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# Robot navigation through cluttered environments: A Lyapunov based control design approach

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***Abstract*— Control design for robotic systems guaranteeing safety and convergence properties in cluttered environments is intrinsically challenging due to their potentially conflicting objectives. While several research streams tackle the problem from different angles, a general solution for nonlinear dynamical systems is still out of reach. In this paper, we tackle the problem through a dynamic extension of a stabilizing the controller, ensuring that the output of a system stays within the vicinity of a predefined path, ensuring safety first and then asymptotic stability of a desired point. The results rely on forward invariance of sublevel sets of Lyapunov functions and on stabilizing dynamically updated reference points. The results are illustrated using the example of a linear double integrator and extended unicycle dynamics.**

## I. Introduction

The advancement of controller design for general nonlinear systems, aiming to ensure both safety and convergence properties—specifically, avoiding collisions with the environment—has garnered significant attention in recent decades. This attention is notably due to the proliferation of autonomous vehicles, spanning mobile robots, drones, submersibles, and robot manipulators. These systems frequently operate in cluttered environments involving static and moving obstacles, accentuating the necessity for reliable controller designs characterized by provable convergence, avoidance, and robustness properties. Although considerable progress has been achieved in obstacle avoidance controller designs, the inherent conflict between the objectives of avoidance and convergence poses a persistent challenge. This challenge remains unsolved, especially within the constrained nonlinear nature of mechanical systems navigating cluttered environments populated by static and moving obstacles.

Classical stabilizing controller designs, incorporating avoidance mechanisms (dating from the late 1980s), include artificial potential fields and navigation functions [15], [16]. These approaches leverage the gradient of a positive definite function to drive the system's state toward a target position while avoiding obstacles. However, their application with guarantees is typically restricted to kinematic models and, at best, almost global stability.

Modern approaches, such as those related to artificial potential fields, combine control Lyapunov functions for convergence with control barrier functions for obstacle avoidance [1], [2]. Despite gaining considerable attention in recent years, similar to artificial potential fields, these approaches often encounter challenges in avoiding deadlock situations—where avoidance is guaranteed but at the expense of convergence. Overcoming deadlock situations to attain global results represents a current topic of interest, particularly given the necessity for discontinuous feedback laws in many applications due to topological obstructions of the problem, making the problem inherently challenging.

General controller designs that directly account for constraints (e.g., obstacles) in the controller design encompass model predictive control (MPC) and reference governors [14], [7], [6]. In MPC, the implicitly defined feedback law relies on the solution of a potentially non-convex optimization problem, while in reference governors, updating an appropriate reference signal is generally nontrivial. Instead of directly addressing the combined control problem (i.e., avoidance and stabilization), most MPC and reference governor approaches partition the problem into path generation or trajectory planning and subsequent controller design for path following and reference tracking. In contrast to the reactive controller designs aforementioned, once a viable path or trajectory has been identified, topological obstructions are circumvented. This is achieved by reducing the problem to a reference tracking or path-following problem, as illustrated in prior research. However, ensuring that the system's state remains within a safe vicinity of the path remains challenging, particularly for underactuated dynamical systems. For a comprehensive overview of robot navigation in complex environments, we refer readers to [8].

This paper presents a controller design to address obstacle avoidance and convergence properties simultaneously by remaining in the vicinity of a predefined path with an endpoint, using Lyapunov arguments and forward invariance. Specifically, we design a dynamic controller assuming that a given path through a cluttered environment and a known Lyapunov function enable convergence to an arbitrary point on the path. This controller guarantees that the system's state remains in a safe neighborhood around the path, with the state asymptotically converging to the path's endpoint. Safety is ensured through forward invariance of sublevel sets of Lyapunov-like functions. At the same time, convergence is guaranteed by gradually shifting reference points on the path forward and through local asymptotic stability arguments of sets. Regarding updating the reference point, the approach resembles controller designs employing reference governors. While the controller avoids deadlocks by design, combined convergence and avoidance properties are only guaranteed locally, i.e., if the closed-loop dynamics is initialized in a

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vicinity of the path. This controller design is closely related to prior work in [10], [11], with distinctions lying in its focus on more general dynamical systems and more general Lyapunov functions.

The paper is structured as follows: Section II introduces the problem of interest and provides illustrative examples; Section III formalizes the controller design and presents the main convergence and safety result; Section IV demonstrates the closed-loop properties of the controller through numerical simulations based on two examples; and Section V concludes the paper with final remarks. Proofs and Lyapunov derivations can be found in [5].

**Notation:** The Euclidean norm is denoted as $|\cdot|$, i.e., for $x \in \mathbb{R}^n$, $|x| = \sqrt{x^\top x}$. For a closed set $\mathcal{A} \subset \mathbb{R}^n$ we define $|\cdot|_\mathcal{A}$ as $|x|_\mathcal{A} = \min_{y \in \mathcal{A}} |x-y|$. The boundary of $\mathcal{A}$ is denoted by $\partial\mathcal{A}$. The identity matrix of appropriate dimension is denoted as $I \in \mathbb{R}^{n\times n}$. A closed sphere of radius $r > 0$ centered around a closed set $\mathcal{A} \subset \mathbb{R}^n$ is denoted by $B_r(\mathcal{A}) = \{x \in \mathbb{R}^n | \; |x|_\mathcal{A} \le r\}$. For two sets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^n$, $\mathcal{A} + \mathcal{B} = \{a + b \in \mathbb{R}^n | a \in \mathcal{A}, b \in \mathcal{B}\}$ denotes the Minkowski sum. Additionally, $\mathcal{K}_\infty$- and $\mathcal{KL}$-functions are used throughout the paper, whose definitions can be found in [12, Chapter 1], for example. The rotation matrix for $\phi \in \mathbb{R}$ is denoted by

$$R(\phi) = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}. \tag{1}$$

## II. Problem Formulation

Consider a dynamical system of the general form

$$\dot{x} = f(x,u), \qquad z = h(x) \tag{2}$$

where $x \in \mathbb{R}^n$ denotes the state of the system, $u \in \mathbb{R}^m$ is the control input and $z \in \mathbb{R}^p$ is the output of interest. The function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous and $h : \mathbb{R}^n \to \mathbb{R}^p$ is a continuously differentiable function,

We assume that the whole state $x$ is either known or can be estimated for use as feedback information in the controller design. As such, the output $z$ does not represent a measured output but rather encodes a signal associated with the stabilization problem. Based on (2) induced equilibria of the $x$-dynamics characterized through a pair $(x_e, u_e) \in \mathbb{R}^{n+m}$ can be defined as

$$\Gamma = \left\{(x_e,u_e) \in \mathbb{R}^{n+m} | 0 = f(x_e,u_e)\right\}. \tag{3}$$

Throughout the paper, we make use of the following assumption, which ensures that by selecting the correct state-input pair, any point $z \in \mathbb{R}^p$ is an equilibrium of the output dynamics $\dot{z} = \frac{d}{dt}h(x) = \frac{dh}{dx}(x)\dot{x} = \frac{dh}{dx}(x)f(x,u)$.

*Assumption 1:* For all constant $z_e \in \mathbb{R}^p$ there exists $(x_e, u_e) \in \Gamma$ such that $z_e = h(x_e)$. Moreover, there exists a Lipschitz continuous set-valued map $G : \mathbb{R}^p \rightrightarrows \Gamma$,

$$G(z_e) = \{(x_e,u_e) \in \mathbb{R}^{n+m} | z_e = h(x_e), \; (x_e,u_e) \in \Gamma\}.$$

$\diamond$

Assumption 1 implicitly ensures that $G(z_e) \neq \emptyset$ for all $z_e \in \mathbb{R}^p$. For a definition of Lipschitz continuity for set-valued maps we refer to [4] or [3, Def. 1.4.5]. In the following, we use $G_1$ and $G_2$ to refer to the first $n$ and the last $m$ components of $G$, i.e., $G(z_e) = \left[G_1(z_e)^\top, G_2(z_e)^\top\right]^\top$.

*Remark 1:* Note that Assumption 1 is restrictive. It can be relaxed using a more general assumption requiring that for all $z_e \in \mathbb{R}^p$ constant, there exist absolutely continuous functions $x_e : \mathbb{R}_{\ge 0} \to \mathbb{R}^n$, $u_e : \mathbb{R}_{\ge 0} \to \mathbb{R}^m$ such that

$$\begin{aligned} \dot{x}_e(t) &= f(x_e(t), u_e(t)) && \text{for almost all } t \in \mathbb{R}_{\ge 0}, \\ z_e &= h(x_e(t)) && \text{for all } t \in \mathbb{R}_{\ge 0}. \end{aligned}$$

This setting encompasses also cases for which $\frac{dh}{dx}(x(t))f(x(t),u(t)) = 0$ holds for almost all $t \in \mathbb{R}_{\ge 0}$ even though $f(x(t),u(t))$ is not necessarily zero ($f(x(t),u(t)) \neq 0$). To streamline the concepts presented in this paper, we focus exclusively on Assumption 1. $\circ$

The set-valued maps $G_1$, $G_2$ allow us to define a pair $(x_e, u_e)$ through the output, which is illustrated on two simple examples.

*Example 1 (Double integrator in 2D space):* Consider an agent moving in the 2-dimensional (2D) space modeled as a double integrator

$$\dot{x} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} u, \qquad z = \begin{bmatrix} I & 0 \end{bmatrix} x \tag{4}$$

with $x \in \mathbb{R}^4$, $u, z \in \mathbb{R}^2$. Here $\Gamma$ defined in (3) is given by

$$\Gamma = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \times \{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\}$$

with $\operatorname{span}\{v_1, v_2\} = \{\lambda_1 v_1 + \lambda_2 v_2 | \lambda_1, \lambda_2 \in \mathbb{R}\}$. With respect to Assumption 1, the set-valued maps $G_1$ and $G_2$ can be defined as Lipschitz continuous functions

$$G_1(z_e) = \left( \begin{bmatrix} z_{e_1} \\ z_{e_2} \\ 0 \\ 0 \end{bmatrix} \right), \quad G_2(z_e) = \left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\right) \tag{5}$$

showing that Assumption 1 is satisfied. $\triangle$

*Example 2 (Unicycle dynamics):* Consider the unicycle dynamics

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} u_1\cos(\theta) \\ u_1\sin(\theta) \\ u_2 \end{bmatrix} \tag{6}$$

where $\Gamma$ is given by $\Gamma = \mathbb{R}^3 \times \{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\}$. Accordingly, by choosing for $z = [p_1, p_2, \theta]^\top$, the pose of the system, one can define the Lipschitz continuous functions $G_1(z_e) = z_e$ and $G_2(z_e) = 0$.

When the output of interest lies solely in the position, i.e., $z = p$ then $\Gamma$ does not change (i.e., $\Gamma = \mathbb{R}^3 \times \{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\}$), but we can redefine $G_1$ and $G_2$ as the set-valued maps

$$G_1(z_e) = \{z_e\} \times \mathbb{R} \quad \text{and} \quad G_2(z_e) = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right].$$

$\triangle$

While Assumption 1 and the examples so far focus on equilibria of the dynamical system (2), as a next step we focus on the existence of control laws stabilizing arbitrary but fixed output signals $z_e \in \mathbb{R}^p$.

*Assumption 2:* There exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, $V_\cdot(\cdot) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_{\ge 0}$, $(x, z_e) \mapsto V_{z_e}(x)$, continuously differentiable with respect to $x$ and locally Lipschitz continuous with respect to $(x, z_e)$, and a locally Lipschitz continuous feedback law $u_\cdot(\cdot) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ such that the solutions of (2) are forward complete for all $(z_e, x) \in \mathbb{R}^p \times \mathbb{R}^n$ and

$$\alpha_1(|x|_{G_1(z_e)}) \leq V_{z_e}(x) \leq \alpha_2(|x|_{G_1(z_e)}) \quad (7)$$
$$\langle \nabla V_{z_e}(x), f(x, u_{z_e}(x))\rangle \leq -\alpha_3(V_{z_e}(x)) \quad (8)$$

holds for all $(z_e, x) \in \mathbb{R}^p \times \mathbb{R}^n$. $\diamond$

Based on Assumption 2, the following result can be stated, which follows from the more general result [17, Thm. 1].

*Proposition 1:* Consider the dynamical system (2) and let $z_e \in \mathbb{R}^p$ be fixed. If Assumptions 1 and 2 are fulfilled, then there exists a feedback law $u_{z_e} : \mathbb{R}^n \to \mathbb{R}^m$ and $\beta \in \mathcal{KL}$ such that

$$|x(t)|_{G_1(z_e)} \leq \beta(|x(0)|_{G_1(z_e)}, t), \quad \forall\, t \in \mathbb{R}_{\geq 0} \quad (9)$$

for all $x_0 \in \mathbb{R}^n$. Moreover, forward invariance

$$x(t) \in \{x \in \mathbb{R}^n | V_{z_e}(x) \leq V_{z_e}(x_0)\}, \quad \forall\, t \in \mathbb{R}_{\geq 0} \quad (10)$$

is satisfied for all $x_0 \in \mathbb{R}^n$. $\lrcorner$

*Remark 2:* Compared to [17, Thm. 1] we use (9) instead of $\alpha_1(|x(t)|_{G_1(z_e)}) \leq \beta(\alpha_2(|x(0)|_{G_1(z_e)}), t)$ for all $t \in \mathbb{R}_{\geq 0}$, which is possible since $\beta(\cdot, \cdot) \in \mathcal{KL}$ if and only if $\alpha_1^{-1}(\beta(\alpha_2(\cdot), \cdot)) \in \mathcal{KL}$. $\circ$

Note that (9) implies $|x(t)|_{G_1(z_e)} \to 0$ for $t \to \infty$, i.e., $h(x(t)) \to z_e$ for $t \to \infty$. We continue with Examples 1 and 2 for an illustration of Assumption 2 and Proposition 1.

*Example 3 (Double integrator continued):* Since system (4) is controllable a feedback gain $K \in \mathbb{R}^{2\times 4}$, $u = Kx$, stabilizing the origin and a corresponding quadratic Lyapunov function exist. The feedback gain as well as a positive definite matrix $P \in \mathbb{R}^{4\times 4}$ defining the Lyapunov function can be obtained as the solution of the Riccati equation

$$A^\top P + PA + I - PBR^{-1}B^\top P = 0, \quad (11)$$

where $R \in \mathbb{R}^{2\times 2}$ denotes an arbitrary positive definite matrix (see [12, Theorem 14.4], for a corresponding result, for example). Since $G_1$ in (5) is a function (i.e., a special set-valued map) one can define:

$$\begin{aligned} V_{z_e}(x) &= (x - G_1(z_e))^\top P(x - G_1(z_e)),\\ \alpha_i(r) &= c_i r^2, \quad i \in \{1, 2\}, \quad \alpha_3(r) = c_3 r,\\ \alpha_3(V_{z_e}(x)) &= c_3 V_{z_e}(x),\\ u_{z_e}(x) &= K(x - G_1(z_e)), \quad K = -R^{-1}B^\top P \end{aligned}$$

for appropriately selected parameters $c_1, c_2, c_3 \in \mathbb{R}_{>0}$, and it follows that Assumption 2 is satisfied. $\triangle$

*Example 4:* (Unicycle continued) We focus on the unicycle dynamics with output $z = p$. Additionally, we define the following positive (semi-)definite matrices

$$P_1 = \left[\begin{smallmatrix} 1 & 1\\ 1 & 2\end{smallmatrix}\right], \; P_2 = \left[\begin{smallmatrix} 1 & 0\\ 0 & 0\end{smallmatrix}\right], \; P_3 = \left[\begin{smallmatrix} 0 & 0\\ 0 & 1\end{smallmatrix}\right].$$

Then it can be shown that a Lyapunov function according to Assumption 2 is given by

$$\begin{aligned} V_{z_e}(x) = &\tfrac{1}{2}(z - z_e)^\top R(\theta) P_1 R(\theta)^\top (z - z_e)\\ &+ \tfrac{1}{4}\left((z - z_e)^\top R(\theta) P_2 R(\theta)^\top (z - z_e)\right)^4 \quad (12)\\ &+ \tfrac{1}{4}\left((z - z_e)^\top R(\theta) P_3 R(\theta)^\top (z - z_e)\right)^4, \end{aligned}$$

where $R(\cdot)$ denotes the rotation matrix defined in (1) and corresponding $\mathcal{K}_\infty$-functions are given by

$$\begin{aligned} &\alpha_1(r) = \tfrac{\lambda_{\min(P_1)}}{2} r^2, \quad \alpha_2(r) = \tfrac{\lambda_{\max(P_1)}}{2} r^2 + \tfrac{1}{2} r^4\\ &\alpha_3(r) = \tfrac{5}{2}(\alpha_2^{-1}(r))^4 \end{aligned}$$

(and where $\lambda_{\min(\cdot)}$ and $\lambda_{\min(\cdot)}$ denote the smallest/largest eigenvalue of a symmetric matrix, respectively). A matching feedback law can be defined as

$$u_{z_e}(x) = 20 \begin{bmatrix} \bar{z}_1 \bar{z}_2^2 + \frac{5}{4}\bar{z}_1^3 + \bar{z}_2^3\\ \bar{z}_1 \bar{z}_2 \end{bmatrix}, \quad (13)$$

which relies on the coordinate transformation

$$\bar{z} = -R(\theta)^\top (z - z_e). \quad (14)$$

The derivations can be found in [5, Appendix A and B]. Here, (9) guarantees that $z$ converges to $z_e$. Convergence of $\theta$ implicitly follows from the fact that $u_{z_e}(x) \to 0$ for $z \to z_e$, but $\theta$ can converge to an arbitrary value $\theta \in \mathbb{R}$. $\triangle$

Proposition 1 provides the framework to ensure the convergence of $z(t) = h(x(t))$ towards a fixed output of interest $z_e \in \mathbb{R}^p$ for $t \to \infty$. As a last step, preceding the main result in the subsequent section, we introduce the concept of a path in $\mathbb{R}^p$ and the corresponding safe neighborhood to design control laws ensuring convergence to the vicinity of the path first and then to the desired point $z_e$.

*Definition 1 (Normalized path with endpoint):* A Lipschitz continuous function $s : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ is called normalized path with endpoint $s^\star \in \mathbb{R}^p$ if it satisfies the following properties

$$\exists\, T \in \mathbb{R}_{\geq 0} \text{ so that } s(t) = s(T) = s^\star \; \forall\, t \geq T$$

and $|\dot{s}(t)| = 1$ for almost all $t \in [0, T]$. $\spadesuit$

*Remark 3:* Note that $|\dot{s}(t)| = 1$ for almost all $t \in [0, T]$ in Definition 1 can be assumed without loss of generality. Lipschitz continuity of $s(\cdot)$ ensures that $s(\cdot)$ is differentiable for almost all $t \in [0, T]$, which can be weakened if necessary. $\circ$

*Definition 2:* (Sublevel sets and safety tubes) Let $V_\cdot(\cdot) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_{\geq 0}$, $(x, z_e) \mapsto V_{z_e}(x)$, be continuously differentiable with respect to $x$ and locally Lipschitz continuous with respect to $(x, z_e)$. Moreover, let $d : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_{>0}$ be a Lipschitz continuous positive function. Then, we define corresponding sublevel sets as

$$\begin{aligned} \mathcal{S}_{(z_e,x)} &= \{x \in \mathbb{R}^n | V_{z_e}(x) \leq d(x, z_e)\},\\ \mathcal{H}_{(z_e,x)} &= \{h(x) \in \mathbb{R}^p | x \in \mathcal{S}_{(z_e,x)}\}. \end{aligned} \quad (15)$$

Let $s(\cdot)$ be a normalized path according to Definition 1 and let $x(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ be an absolutely continuous function. Then

$$\mathcal{S} = \bigcup_{t \in \mathbb{R}_{\geq 0}} \mathcal{S}_{(s(t),x(t))}, \qquad \mathcal{H} = \{h(x) | x \in \mathcal{S}\}, \quad (16)$$

define corresponding safety tubes. $\spadesuit$

To illustrate the role of the function $d(\cdot, \cdot)$, consider a closed unsafe set $\mathcal{O} \subset \mathbb{R}^p$ representing obstacles. Then, intuitively, $\mathcal{H}_{(z_e,x)} \cap \mathcal{O} = \emptyset$ ensures that a Lyapunov function's forward invariant sublevel set does not contain parts of the unsafe set. Accordingly, under Assumption 2, if $x(0) \in \mathcal{S}_{(z_e,x(0))}$ and $\mathcal{H}_{(z_e,x(0))} \cap \mathcal{O} = \emptyset$, then $h(x(t)) \notin \mathcal{O}$ can be guaranteed for all $t \in \mathbb{R}_{\geq 0}$ by appropriately selecting the input $u$. Note that in this context, $d(\cdot,\cdot)$ depends on the reference point of interest $z_e$ and the current state $x$ of the system. Similarly, $\mathcal{H} \cap \mathcal{O} = \emptyset$ guarantees that there exists an input $u$ safely steering the states of a system through an unsafe environment with obstacles. A path according to Definition 1 through a cluttered environment can for example be obtained through convex lifting techniques [9], [13]. Here, we focus on a controller design with the properties above under the assumption that a path is known.

## III. Controller design

In this section, we design and investigate the properties of a dynamic feedback controller that asymptotically stabilizes the output of the dynamical system (2) at an arbitrary position $z \in \mathbb{R}^p$ corresponding to the endpoint of a path. To illustrate the control design, we consider the dynamical system (2) and assume that Assumptions 1 and 2 are satisfied. Let $s : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ be a path defined according to Definition 1 and define $d(\cdot,\cdot)$, $\mathcal{S}_{z_e,x}$ and $\mathcal{S}$ according to Definition 2. Then we extend the static feedback controller from Assumption 2 to the dynamic feedback controller

$$\kappa = u_{s(\tau)}(x) \tag{17}$$
$$\dot{\tau} = \max\{0, \min\{c(V_{s(\tau)}(x) - d(x, s(\tau))), 2T - \tau\}\} \tag{18}$$

for $c \in \mathbb{R}_{>0}$.

*Remark 4:* The term $2T - \tau$ in the dynamics (18) is used to ensure that $\tau$ remains bounded for appropriately selected initial conditions. The right-hand side in (18) implies that $\dot{\tau} \geq 0$ and thus $\tau(\cdot)$ is monotonically increasing by construction. ○

The controller leads to the closed-loop dynamics

$$\begin{aligned}\begin{bmatrix}\dot{x}\\ \dot{\tau}\end{bmatrix} &= F(x,\tau) \qquad (19)\\ &:= \begin{bmatrix} f(x, u_{s(\tau)}(x)) \\ \max\{0, \min\{c(d(x, s(\tau)) - V_{s(\tau)}(x)), 2T - \tau\}\}\end{bmatrix}.\end{aligned}$$

*Lemma 1:* Let Assumptions 1 and 2 be satisfied, and let $V$ and $u$ come from Assumption 2. Let $s$ be a path according to Definition 1 and let $d(\cdot,\cdot)$, $\mathcal{S}_{z_e}$, $\mathcal{S}$ be defined according to Definition 2. Then the function $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ in (19) is locally Lipschitz continuous on $\mathbb{R}^n \times \mathbb{R}$. ⌟

*Proof:* The proof follows immediately from the Lipschitz continuity of $f$, $u$, $s$, $d$, $\min\{\cdot,\cdot\}$, and $\max\{\cdot,\cdot\}$. ■

Lemma 1 is introduced to ensure that solutions of (19) exist and are unique.

*Lemma 2:* Let the assumptions of Lemma 1 be satisfied. Consider an arbitrary initial condition $(x_0, \tau_0) \in \mathcal{S}_{(s(\tau_0),x_0)} \times [0, 2T]$. Then the solution of (19) satisfies

$$(x(t), \tau(t)) \in \bigcup_{t \in \mathbb{R}_{\geq 0}} \mathcal{S}_{(s(\tau(t)),x(t))} \times [0, 2T]$$

for all $t \in \mathbb{R}_{\geq 0}$. ⌟

The proof can be found in the preprint [5].

From the above lemmas, we present hereafter the main result of the paper proving the convergence

$$z(t) = h(x(t), u_{s(\tau(t))}(x(t))) \to s^\star \tag{20}$$

of the closed-loop system as $t \to \infty$.

*Theorem 1:* Let $\mathcal{C} \subset \mathbb{R}^n$ be a compact set such that all solutions of the closed-loop system (19) with $(x_0, \tau_0) \in \mathcal{C} \cap \mathcal{S}_{(s(\tau_0),x_0)} \times [0, 2T]$ are bounded. If the assumptions of Lemma 1 are satisfied then, for any arbitrary initial condition $(x_0, \tau_0) \in \mathcal{C} \cap \mathcal{S}_{(s(\tau_0),x_0)} \times [0, 2T]$, the solution of (19) ensures that $\tau(t) \to 2T$ and (20) holds as $t \to \infty$. ⌟

The proof can be found in the preprint [5].

*Remark 5:* Locally minimally invasive controller designs, typically used in the context of control barrier functions and safety [1], [2], have become popular over the last years. A similar approach is also possible in this paper. For instance, if the Lyapunov-like function in Assumption 2 is replaced by a control Lyapunov-like function, then instead of predefining the control law as in Assumption 2, one can replace condition (8) with $\forall (z_e, x) \in \mathbb{R}^p \times \mathbb{R}^n$ there exists $u \in \mathbb{R}^m$ such that

$$\langle \nabla V_{z_e}(x), f(x,u)\rangle \leq -\alpha_3(V_{z_e}(x)).$$

Then, one can define a minimally invasive control law implicitly through the optimization problem

$$\begin{aligned} u_{z_e}(x) \in \operatorname{argmin}_{u \in \mathbb{R}^m} & |u - \nu(x)|^2 \\ \text{s.t.} \quad & \langle \nabla V_{z_e}(x), f(x,u)\rangle \leq -\alpha_3(V_{z_e}(x))\end{aligned}$$

where $\nu : \mathbb{R}^n \to \mathbb{R}^m$ denotes an arbitrary Lipschitz continuous feedback law guaranteeing $h(x(t)) \to s^\star$ for $t \to \infty$ without avoidance guarantees. The analysis of minimally invasive controller designs is left for future work. ○

## IV. Numerical simulations

We illustrate the design and the performance of the controller introduced in Section III based on two examples. We consider the simple double integrator discussed in Example 1, and the dynamic extension of the unicycle in Example 2.

### A. Path in a cluttered environment

We consider the following path with endpoint $s : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$ defined as

$$s(t) = \begin{cases} \frac{1}{\sqrt{2}}\begin{bmatrix} t \\ t \end{bmatrix} & \text{if } t \in [0, \sqrt{2}], \\ \frac{1}{\sqrt{2}}\begin{bmatrix} t \\ 2\sqrt{2} - t \end{bmatrix} & \text{if } t \in [\sqrt{2}, 2\sqrt{2}], \\ \frac{1}{\sqrt{2}}\begin{bmatrix} t \\ t - 2\sqrt{2} \end{bmatrix} & \text{if } t \in [2\sqrt{2}, 3\sqrt{2}], \\ \frac{1}{\sqrt{2}}\begin{bmatrix} t \\ 4\sqrt{2} - t \end{bmatrix} & \text{if } t \in [3\sqrt{2}, 4\sqrt{2}], \\ \frac{1}{\sqrt{2}}\begin{bmatrix} t \\ t - 4\sqrt{2} \end{bmatrix} & \text{if } t \in [4\sqrt{2}, 5\sqrt{2}], \\ \begin{bmatrix} 5 \\ 1 \end{bmatrix} & \text{if } t \in [5\sqrt{2}, \infty). \end{cases} \tag{21}$$

The scaling, i.e., the multiplication with $\frac{1}{\sqrt{2}}$ is necessary to ensure that the condition $\dot{s}(t) = 1$ is satisfied for almost all $t \in \mathbb{R}_{\geq 0}$. In addition to the path we consider that the set of obstacles consists of the union $N \in \mathbb{N}$ circular obstacles

$$\mathcal{O} = \bigcup_{i=1}^{N} B_{r_i}(c_i), \quad r_i \in \mathbb{R}_{>0},\ c_i \in \mathbb{R}^2. \tag{22}$$

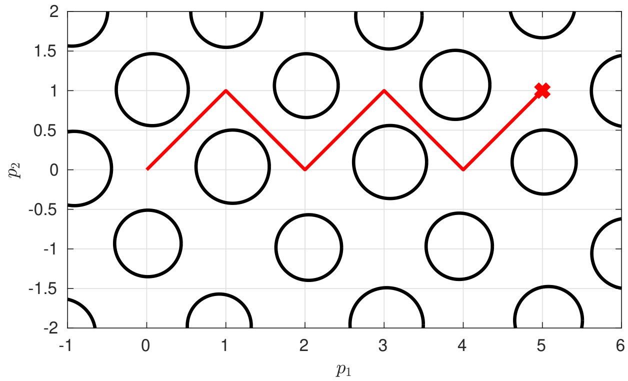


Fig. 1. Example of a path with endpoint $s^\star = [5, 1]^\top$ (red) in a cluttered environment with circular obstacles (black).

The path with endpoint together with the set $\mathcal{O}$ is visualized in Figure 1. For a given Lyapunov function $V_\cdot(\cdot)$, the function $d(\cdot, \cdot)$ defining safe sublevel sets (see Definition 2) is computed online based on a discretization of the boundary of $\mathcal{O}$, i.e., we represent the obstacle through a set of points

$$q^i \in \partial\mathcal{O}, \quad i \in \{1, , \dots, M\} \subset \mathbb{N} \tag{23}$$

for $M \in \mathbb{N}$. The specific implementation and definition of $d(\cdot, \cdot)$ is made precise in the next sections based on the individual examples.

### B. Double integrator controller illustration

Recall dynamics (4). To define a control law and a corresponding Lyapunov function we solve the Riccati equation (11) for $R = 10^{-3} I$ leading to the controller

$$u_{z_e}(x) = K \begin{bmatrix} x_1 - z_{e,1} \\ x_2 - z_{e,2} \\ x_3 \\ x_4 \end{bmatrix}, \quad K = \begin{bmatrix} -31.6 & 0 & -32.6 & 0 \\ 0 & -31.6 & 0 & -32.6 \end{bmatrix}$$

and the Lyapunov function

$$V_{z_e}(x) = \begin{bmatrix} x_1 - z_{e,1} \\ x_2 - z_{e,2} \\ x_3 \\ x_4 \end{bmatrix}^\top P \begin{bmatrix} x_1 - z_{e,1} \\ x_2 - z_{e,2} \\ x_3 \\ x_4 \end{bmatrix}$$

$$P = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{12}^\top & P_{22} \end{array}\right] = \left[\begin{array}{cc|cc} 1.03 & 0 & 0.03 & 0 \\ 0 & 1.03 & 0 & 0.03 \\ \hline 0.03 & 0 & 0.03 & 0 \\ 0 & 0.03 & 0 & 0.03 \end{array}\right].$$

To define appropriate sublevel sets $d(\cdot, \cdot)$ such that safety is guaranteed, we first rewrite the Lyapunov function

$$\begin{aligned} W_{z_e}(\chi_1, \chi_2) = (\chi_1 - z_e)^\top P_{11} (\chi_1 - z_e) \\ + 2(\chi_1 - z_e)^\top P_{12} \chi_2 + \chi_2^\top P_{22} \chi_2 \end{aligned} \tag{24}$$

and we consider the optimization problem

$$\eta_{z_e}(\chi_1) = \min_{\chi_2 \in \mathbb{R}^2} W_{z_e}(\chi_1, \chi_2). \tag{25}$$

Here, $\eta$ defines the smallest value of the Lyapunov function for $\chi_1$ and $z_e$ fixed and $\chi_2$ arbitrary. Based on $\eta$ and the disretization of the boundary of the obstacle in (23) we define

$$d(x, z_e) = \min_{i \in \{1, \dots, M\}} \eta_{z_e}(q^i). \tag{26}$$

Note that in this particular example $\tilde{d}(z_e) = d(z_e, x)$ is only a function of $z_e$, but not a function of the state $x$.[1] Since $W_{z_e}(\chi_1, \cdot)$ is a quadratic function, the minimum in (25) can be calculated explicitly and it holds that

$$\eta_{z_e}(\chi_1) = W_{z_e}(\chi_1, -P_{22}^{-1} P_{12}^\top (\chi_1 - z_e)). \tag{27}$$

Accordingly, if $M$ in (23) is selected reasonably small, $\tilde{d}(z_e)$ can be calculated online.

Figure 2 shows the closed-loop solution $z(x(t)) = Cx(t)$ of the double integrator (in blue) initialized through $x_0 = [-0.25, 0.25, 0, 0]^\top$, together with the input corresponding to the closed loop solution. As expected through the theoretical results discussed in Section III, $z(t)$ asymptotically converges to $s^\star = [5, 1]^\top$ while avoiding the circular obstacles. In addition, level sets of the projected Lyapunov function

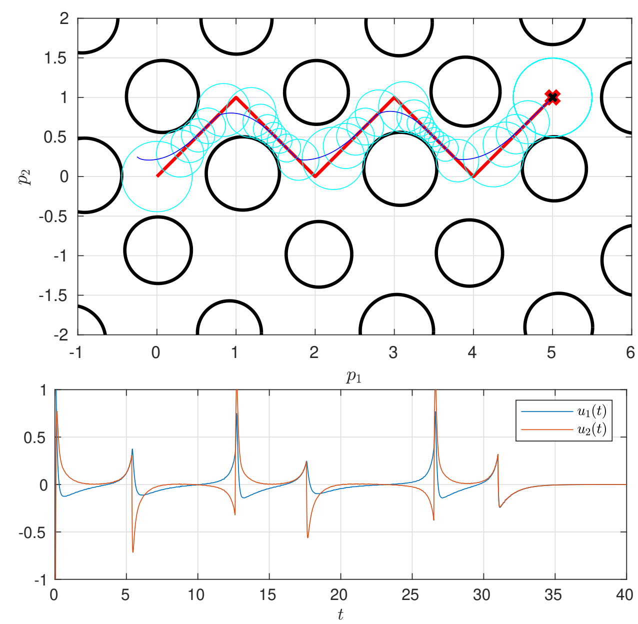


Fig. 2. Top: Closed loop solution of the double integrator dynamics (blue) avoiding circular obstacles (black) staying in safety neighborhoods (cyan) centered around a path (red). Bottom: Input of the avoidance controller with convergence guarantees corresponding to the closed-loop solution.

$\eta_{z_e}(\cdot)$ in (27) are shown in cyan. While the safety neighborhoods $\mathcal{S}_{(z_e,x)}$ are updated continuously, only snapshots taken once every couple of second are shown in Figure 2. For the simulations, the parameter $c = 10$ is used in (18) to define the $\tau$-dynamics.

### C. Unicycle controller illustration

As a second example consider the extended unicycle dynamics

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \\ w \end{bmatrix}, \quad \begin{bmatrix} \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} a \\ q \end{bmatrix} \tag{28}$$

[1] The function $\tilde{d}(z_e)$ can be modified to $\tilde{d}(z_e) - \varepsilon$, $\varepsilon > 0$, to take the error from the discretization of the boundary of the obstacles into account. For simplicity of presentation the discretization error is ignored here.

with state $x = [p_1, p_2, \theta, v, w]^\top$ input $u = [a, q]^\top$ and output of interest $p = [p_1, p_2]^\top$. Additionally, we use the notation $\chi_1 = [p_1, p_2, \theta]^\top$ and $\chi_2 = [v, w]^\top$ to define the function $d$.

As shown in [5, Appendix C] a function according to Assumption 2 is given by

$$W_{z_e}(\chi_1, \chi_2) = V_{z_e}(\chi_1) + \tfrac{1}{2}(v - u_{z_e,1}(\chi_1))^2 + \tfrac{1}{2}(w - u_{z_e,2}(\chi_1))^2,$$

extending the Lyapunov-like function (12) and where $u_{z_e}(\chi_1)$ denotes the input in (13). Similar to the example of the double integrator we define the function $d(\cdot,\cdot)$ by considering the projection

$$\eta_{z_e}(\chi_1) = \min_{\chi_2 \in \mathbb{R}^2} W_{z_e}(\chi_1, \chi_2) = V_{z_e}(\chi_1). \tag{29}$$

Based on $\eta_{ze}(\chi_1)$, safe sublevel sets are characterized through the definition

$$d(x, z_e) = \min_{i \in \{1,\dots,M\}} \eta_{z_e}\left(\begin{bmatrix} q_1^i \\ q_2^i \\ \theta \end{bmatrix}\right). \tag{30}$$

In contrast to (26), the function $d$ in (30) explicitly depends on the state $x$ (or more precisely on the state $\theta$).

Figure 3 shows an illustration analogue to Figure 2 initialized through $x_0 = [-0.25, 0.25, 0, 0, 0]^\top$. Through the particular selection of $x_0$, safety is initially not guaranteed, i.e., for $t = 0$ it holds that $W_{z_e(\tau(0))}(x(0), 0) > d(x(0), z_e(\tau(0))$. However, once the condition $W_{z_e(\tau(\tilde{t}))}(\chi_1(\tilde{t}), \chi_2(\tilde{t})) \leq d(x(\tilde{t}), z_e(\tau(\tilde{t}))$ is satisfied for $\tilde{t} \in \mathbb{R}_{\geq 0}$ the condition holds for all $t \geq \tilde{t}$.

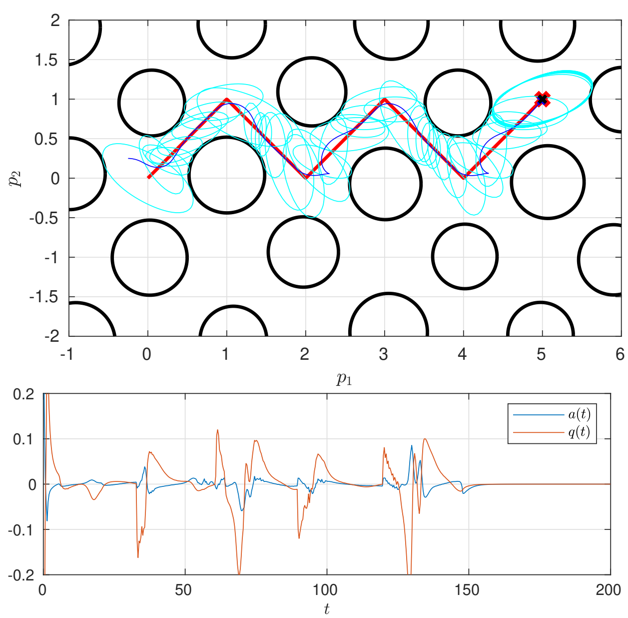


Fig. 3. Top: Closed loop solution of the extended unicycle dynamics (28) using the same setting as in Figure 2. Bottom: Input corresponding to the closed loop solution of the extended unicycle dynamics (28).

The feedback law is defined as

$$\begin{aligned} a(x) &= -v + u_{z_e,1}(x) + \left(75\bar{p}_1^2 + 20\bar{p}_2^2\right)(-v + w\bar{p}_2) \\ &\quad - \left(60\bar{p}_2^2 + 40\bar{p}_1\bar{p}_2\right) w\bar{p}_1, \\ q(x) &= -w + u_{z_e,2}(x) + 20\bar{p}_2(-v + w\bar{p}_2) - 20\bar{p}_1^2 w, \end{aligned}$$

which relies on the feedback law (13) and on the coordinate transformation (14) and is derived in [5, Appendix C].

## V. Conclusions

In this paper, we have introduced a general controller for nonlinear dynamical systems with safety and convergence properties, navigating the output of interest of a system through a cluttered environment based on a predefined path. While safety is guaranteed through forward invariance of sublevel sets of Lyapunov functions, convergence is guaranteed through an appropriate dynamic update of reference point along the path. Future work will focus on the dynamic construction of Lyapunov functions along the path and on improved performance of closed-loop solutions.

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