i=1}^{N}\sum_{k=1}^{t} \left\| r(\Sigma, k; Y^i|\eta^i, \nu^i) \right\|^2 \\ s.t.\ Q \in \mathbb{S}_+^n, \ \|Q\| \le \alpha_1, \ \|S\| \le \alpha_2, \end{aligned} \tag{17}$$

where $\Psi^1, \cdots, \Psi^n$ in $r(\Sigma, k; Y^i|\eta^i, \nu^i)$ are computed via (14) *as a function of Σ. Then the least squares estimator is statistically consistent, namely, $\Sigma_N^* \xrightarrow{w.p.1} \bar{\Sigma}$ as $N \to \infty$.*

Proof. See Appendix A. □

IV. Identification of the Canonical Class in Inverse SSKFs

In this section, the inverse SSKF in Problem 3 is studied. By showing the ill-posedness of such a problem, a canonical class of the covariance matrix is constructed, based on which a well-posed inverse problem is solved instead.

In the sequel, we assume that the covariance matrix lies in the parameter space of (18). Here the detectability condition is widely used to guarantee the stability of Kalman filters and the existence of SSKF [17], namely, the closed-loop matrix $A_+ = A - KC$ is stable.

$$\begin{aligned} \mathbb{J} = \{\Sigma = \begin{bmatrix} Q & S \\ S^{\mathrm{T}} & R \end{bmatrix} : Q \succeq 0, \ Q - SR^{-1}S^{\mathrm{T}} \succeq 0 \\ (\tilde{A}, \tilde{Q}^{\frac{1}{2}}) \text{ detectable}\}, \end{aligned} \tag{18}$$

where $\tilde{A} = A^{\mathrm{T}} - C^{\mathrm{T}}R^{-1}S^{\mathrm{T}}$ and $\tilde{Q} = Q - SR^{-1}S^{\mathrm{T}}$.

Firstly, the equivalence of covariance matrices is studied.

Definition 2. *Two covariance matrices $\Sigma_1, \Sigma_2 \in \mathbb{J}$ are defined as equivalent (denoted by $\Sigma_1 \sim \Sigma_2$) if the same steady-state Kalman gain K can be generated. In another word, for any given sensor measurements, the same posterior estimates in* (7) *are obtained.*

In the following, we will show that the problem of recovering (Q, S) is ill-posed, where the equivalence set of Σ always consists of nonunique elements.

Proposition 2. *For any $\Sigma \in \mathbb{J}$, let K be the corresponding steady-state Kalman gain. Then the following matrix*

$$\bar{\Sigma} = \begin{bmatrix} KRK^{\mathrm{T}} & KR \\ RK^{\mathrm{T}} & R \end{bmatrix}, \tag{19}$$

is equivalent to Σ, namely, $\bar{\Sigma} \in \mathbb{J}$ and $\bar{\Sigma} \sim \Sigma$.

Proof. Denote $\bar{Q} = KRK^{\mathrm{T}}$ and $\bar{S} = KR$. It is obvious that $\bar{Q} \succeq 0$ and $\bar{Q} - \bar{S}R^{-1}\bar{S}^{\mathrm{T}} = 0 \succeq 0$. As $\tilde{A} = A^{\mathrm{T}} - C^{\mathrm{T}}R^{-1}\bar{S}^{\mathrm{T}} = A^{\mathrm{T}} - C^{\mathrm{T}}K^{\mathrm{T}} = A_+^{\mathrm{T}}$ is stable, it holds that $(\tilde{A}, \tilde{Q}^{\frac{1}{2}})$ is detectable and $\bar{\Sigma} \in \mathbb{J}$. Let $\bar{P}$ be the unique positive semi-definite solution to the ARE associated with $\bar{\Sigma}$, i.e.,

$$\begin{aligned} \bar{P} &= A\bar{P}_+A^{\mathrm{T}} - (A\bar{P}C^{\mathrm{T}} + \bar{S})\bar{\Gamma}^{-1}(C\bar{P}A^{\mathrm{T}} + \bar{S}^{\mathrm{T}}) + \bar{Q}, \\ &= A_+\bar{P}A_+^{\mathrm{T}} - A_+\bar{P}C^{\mathrm{T}}(R + C\bar{P}C^{\mathrm{T}})^{-1}C\bar{P}A_+^{\mathrm{T}}. \end{aligned}$$

Since A_+ is stable, $\bar{P} = 0$ is the unique stabilizing solution, which yields the corresponding steady-state Kalman gain as $\bar{K} = (A\bar{P}C^{\mathrm{T}} + \bar{S})(C\bar{P}C^{\mathrm{T}} + R)^{-1} = K$, thus $\bar{\Sigma} \sim \Sigma$. □

To formulate a well-defined inverse problem, it is necessary to identify each equivalence class of solutions with a unique representative covariance matrix. Inspired by results on continuous-time IOC [18], one possible rationale is to construct a canonical class of Σ that is parametrized by K:

$$\mathbb{J}_c = \left\{\Sigma_K := \begin{bmatrix} KRK^{\mathrm{T}} & KR \\ RK^{\mathrm{T}} & R \end{bmatrix} : A - KC \text{ is stable}\right\}. \tag{20}$$

Proposition 2 indicates that each covariance matrix pair (Q, S) can be parametrized by a canonical form (KRK^{T}, KR) as in (20). By identifying parameters in the canonical class $\mathbb{J}_c$ instead of in the original parameter space $\mathbb{J}$, a well-posed inverse filtering problem can be formulated.

Problem 4. *Consider a Kalman filter with covariance matrix Σ. Given sensor measurements Y_{t-1} and the corresponding noisy observations of the posterior estimates, the task is to reconstruct a canonical form $\Sigma_K \in \mathbb{J}_c$ such that $\Sigma_K \sim \Sigma$.*

We further consider the space of stabilizing K as the parameter space and re-define the corresponding identification model $\mathcal{M}_t(K)$, where the covariance matrix is constructed according to (20). In the sequel we will show that the parameter space in (18) can be partitioned into disjoint equivalence sets, each of which characterizes a filtering model and can be identified with a unique canonical form in $\mathbb{J}_c$.

Proposition 3. *For any $\Sigma \in \mathbb{J}$, there exists a unique canonical form $\Sigma_K \in \mathbb{J}_c$ such that $\Sigma \sim \Sigma_K$. Furthermore, the model structure $\mathcal{M}_t(K)$ is strictly globally identifiable.*

Proof. The existence and uniqueness of the canonical form follow by Proposition 2. Strictly global identifiability of the model structure can be obtained by observing that $\mathcal{M}_k(K)$ has the same structure as (10) by replacing K_i with K. □

Next, we further look into more details on the equivalence sets in $\mathbb{J}$.

Theorem 3. *For each canonical form $\Sigma_K \in \mathbb{J}_c$ with parameter K. Denote $\mathcal{J}(K)$ as the equivalence set in $\mathbb{J}$ that is identified by Σ_K, namely, $\mathcal{J}(K) = \{\Sigma \in \mathbb{J} : \Sigma \sim \Sigma_K\}$. For any $\Delta P \in \mathbb{S}_+^n$, denote*

$$Q_K(\Delta P) = KRK^{\mathrm{T}} + \Delta P - A\Delta PA^{\mathrm{T}} + KC\Delta PC^{\mathrm{T}}K^{\mathrm{T}},$$
$$S_K(\Delta P) = KR - (A - KC)\Delta PC^{\mathrm{T}}.$$

Then the following statements hold:

1) *$\mathcal{J}(K)$ is equivalent to*

$$\mathcal{J}(K) = \Big\{\Sigma(\Delta P) = \begin{bmatrix} Q_K(\Delta P) & S_K(\Delta P) \\ S_K(\Delta P)^{\mathrm{T}} & R \end{bmatrix} : \Delta P \in \mathbb{S}_+^n \text{ satisfies } \Sigma(\Delta P) \in \mathbb{J}\Big\}. \quad (21)$$

2) *$\mathbb{J} = \bigcup_{K:\Sigma_K \in \mathbb{J}_c} \mathcal{J}(K)$ and $\mathbb{J}_c$ contains exactly one element in each equivalence set.*

Proof. See Appendix B. □

Similar to the results in Section III-B, Problem 4 can also be formulated as a system identification problem. A sequence of consistent least squares estimators can be designed, where the requirement on the number of observations is relaxed.

Proposition 4. *Let $\bar{\Sigma}$ be the true covariance matrix of an SSKF, based on which the observations $\{\tilde{x}^i_{1:t}, Y^i_{t-1}\}_{i=1}^N$ are generated under Assumption 2. Denote $\bar{K}$ as the parameter matrix in the canonical form such that $\Sigma_{\bar{K}} \sim \bar{\Sigma}$. Suppose K_N^* is the local minimizer to the following least squares estimation problem,*

$$K_N^* = \arg\min_K \frac{1}{tN}\sum_{i=1}^N\sum_{k=1}^t \left\|\tilde{x}_k - \mathcal{M}_k(K)Y^i_{k-1}\right\|^2 \quad \text{s.t. } \rho(A - KC) \le 1,\ \|K\| \le \alpha_3, \quad (22)$$

where α_3 is given. Then the empirical estimate of the canonical form is obtained as

$$\begin{bmatrix} K_N^* R (K_N^*)^{\mathrm{T}} & K_N^* R \\ R(K_N^*)^{\mathrm{T}} & R \end{bmatrix} \succeq 0,$$

whose statistical consistency is guaranteed by $K_N^* \xrightarrow{w.p.1} \bar{K}$ as $N \to \infty$.

Proof. Following Theorem 2, the statement can be proved in a similar way, which is omitted due to page limitation. □

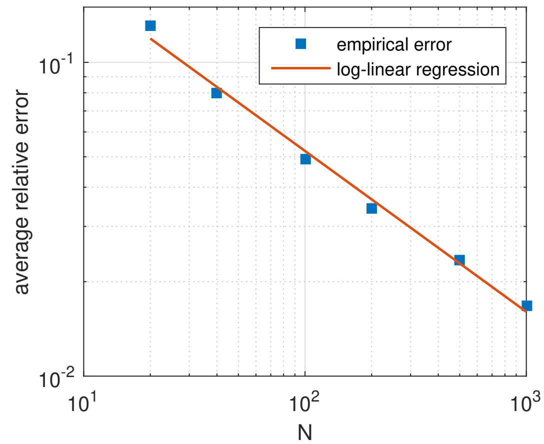

Fig. 1: Convergence of relative errors.

V. Numerical Simulations

In this section, numerical simulations are provided to demonstrate the performances of the proposed empirical estimator in (17). Results in Section IV can be tested in a similar way and are omitted due to page limitation. Random system matrices are generated as follows

$$A = \begin{bmatrix} 1 & 0.1 \\ -0.1034 & 1.0492 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 0.0070 & 0 \\ 0 & 0.1593 \end{bmatrix},$$
$$C = \begin{bmatrix} 1.1182 & 1.5792 \end{bmatrix}.$$

Asymptotic convergence of the proposed estimator is shown via Monte Carlo studies. Let $N = 20, 40, 100, 200, 500, 1000$, respectively. For each choice of N, independent experiments are carried out on time interval $[0, 30]$ for 200 times. In each experiment, the true parameter $\bar{\Sigma}$ is chosen randomly with $\alpha_1 = \alpha_2 = 5$, and the covariances of v_k and ν_k are chosen as random positive definite matrices with a spectral radius less than 1. Driven by zero-mean random processes x_0, w_k, v_k, and ν_k, N posterior sequences of the Kalman filter are generated. The empirical estimate Σ_N^* is computed by solving (17), and the average relative error is shown by the log–log plot in Fig. 1. In all the scenarios, the unknown covariance matrix is reconstructed with rather small relative errors. A linear regression model is adopted to fit the relative errors in the logarithmic scale. We observe that $\left\|\Sigma_N^* - \bar{\Sigma}\right\| \approx o_p(N^{-0.51})$, which is consistent with many least squares estimators where $\sqrt{N}\left\|\Sigma_N^* - \bar{\Sigma}\right\| = o_p(1)$.

VI. Conclusion

In this paper, the inverse Kalman filtering problem is studied based on a novel duality-based approach, where correlated noises are involved. The unknown covariance matrix is reconstructed by solving a sequence of well-posed least squares estimation problems, whose solutions converge almost surely to the true parameter. Furthermore, a canonical class underlying the filtering model is proposed for sub-optimal filters with the steady-state Kalman gain. An empirical estimator is also designed that is statistically consistent. Finally, the performances of the proposed methods are demonstrated by Monte Carlo simulations.

APPENDIX

A. Proof of Theorem 2

Firstly, we show that $\bar{\Sigma}$ is the unique minimizer to $\mathcal{L}(\Sigma)$. Note that the risk function can be rewritten as

$$\mathcal{L}(\Sigma) = \underbrace{\frac{1}{t}\sum_{k=1}^{t}\mathbb{E}_\eta[\|\tilde{r}(\Sigma,k;Y|\eta)\|^2]}_{\mathcal{V}(\Sigma)} + \frac{1}{t}\sum_{k=1}^{t}\mathbb{E}_\nu[\|\nu_k\|^2],$$

where $\tilde{r}(\Sigma,k;Y|\eta) = r(\Sigma,k;Y|\eta,\nu) - \nu_k$. Since $\mathbb{E}_\nu[\|\nu_k\|^2]$ is independent of Σ, it suffices to show that $\bar{\Sigma}$ is the unique minimizer to $\mathcal{V}(\Sigma)$. By (15) we know that $\mathcal{V}(\bar{\Sigma}) = 0$. Let Σ (with parameters Q and S) and $\{\Psi^i\}_{i=1}^n$ be arbitrary matrices satisfying the constraints in (16). It indicates that Ψ^i is the adjoint solution to (6) with penalty matrices (Q,S) and terminal state e_i, namely,

$$z_i^j = \Phi_\Sigma(t,i)^{\mathrm{T}} z_t^j, \quad j = 1,\cdots,n \text{ and } i = 0,\cdots,t,$$

where $\Phi_\Sigma(t+1,s) = (A - K_t(\Sigma)C)\Phi_\Sigma(t,s)$ for any t,s, and $\Phi_\Sigma(t,t) = I$. Under Assumption 2, Proposition 1 implies that there exists some time instant t' and $Y_{t'-1}$ such that

$$\tilde{r}(\Sigma,t';Y|\eta) = \begin{bmatrix}\Delta\Phi(\Sigma,1) & \cdots & \Delta\Phi(\Sigma,t')\end{bmatrix} Y_{t'-1} \neq 0,$$

where $\Delta\Phi(\Sigma,j) = \Phi_{\bar{\Sigma}}(t',j)K_{j-1}(\bar{\Sigma}) - \Phi_\Sigma(t',j)K_{j-1}(\Sigma)$ for $j = 1,2,\cdots,t'$. Moreover, $\tilde{r}(\Sigma,t';Y|\eta) \neq 0$ indicates that there must exist some $\tilde{t} \leq t'$ such that $\Delta\Phi(\Sigma,\tilde{t}) \neq 0$. Recall that $R \succ 0$, we then have

$$\begin{aligned}\mathcal{V}(\Sigma) &\geq \frac{1}{t}\mathbb{E}_\eta[\|\tilde{r}(\Sigma,t';Y|\eta)\|^2],\\ &\geq \frac{1}{t}\operatorname{Tr}\left(\Delta\Phi(\Sigma,\tilde{t})R\Delta\Phi(\Sigma,\tilde{t})^{\mathrm{T}}\right) > 0.\end{aligned}$$

Hence, $\bar{\Sigma}$ is the unique minimizer to $\mathcal{V}(\Sigma)$. Following our early work [9], we can also show that the sample mean of $f(\Sigma;Y|\eta,\nu) = \frac{1}{t}\sum_{k=1}^{t}\|r(\Sigma,k;Y|\eta,\nu)\|^2$ converges almost surely to its expectation, which happens uniformly at Σ within the considered compact parameter space. Together with the fact that $\bar{\Sigma}$ is the unique minimizer to $\mathcal{L}(\Sigma)$, consistency of Σ_N^* is guaranteed [19].

B. Proof of Theorem 3

To begin with, we show the analytic expression of $\mathcal{J}(K)$ in (21). Firstly we prove the necessity. Recall that Σ_K is defined by the triple $(\bar{Q},R,\bar{S}) := (KRK^{\mathrm{T}},R,KR)$ and $\bar{P} = 0$ is the unique stabilizing solution to ARE. Let Σ be an arbitrary element in $\mathcal{J}(K)$ defined by matrix triple (Q,R,S) and ΔP be the corresponding positive semi-definite solution to ARE. By Proposition 3, $\Sigma \sim \Sigma_K$ if and only if the same steady-state Kalman gain can be obtained, namely,

$$\begin{aligned}&K = (A\Delta PC^{\mathrm{T}} + S)(C\Delta PC^{\mathrm{T}} + R)^{-1},\\ \Longrightarrow\ &KR = (A-KC)\Delta PC^{\mathrm{T}} + S \implies S = S_K(\Delta P).\end{aligned}$$

Denote $\Delta Q := Q - \bar{Q}$ and $\Delta S := S_K(\Delta P) - \bar{S}$. Substituting $\Delta S = -(A-KC)\Delta PC^{\mathrm{T}}$, the ARE associated with Σ can be rewritten as

$$\begin{aligned}\Delta P &= A\Delta PA^{\mathrm{T}} - (A\Delta PC^{\mathrm{T}} + \Delta S)K^{\mathrm{T}} + \Delta Q,\\ &= A\Delta PA^{\mathrm{T}} - KC\Delta PC^{\mathrm{T}}K^{\mathrm{T}} + \Delta Q,\end{aligned}$$

which implies $Q = Q_K(\Delta P)$. In addition, the conditions in (21) are naturally satisfied since $J \in \mathbb{J}$.

Next, the sufficiency will be proved. For any ΔP satisfying the condition in (21), straightforward computations show that ΔP is the unique positive semi-definite solution to the ARE associated with weighting matrix $(Q_K(\Delta P),R,S_K(\Delta P))$. Substituting the expression of $S_K(\Delta P)$, the corresponding Kalman gain is computed by

$$K(\Delta P) = (A\Delta PC^{\mathrm{T}} + S_K(\Delta P))(C\Delta PC^{\mathrm{T}} + R)^{-1} = K,$$

which implies that $\Sigma \sim \Sigma_K$ and the first statement is proved.

Moreover, the second statement follows directly from Proposition 3.

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