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Linear Last-Iterate Convergence for Continuous Games with Coupled Inequality Constraints

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Abstract— **In this paper, the generalized Nash equilibrium (GNE) seeking problem for continuous games with coupled affine inequality constraints is investigated in a partial-decision information scenario, where each player can only access its neighbors' information through local communication although its cost function possibly depends on all other players' strategies. To this end, a novel decentralized primal-dual algorithm based on consensus and dual diffusion methods is devised for seeking the variational GNE of the studied games. This paper also provides theoretical analysis to show that the designed algorithm converges linearly for the last-iterate, which, to our best knowledge, is the first to propose a linearly convergent GNE seeking algorithm under coupled affine inequality constraints. Finally, a numerical example is presented to demonstrate the efficiency of the obtained theoretical results.**

I. INTRODUCTION

Game theory, which is the study of mathematical models of strategic interactions among rational agents, has been commonly applied in artificial intelligence (AI) [1], including multi-agent AI systems [2], imitation and reinforcement learning [3], and adversarial training in generative adversarial networks [4]. A principal concept in noncooperative games is Nash equilibrium (NE), representing a stable state from which no one has incentive to deviate. The types of convergence involved in existing NE seeking algorithms can be classified into the empirical distribution (i.e., time-average) of no-regret play and day-to-day play (i.e., last-iterate convergence) [5]. In general, a no-regret learning algorithm can achieve sublinear regret bounds and generally ensure the convergence in the sense of empirical distribution, while may often fail to guarantee the last-iterate convergence, except for quite a few scenarios, such as two-player zero-sum games [6]. As such, this paper focuses on the last-iterate convergence because of its significance in many practical applications, such as generative adversarial networks [4].

On the other hand, decentralized Nash equilibrium (NE) and generalized NE (GNE) seeking problems have received considerable attention owing to the advantages of decentralized algorithms such as in scalability, reliability, robustness and efficiency. Roughly speaking, the existing decentralized algorithms for seeking NEs of noncooperative games can be divided into three categories according to the types of constraints on the players' strategy sets. Specifically, the first class is for unconstrained games [7]–[9], i.e., each player can take actions from the whole set of real numbers or vectors. The second category is for games with local and uncoupled strategy set constraints [10]–[15], while the third type is for games with coupled constraints including affine/nonlinear equality/inequality constraints [16]–[19]. Most of these existing works focus on the design and the asymptotic convergence analysis of decentralized NE seeking algorithms, yet the convergence rate of the proposed algorithms is less discussed.

In fact, the convergence rate of an algorithm is extremely important in providing useful insights into how quickly the generated sequence approaches the target point or (generalized) NEs. Therefore, proposing a decentralized NE seeking algorithm with a faster convergence rate is necessary.

Our Contribution. In this paper, we consider the GNE seeking problem for continuous games with coupled affine inequality constraints, where individual players aim to privately minimize their own cost functions by selecting a strategy profile satisfying the coupled inequality. It is noted that the cost function of each player may depend on all other players' strategies, and it is possibly impractical for all information sharing especially in large-scale networks. Hence, a partial-decision information scenario, i.e., each player can only access its own strategy and cost function, as well as neighbors' strategies through communicating via a connected graph, is considered as done in some existing works [12], [15]–[23]. In such setting, a novel decentralized primal-dual algorithm, where each player is equipped with additional variables to estimate all the other players' actions and the global dual variable, is proposed to learn the unique variational GNE of the considered game and is also rigorously proved to be linearly convergent for the last-iterate. Finally, a numerical example on a Nash-Cournot game is presented to illustrate the effectiveness of theoretical results. The main contributions of this paper are summarized as follows.

1) To the best of our knowledge, this paper is the first to propose a linearly last-iterate convergent decentralized GNE seeking algorithm for continuous games with general cost functions and coupled affine inequality constraints. It can also be seen from simulations that our designed algorithm outperforms the forward-backward (FB) algorithm in [16] in terms of the convergence rate. Furthermore, the upper bounds for required stepsizes are explicitly provided, which depend on the number

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of players, the communication structure, and the affine inequality along with the Lipschitz and monotonicity constants of the pseudo-gradient function.

2) The linearly convergent algorithm is creatively designed by modifying the typical primal-dual algorithm and drawing into a positive semi-definite matrix related to the communication topology, and simultaneously following the consensus and dual diffusion methods.

Related Work. To date, it has been proven that decentralized gradient-based algorithms to seek NEs for continuous games without constraints converge linearly [8], [9], [24]. For continuous games with local strategy set constraints, an inexact-ADMM algorithm was proposed in [15] for finding NEs, which was proved to converge relatively fast with a sublinear convergence rate $O(\frac{1}{k})$, where k is the iteration time. Then, decentralized algorithms based on gradient-play method were devised for games with local set constraints in [20], [21], [25] and were shown to possess a linear convergence rate. Nevertheless, there are few works on linearly convergent GNE seeking algorithms for games with coupled constraints. A recent study [22] proposed a proximal-point algorithm by designing a novel preconditioning matrix for learning GNE of games with strategy set constraints and affine inequality constraints, which can improve the convergence speed compared with gradient-play methods, yet the linear convergence is not established. In [23], a continuous-time GNE learning algorithm in continuous-time for continuous games with strategy set constraints and affine equality constraints was studied and theoretically shown to have an exponential convergence rate when no local strategy set constraints are involved. Recently, in [26], the method of singular perturbations analysis was leveraged to establish linear convergence of the designed fully distributed iterative GNE seeking algorithm for aggregative games with affine coupled constraints. Our paper is the first to design a linearly convergent GNE seeking algorithm for games under coupled constraints with general cost functions, while the authors in [26] studied aggregative games although set constraints were considered.

Notations. Denote by $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ the sets of real numbers, real n-dimensional column vectors and real $n\times n$-matrices, respectively. Let $\mathbb{R}^n_+$ be the set of nonnegative real vectors. The symbol $[m]$ for an integer $m>0$ represents the set $\{1,2,\ldots,m\}$. $\mathbf{1}_n \in \mathbb{R}^n$ (resp. $\mathbf{0}_n \in \mathbb{R}^n$) is a vector with all elements being 1 (resp. 0). For a vector or matrix A, its transpose is denoted as $A^\top$. $col(x_1,\ldots,x_n) := (x_1^\top,\ldots,x_n^\top)^\top$. The Kronecker product of matrices A and B is denoted as $A\otimes B$. $P_\Omega[x] := \arg\min_{y\in\Omega}\|y-x\|$ denotes the projection of the vector $x\in\mathbb{R}^n$ onto the closed convex set Ω and $N_\Omega(x) := \{v|v^\top(y-x)\le 0,\ \forall y\in\Omega\}$ is the normal cone to Ω at $x\in\Omega$. $\mathrm{diag}\{a_1,a_2,\ldots,a_n\}$ represents a diagonal matrix with a_i, $i\in[n]$, on its diagonal. Denote by $\lambda_{\min}(A)$, $\lambda_{\max}(A)$ and $\rho(A)$ the smallest eigenvalue, largest eigenvalue and spectral radius of a square matrix A, respectively. $\underline{\sigma}(A)$ is the smallest nonzero singular value of A. For a positive semi-definite matrix $M\in\mathbb{R}^{n\times n}$, denote $\|x\|_M := \sqrt{x^\top M x}$ with $x\in\mathbb{R}^n$.

II. PROBLEM FORMULATION

Consider a normal-form continuous game with N players, where each player takes its strategy (decision, action) $x_i \in \mathbb{R}^{n_i}$. Denote by $x = col(x_1,\ldots,x_N)$ and $x_{-i} := col(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_N)$ the joint action of all the players and the joint action of all the players except i, respectively. The private cost function of player i is $f_i(x_i,x_{-i})$, depending on both local variable x_i and other players' decisions x_{-i}. The objective of player i is to selfishly minimize its own cost function $f_i(x_i,x_{-i})$ subject to a coupled constraint $\Omega := \{x\in\mathbb{R}^n | Ax\le b\}$, where $A := [A_1,A_2,\ldots,A_N]$, $b := \sum_{i=1}^N b_i$, $n := \sum_{i=1}^N n_i$, $A_i\in\mathbb{R}^{m\times n_i}$ and $b_i\in\mathbb{R}^m$. Here, A_i and b_i can only be privately accessible to player i. In this setting, a strategy profile $x^* = (x_i^*, x_{-i}^*)$ is called a GNE if for each $i\in[N]$, the following inequality holds:

$$f_i(x_i^*, x_{-i}^*) \le f_i(x_i, x_{-i}^*),\ \forall x_i \in X_i(x_{-i}^*), \tag{1}$$

where $X_i(x_{-i}^*) := \{x_i\in\mathbb{R}^{n_i} | A_i x_i \le b - \sum_{j\ne i} A_j x_j^*\}$.

Some standard assumptions on this game are listed below.

Assumption 1: For each $i\in[N]$, the function $f_i(x_i,x_{-i})$ is continuously differentiable and convex with respect to x_i given any x_{-i}. For each $i\in[N]$, $\nabla_i f_i(x_i,x_{-i}) := \frac{\partial f_i(x_i,x_{-i})}{\partial x_i}$ is L-Lipschitz continuous for some $L>0$, i.e.,

$$\|\nabla_i f_i(x) - \nabla_i f_i(y)\| \le L\|x-y\|,\ \forall x,y\in\mathbb{R}^n.$$

Moreover, the constraint set Ω is nonempty and satisfies Slater's constraint qualification.

Define the pseudo-gradient $F:\mathbb{R}^n\to\mathbb{R}^n$ of game (1) as

$$F(x) := col(\nabla_1 f_1(x_1,x_{-1}),\ldots,\nabla_N f_N(x_N,x_{-N})) \tag{2}$$

for $x = col(x_1,\ldots,x_N)$ with $x_i\in\mathbb{R}^{n_i}$, $i\in[N]$.

Assumption 2: The pseudo-gradient F defined in (2) is μ-strongly monotone for some $\mu>0$, that is, there holds

$$(F(x)-F(y))^\top(x-y) \ge \mu\|x-y\|^2,\ \forall x,y\in\mathbb{R}^n.$$

From the definition of GNE in (1), the strategy profile $x^* = (x_i^*, x_{-i}^*)$ is a GNE, if and only if x_i^* is an optimal solution to the following optimization problem

$$\min_{x_i\in\mathbb{R}^{n_i}} \quad f_i(x_i, x_{-i}^*)$$
$$\text{s.t.} \quad A_i x_i \le b - \sum_{j\ne i} A_j x_j^*.$$

For this optimization problem, the Lagrangian function for each player $i\in[N]$ is given as

$$\mathcal{L}_i(x_i,\lambda_i;x_{-i}) := f_i(x_i,x_{-i}) + \lambda_i^\top(Ax-b), \tag{3}$$

where $\lambda_i\in\mathbb{R}^m_+$ is the dual variable. If the profile $x^* = (x_i^*, x_{-i}^*)$ is a GNE, then by Karush-Kuhn-Tucker (KKT) conditions of this optimization problem, there exist dual variables $\lambda_i^*\in\mathbb{R}^m_+$, $i\in[N]$, such that

$$\mathbf{0}_{n_i} = \nabla_i f_i(x_i^*, x_{-i}^*) + A_i^\top \lambda_i^*, \tag{4}$$

$$\mathbf{0}_m \in -(Ax^*-b) + N_{\mathbb{R}^m_+}(\lambda_i^*),\ i\in[N]. \tag{5}$$

If $\lambda_1^* = \lambda_2^* = \cdots = \lambda_N^* = \lambda^*$, then the GNE with dual variable λ^* is called a variational GNE, as discussed in many existing references [16]–[19], which is economically justifiable. Here, it is assumed that a variational GNE of the considered game exists as we focus on the design and linear convergence analysis of the GNE seeking algorithm rather than the existence of GNEs. From the variational inequality perspective, a variational GNE x^* is a solution to the following variational inequality:

$$F^\top(x^*)(x - x^*) \geq 0, \ \forall x \in \Omega.$$

Then, under Assumption 2, the variational GNE of game (1) is unique [27].

Note that there is no central node that bidirectly communicates with all players to provide them with global information, i.e., a partial-decision information setting, and NE seeking under partial-decision information has been investigated only recently [14], [16]. As designing NE seeking algorithms at each player needs opponents' decisions to compute the value or gradient of its cost function when not available directly, in this paper, we assume that players can only have partial-decision information, received by communicating with their neighbours over an undirected graph, denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, where $\mathcal{V} = [N]$ is the node set corresponding to all players, $\mathcal{E}$ is the edge set, and $W = (w_{ij}) \in \mathbb{R}^{N\times N}$ is the adjacency matrix. $w_{ij} > 0$ if $(i, j) \in \mathcal{E}$ meaning that players i and j can communicate directly with each other, and otherwise $w_{ij} = 0$. It is assumed that $w_{ii} > 0$ for all $i \in [N]$ in this paper. Furthermore, $w_{ij} > 0$ also means the impact of player j on i. A standard assumption is made on graph $\mathcal{G}$.

Assumption 3: $\mathcal{G}$ is connected. The adjacency matrix W is symmetric and doubly stochastic, that is, $W^\top = W$, $W\mathbf{1}_N = \mathbf{1}_N$ and $\mathbf{1}_N^\top W = \mathbf{1}_N^\top$.

Based on Assumption 3, which is a mild condition in decentralized algorithms, one has [28]

$$\sigma := \|W - \mathbf{1}_N\mathbf{1}_N^\top/N\| \in [0, 1). \tag{6}$$

Assumption 4: For each $i \in [N]$, A_i has full row rank.

Under Assumption 4, it can be ensured that $\lambda_{\min}(A_iA_i^\top) > 0$. Assumption 4 is essential in proving the linear convergence of the designed algorithm, which is often made for linear convergent algorithms for decentralized optimization with affine constraints (see [29] and references therein).

In summary, in this setting, the aim of this paper is to design a decentralized algorithm to find the unique variational GNE of game (1) and also establish the linear convergence rate of the designed algorithm.

III. THE DEVELOPED ALGORITHM AND CONVERGENCE ANALYSIS

In this section, a decentralized primal-dual algorithm is first proposed to learn the unique variational GNE for coupled constrained games in a partial-decision information scenario and then the linear convergence result is provided.

Note that each player cannot access the information of all other players in the partial-decision information setting. Then, each player is assigned two additional variables to estimate the decisions of other players and the global dual variable, respectively, through local communication via the graph $\mathcal{G}$. It can be obtained that the weighted matrix W is primitive under Assumption 3. Then, W has a simple eigenvalue 1 and other eigenvalues being in $(-1, 1)$. Therefore, $I_N - W$ is a positive semi-definite matrix, and $(I_{Nm} - W \otimes I_m)\boldsymbol{\lambda} = 0$ for $\boldsymbol{\lambda} \in \mathbb{R}^{Nm}$ if and only if $\boldsymbol{\lambda} = \mathbf{1}_N \otimes \lambda$ for some $\lambda \in \mathbb{R}^m$. Define a symmetric matrix $B \in \mathbb{R}^{N\times N}$ such that $B^2 = \frac{1}{2}(I_N - W)$, then $\lambda_{\min}(B^2) = 0$, $\lambda_{\max}(B^2) = \lambda^2_{\max}(B) < 1$, and it holds that $(B \otimes I_m)\boldsymbol{\lambda} = 0$ for $\boldsymbol{\lambda} \in \mathbb{R}^{Nm}$ is equivalent to $\boldsymbol{\lambda} = \mathbf{1}_N \otimes \lambda$ for some $\lambda \in \mathbb{R}^m$.

Denote $\Pi := \mathrm{diag}\{A_1, \ldots, A_N\}$, $\mathbf{b} := col(b_1, \ldots, b_N)$ and $\mathcal{B} := B \otimes I_m$. At iteration k, player i is endowed with variables $x_{i,k}$, $x_{ji,k}$ and $\lambda_{i,k}$ to represent its decision, the estimate of player j' decision and the estimate of the global dual variable, respectively. $x_{ii,k} = x_{i,k}$. In view of $\mathcal{B}(\mathbf{1}_N \otimes \lambda^*) = 0$, a modified primal-dual algorithm is proposed as follows:

$$\begin{aligned} x_{i,k+1} &= \sum_{j=1}^N w_{ij}x_{ij,k} - \alpha\nabla_i f_i(x_{i,k}, \mathbf{x}_{-i,k}) \\ &\quad - \alpha A_i^\top \lambda_{i,k}, \end{aligned} \tag{7a}$$

$$\mathbf{x}_{-i,k+1} = \sum_{j=1}^N w_{ij}\mathbf{x}^j_{-i,k}, \tag{7b}$$

$$\mathbf{v}_{k+1} = \boldsymbol{\lambda}_k - \mathcal{B}^2\boldsymbol{\lambda}_k + \beta(\Pi x_{k+1} - \mathbf{b}) + \mathcal{B}\mathbf{y}_k, \tag{7c}$$

$$\mathbf{y}_{k+1} = \mathbf{y}_k - \gamma\mathcal{B}\mathbf{v}_{k+1}, \tag{7d}$$

$$\boldsymbol{\lambda}_{k+1} = P_{\mathbb{R}^{Nm}_+}[\mathbf{v}_{k+1}], \tag{7e}$$

where $\mathbf{x}_{-i,k} := col(x_{1i,k}, ..., x_{(i-1)i,k}, x_{(i+1)i,k}, ..., x_{Ni,k})$, $\mathbf{x}^j_{-i,k} := col(x_{1j,k}, \ldots, x_{(i-1)j,k}, x_{(i+1)j,k}, \ldots, x_{Nj,k})$, $x_k = col(x_{1,k}, \ldots, x_{N,k})$, $\boldsymbol{\lambda}_k := col(\lambda_{1,k}, \ldots, \lambda_{N,k})$, $\mathbf{v}_k, \mathbf{y}_k \in \mathbb{R}^{Nm}$ are auxiliary variables, and $\alpha, \beta, \gamma > 0$ are stepsizes to be determined. Moreover, one can initialize $x_{i,0} \in \mathbb{R}^{n_i}$, $\mathbf{x}_{-i,0} \in \mathbb{R}^{n-n_i}$, $\boldsymbol{\lambda}_0 \in \mathbb{R}^{Nm}$ arbitrarily, and $\mathbf{y}_0 = \mathbf{0}_{Nm}$. Iteration (7) cannot be implemented in a fully decentralized manner since the matrix B is involved. In what follows, let us equivalently transfer iteration (7) into a fully decentralized algorithm. By (7c), one has

$$\begin{aligned} \mathbf{v}_{k+1} - \mathbf{v}_k &= (I_{Nm} - \mathcal{B}^2)(\boldsymbol{\lambda}_k - \boldsymbol{\lambda}_{k-1}) + \beta\Pi(x_{k+1} - x_k) \\ &\quad + \mathcal{B}(\mathbf{y}_k - \mathbf{y}_{k-1}). \end{aligned} \tag{8}$$

Substituting (7d) into (8) yields that for $k \geq 0$,

$$\begin{aligned} \mathbf{v}_{k+1} &= (I_{Nm} - \gamma\mathcal{B}^2)\mathbf{v}_k + (I_{Nm} - \mathcal{B}^2)(\boldsymbol{\lambda}_k - \boldsymbol{\lambda}_{k-1}) \\ &\quad + \beta\Pi(x_{k+1} - x_k), \end{aligned} \tag{9}$$

where $\mathbf{v}_0$, $\boldsymbol{\lambda}_{-1}$, $\mathbf{y}_{-1}$ and x_0 are set to be $\mathbf{v}_0 = \mathbf{0}_{Nm}$, $\boldsymbol{\lambda}_{-1} = \mathbf{0}_{Nm}$, $\mathbf{y}_{-1} = \mathbf{0}_{Nm}$ and $\Pi x_0 = \mathbf{b}$, respectively. Let $C = (c_{ij})_{N\times N} := B^2 = \frac{1}{2}(I_N - W)$. Then iteration (7) can be rewritten as a fully decentralized algorithm (cf. Algorithm 1).

Remark 1: Algorithm (7) is inspired by [29], [30], where the aim is to minimize a global cost function. However, the

Algorithm 1 Decentralized Primal-Dual Algorithm

Each player i maintains vector variables $x_{i,k} \in \mathbb{R}^{n_i}$, $x_{ji,k} \in \mathbb{R}^{n_j}$, $v_{i,k} \in \mathbb{R}^m$ and $\lambda_{i,k} \in \mathbb{R}^m$ at iteration k.
Initialization: For any $i \in [N]$, initialize $x_{ji,0}$ and $\lambda_{i,0}$ arbitrarily, and set $v_{i,0} = \mathbf{0}_m$, $\lambda_{i,-1} = \mathbf{0}_m$ and $A_i x_{i,0} = b_i$.
Iteration: For every player i, repeat for $k \geq 0$:

$$x_{i,k+1} = \sum_{j=1}^{N} w_{ij} x_{ij,k} - \alpha \nabla_i f_i(x_{i,k}, \mathbf{x}_{-i,k}) - \alpha A_i^\top \lambda_{i,k}, \tag{10a}$$

$$\mathbf{x}_{-i,k+1} = \sum_{j=1}^{N} w_{ij} \mathbf{x}_{-i,k}^{j}, \tag{10b}$$

$$v_{i,k+1} = v_{i,k} - \gamma \sum_{j=1}^{N} c_{ij} v_{j,k} - \sum_{j=1}^{N} c_{ij}(\lambda_{j,k} - \lambda_{j,k-1}) + \lambda_{i,k} - \lambda_{i,k-1} + \beta A_i(x_{i,k+1} - x_{i,k}), \tag{10c}$$

$$\lambda_{i,k+1} = P_{\mathbb{R}_+^m}[v_{i,k+1}]. \tag{10d}$$

problem studied here is different, i.e., continuous games, which, together with the considered partial-decision information setting, leads to that the theoretical analysis in this paper is significantly distinctive from that in [29], [30].

Remark 2: Algorithm 1, which is equivalent to iteration (7), is not a typical primal-dual method since the auxiliary variable $\mathbf{v}_k$, instead of the dual variable $\boldsymbol{\lambda}_k$, is used in (7d). This, together with the term $-\mathcal{B}^2\boldsymbol{\lambda}_k$ used in (7c), plays a critical role in establishing linear convergence when dealing with coupled inequality constraints. In fact, iteration (7) is devised based on an augmented Lagrangian function obtained by adding $-1/2\boldsymbol{\lambda}^\top\mathcal{B}^2\boldsymbol{\lambda} - \mathbf{y}^\top\mathcal{B}\boldsymbol{\lambda}$ to the Lagrangian function (3) since $\mathcal{B}\boldsymbol{\lambda}^* = \mathbf{0}_{Nm}$. Moreover, note that the projection step in (7e) is to tackle the inequality constraints, then Algorithm 1 can reduce to the case of affine equality constraints by letting $\boldsymbol{\lambda}_k = \mathbf{v_k}$ and the results obtained in this paper are applicable to solving the distributed GNE seeking problem with coupled affine equality constraints.

Next, we rewrite (10a) and (10b) into a compact form by introducing two matrices $\mathcal{R}_i$ and $\mathcal{S}_i$ for player i to manipulate its decision variable $x_{i,k}$ and the estimate $\mathbf{x}_{-i,k}$ of other players' decisions. Let

$$\mathcal{R}_i := \begin{bmatrix} \mathbf{0}_{n_i \times n_{<i}} & I_{n_i} & \mathbf{0}_{n_i \times n_{>i}} \end{bmatrix}, \tag{11}$$

$$\mathcal{S}_i := \begin{bmatrix} I_{n_{<i}} & \mathbf{0}_{n_{<i} \times n_i} & \mathbf{0}_{n_{<i} \times n_{>i}} \\ \mathbf{0}_{n_{>i} \times n_{<i}} & \mathbf{0}_{n_{>i} \times n_i} & I_{n_{>i}} \end{bmatrix}, \tag{12}$$

where $n_{<i} := \sum_{j=1}^{i-1} n_j$ and $n_{>i} := \sum_{j=i+1}^{N} n_j$. Then it can be easily verified that $\mathcal{R} := \mathrm{diag}\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ and $\mathcal{S} := \mathrm{diag}\{\mathcal{S}_1, \ldots, \mathcal{S}_N\}$ satisfy

$$\mathcal{R}\mathcal{R}^\top = I_n,\ \mathcal{S}\mathcal{S}^\top = I_{Nn-n},\ \mathcal{R}^\top\mathcal{R} + \mathcal{S}^\top\mathcal{S} = I_{Nn},$$
$$\mathcal{R}\mathcal{S}^\top = \mathbf{0},\ \mathcal{S}\mathcal{R}^\top = \mathbf{0}.$$

Denote

$$\mathbf{x}_{i,k} := col(x_{1i,k}, \ldots, x_{Ni,k}),$$
$$\mathbf{x}_k := col(\mathbf{x}_{1,k}, \ldots, \mathbf{x}_{N,k}).$$

One has $\mathcal{R}_i\mathbf{x}_{i,k} = x_{i,k}$, $\mathcal{S}_i\mathbf{x}_{i,k} = \mathbf{x}_{-i,k}$. Hence, $x_k = \mathcal{R}\mathbf{x}_k$ and $col(\mathbf{x}_{-1,k}, \ldots, \mathbf{x}_{-N,k}) = \mathcal{S}\mathbf{x}_k$. With these notations, (10a) and (10b) can be rewritten as

$$x_{k+1} = \mathcal{R}\mathcal{W}\mathbf{x}_k - \alpha\mathbf{F}(\mathbf{x}_k) - \alpha\Pi^\top\boldsymbol{\lambda}_k, \tag{13}$$

$$\mathcal{S}\mathbf{x}_{k+1} = \mathcal{S}\mathcal{W}\mathbf{x}_k, \tag{14}$$

where $\mathcal{W} := W \otimes I_n$ and

$$\mathbf{F}(\mathbf{x}_k) := col(\nabla_1 f_1(x_{1,k}, \mathbf{x}_{-1,k}), \ldots, \nabla_N f_N(x_{N,k}, \mathbf{x}_{-N,k})). \tag{15}$$

In view of $\mathbf{x}_k = \mathcal{R}^\top x_k + \mathcal{S}^\top\mathcal{S}\mathbf{x}_k$, one has

$$\mathbf{x}_{k+1} = \mathcal{W}\mathbf{x}_k - \alpha\mathcal{R}^\top\mathbf{F}(\mathbf{x}_k) - \alpha\mathcal{R}^\top\Pi^\top\boldsymbol{\lambda}_k. \tag{16}$$

Before presenting the main result, some auxiliary lemmas are first provided. All the proofs are omitted here due to limited space, which will be provided in the full-length paper.

Lemma 1: Iteration (10) or equivalently (7) has a fixed point $(\mathbf{x}^*, \mathbf{v}^*, \mathbf{y}^*, \boldsymbol{\lambda}^*)$ satisfying

$$\mathbf{x}^* = \mathcal{W}\mathbf{x}^* - \alpha\mathcal{R}^\top\mathbf{F}(\mathbf{x}^*) - \alpha\mathcal{R}^\top\Pi^\top\boldsymbol{\lambda}^*, \tag{17}$$

$$\mathbf{v}^* = \boldsymbol{\lambda}^* + \beta(\Pi x^* - \mathbf{b}) + \mathcal{B}\mathbf{y}^*, \tag{18}$$

$$\mathcal{B}\mathbf{v}^* = \mathbf{0}_{Nm}, \tag{19}$$

$$\boldsymbol{\lambda}^* = P_{\mathbb{R}_+^{Nm}}[\mathbf{v}^*], \tag{20}$$

and $\mathcal{B}^2\boldsymbol{\lambda}^* = \mathbf{0}_{Nm}$. For any fixed point $(\mathbf{x}^*, \mathbf{v}^*, \mathbf{y}^*, \boldsymbol{\lambda}^*)$, it holds that $\mathbf{x}^* = \mathbf{1}_N \otimes x^*$ for $x^* \in \mathbb{R}^n$ and $\boldsymbol{\lambda}^* = \mathbf{1}_N \otimes \lambda^*$ for $\lambda^* \in \mathbb{R}^m$ with x^* being the variational GNE of game (1) and λ^* being the optimal global dual variable. ■

Based on (17) and the uniqueness of $\mathbf{x}^*$, it can be seen that the optimal dual solution $\boldsymbol{\lambda}^*$ is also unique. For the fixed point of iteration (7), $(\mathbf{x}^*, \mathbf{v}^*, \mathbf{y}^*, \boldsymbol{\lambda}^*)$ with $\mathbf{x}^* = \mathbf{1}_N \otimes x^*$, $\mathbf{v}^* = \mathbf{1}_N \otimes v^*$, $\boldsymbol{\lambda}^* = \mathbf{1}_N \otimes \lambda^*$ and $\mathbf{y}^*$ being in the range space of $\mathcal{B}$, define the error variables:

$$\tilde{\mathbf{x}}_k := \mathbf{x}_k - \mathbf{x}^*,\quad \tilde{\mathbf{v}}_k := \mathbf{v}_k - \mathbf{v}^*, \tag{21}$$

$$\tilde{\mathbf{y}}_k := \mathbf{y}_k - \mathbf{y}^*,\quad \tilde{\boldsymbol{\lambda}}_k := \boldsymbol{\lambda}_k - \boldsymbol{\lambda}^*. \tag{22}$$

Based on (7) and (16)–(20), the error variables evolve as

$$\tilde{\mathbf{x}}_{k+1} = \mathcal{W}\tilde{\mathbf{x}}_k - \alpha\mathcal{R}^\top(\mathbf{F}(\mathbf{x}_k) - \mathbf{F}(\mathbf{x}^*)) - \alpha\mathcal{R}^\top\Pi^\top\tilde{\boldsymbol{\lambda}}_k, \tag{23}$$

$$\tilde{\mathbf{v}}_{k+1} = \tilde{\boldsymbol{\lambda}}_k - \mathcal{B}^2\tilde{\boldsymbol{\lambda}}_k + \beta\Pi(x_{k+1} - x^*) + \mathcal{B}\tilde{\mathbf{y}}_k, \tag{24}$$

$$\tilde{\mathbf{y}}_{k+1} = \tilde{\mathbf{y}}_k - \gamma\mathcal{B}\tilde{\mathbf{v}}_{k+1}, \tag{25}$$

$$\tilde{\boldsymbol{\lambda}}_{k+1} = P_{\mathbb{R}_+^{Nm}}[\mathbf{v}_{k+1}] - P_{\mathbb{R}_+^{Nm}}[\mathbf{v}^*]. \tag{26}$$

Denote $\mathcal{W}_\infty := \mathbf{1}_N\mathbf{1}_N^\top/N \otimes I_n$ and $\mathbf{x}_{\perp,k} := (I_{Nn} - \mathcal{W}_\infty)\mathbf{x}_k$, then $\mathbf{x}_k = \mathcal{W}_\infty\mathbf{x}_k + \mathbf{x}_{\perp,k}$, $(\mathbf{x}_{\perp,k})^\top\mathcal{W}_\infty\mathbf{x}_k = 0$, and $\mathcal{W}\mathcal{W}_\infty = \mathcal{W}_\infty\mathcal{W} = \mathcal{W}_\infty\mathcal{W}_\infty = \mathcal{W}_\infty$. Hence,

$$\|\mathcal{W}\mathbf{x}_k - \mathbf{x}^*\|^2 = \|\mathcal{W}\mathcal{W}_\infty\mathbf{x}_k - \mathbf{x}^* + \mathcal{W}\mathbf{x}_{\perp,k}\|^2 = \|\mathcal{W}_\infty\mathbf{x}_k - \mathbf{x}^*\|^2 + \|\mathcal{W}\mathbf{x}_{\perp,k}\|^2, \tag{27}$$

$$\|\mathcal{W}\mathbf{x}_{\perp,k}\| = \|(\mathcal{W} - \mathcal{W}_\infty)\mathbf{x}_{\perp,k}\| \leq \sigma\|\mathbf{x}_{\perp,k}\|, \tag{28}$$

where the inequality is derived based on (6).

Lemma 2: Under Assumptions 1–3, the error variable $\tilde{\mathbf{x}}_k$ generated by Algorithm 1 satisfies

$$\begin{aligned}\|\tilde{\mathbf{x}}_{k+1}\|^2 \le \rho(M_\alpha)\|\tilde{\mathbf{x}}_k\|^2 - \frac{\mu}{N}\alpha\|\mathcal{W}_\infty \mathbf{x}_k - \mathbf{x}^*\|^2 \\ - \sigma L\alpha\|\mathbf{x}_{\perp,k}\|^2 - \alpha^2\|\mathcal{R}^\top \Pi^\top \tilde{\boldsymbol{\lambda}}_k\|^2 \\ + 2\alpha(x^* - x_{k+1})^\top \Pi^\top \tilde{\boldsymbol{\lambda}}_k,\end{aligned} \tag{29}$$

where

$$M_\alpha := \begin{bmatrix} 1 - \frac{\mu}{N}\alpha + L^2\alpha^2 & (\sigma+1)L\alpha \\ (\sigma+1)L\alpha & \sigma^2 + 3\sigma L\alpha + L^2\alpha^2 \end{bmatrix}. \tag{30}$$

■

Lemma 3: The error variables $\tilde{\mathbf{v}}_k$, $\tilde{\mathbf{y}}_k$ and $\tilde{\boldsymbol{\lambda}}_k$ generated by Algorithm 1 satisfy

$$\begin{aligned}&\|\tilde{\mathbf{v}}_{k+1}\|^2_{I_{Nm}-\gamma\mathcal{B}^2} + \gamma^{-1}\|\tilde{\mathbf{y}}_{k+1}\|^2 \\ &= \|\tilde{\boldsymbol{\lambda}}_k - \mathcal{B}^2\tilde{\boldsymbol{\lambda}}_k + \beta\Pi(x_{k+1} - x^*)\|^2 - \|\mathcal{B}\tilde{\mathbf{y}}_k\|^2 \\ &\quad + \gamma^{-1}\|\tilde{\mathbf{y}}_k\|^2,\end{aligned} \tag{31}$$

where $\gamma > 0$ is chosen as $\gamma < \lambda_{\max}^{-2}(\mathcal{B})$. ■

Next, it is ready to present the main convergence result on Algorithm 1.

Theorem 1: Under Assumptions 1–4, the sequences $\{\mathbf{x}_k\}$, $\{\mathbf{v}_k\}$, $\{\mathbf{y}_k\}$ and $\{\boldsymbol{\lambda}_k\}$ generated by Algorithm 1 satisfy

$$\begin{aligned}E_{k+1} \le \rho(M_\alpha)\|\tilde{\mathbf{x}}_k\|^2 - \min\{\mu/N, \sigma L\}\alpha\|\tilde{\mathbf{x}}_k\|^2 \\ - \alpha^2\|\Pi^\top\tilde{\boldsymbol{\lambda}}_k\|^2 + \alpha\beta^{-1}\|\tilde{\boldsymbol{\lambda}}_k\|^2 + 2\alpha\beta^{-1}\|\mathcal{B}^2\tilde{\boldsymbol{\lambda}}_k\|^2 \\ - 2\alpha\beta^{-1}\|\mathcal{B}\tilde{\boldsymbol{\lambda}}_k\|^2 - \alpha\beta^{-1}\|\mathcal{B}\tilde{\mathbf{y}}_k\|^2 \\ + \alpha\beta^{-1}\gamma^{-1}\|\tilde{\mathbf{y}}_k\|^2,\end{aligned} \tag{32}$$

where $E_{k+1} := \|\tilde{\mathbf{x}}_{k+1}\|^2_{I_{Nn}-2\alpha\beta\mathcal{R}^\top\Pi^\top\Pi\mathcal{R}} + \alpha\beta^{-1}\|\tilde{\boldsymbol{\lambda}}_{k+1}\|^2_{I_{Nm}-\gamma\mathcal{B}^2} + \alpha\beta^{-1}\gamma^{-1}\|\tilde{\mathbf{y}}_{k+1}\|^2$.

In addition, if $\alpha, \beta, \gamma > 0$ satisfy $\rho(M_\alpha) < 1$ and

$$\begin{aligned}\beta &< \min\left\{\frac{\mu}{2N\lambda_{\max}(\Pi^\top\Pi)}, \frac{\sigma L}{2\lambda_{\max}(\Pi^\top\Pi)}, \frac{1}{\alpha\lambda_{\min}(\Pi\Pi^\top)}\right\}, \\ \gamma &< \min\left\{\frac{2-2\lambda^2_{\max}(\mathcal{B})}{1-\alpha\beta\lambda_{\min}(\Pi\Pi^\top)}, \frac{1}{\lambda^2_{\max}(\mathcal{B})}\right\},\end{aligned} \tag{33}$$

then, $\mathbf{x}_k$ and $\boldsymbol{\lambda}_k$ generated by Algorithm 1 linearly converge to $\mathbf{x}^*$ and $\boldsymbol{\lambda}^*$, respectively. Specifically,

$$E_{k+1} \le aE_k, \tag{34}$$

where $a := \max\{\rho(M_\alpha), 1-\alpha\beta\lambda_{\min}(\Pi\Pi^\top), 1-\underline{\sigma}^2(\mathcal{B})\gamma\} < 1$. ■

Theorem 1 presents the linear last-iterate convergence result on the designed distributed GNE seeking algorithm in Algorithm 1 and also provides the explicit convergence rate by appropriately selecting stepsizes α, β and γ. However, how to find a feasible stepsize α is still not clear. In what follows, a proposition on α is derived to show the range of feasible α ensuring $\rho(M_\alpha) < 1$.

Proposition 1: If the positive constant α satisfies $\alpha < \min\left\{\frac{1-\sigma^2}{9\sigma L}, \frac{\sqrt{1-\sigma^2}}{\sqrt{3}L}, \frac{\mu(1-\sigma)}{6NL^2}\right\}$, then $\rho(M_\alpha) < 1$. ■

Remark 3: From Theorem 1 and Proposition 1, it can be observed that the upper bounds of stepsizes α, β and γ depend on the number of the players, the communication structure, and the affine inequality, as well as the Lipschitz and monotonicity constants of the pseudo-gradient function. In fact, from the proofs, one can find that these bounds on the stepsizes are not tight and larger stepsizes may be selected to ensure better convergence results.

IV. SIMULATIONS

In this section, consider a Nash-Cournot game [16], where there are N firms producing a commodity that is sold to m markets. Each firm $i \in [N]$ participates in n_i $(\le m)$ of the m markets and determines its production quantities $x_i \in \mathbb{R}^{n_i}$ to be delivered to the n_i markets. Since each market has a maximal capacity, which results in a coupled affine inequality $Ax \le b$, where $A = [A_1, \ldots, A_N]$ with $A \in \mathbb{R}^{m\times n_i}$, $x = col(x_1, \ldots, x_N)$, $b = col(b_1, \ldots, b_m)$ with $b_i > 0$ being the capacity of the market i. The aim of each firm is to minimize its cost function $f_i(x_i, x_{-i}) = c_i(x_i) - p(Ax)^\top A_i x_i$, where $c_i(x_i) = x_i^\top Q_i x_i + q_i^\top x_i$ is the production cost of firm i with $Q_i \in \mathbb{R}^{n_i\times n_i}$ being a positive definite matrix, $q_i \in \mathbb{R}^{n_i}$ and $p : \mathbb{R}^m \to \mathbb{R}^m$ being a price vector function associating with the markets. Specifically, the price for market i is the ith element of $p(Ax)$, i.e., $[p(Ax)]_i = L_i - w_i[Ax]_i$, where $L_i > 0$ and $w_i > 0$. Set $N = 50$, $m = n_i = 5$, and $A_i = I_m$, $i \in [N]$. Choose Q_i to be a diagonal matrix. For each $i \in [N]$, randomly select the diagonals of Q_i, b_i, L_i and w_i from $[1, 8]$, $[1, 2]$, $[10, 20]$, $[1, 3]$ and $[5, 10]$ with uniform distributions, respectively. The communication graph is randomly created as shown in Figure 1. This setup satisfies all our theoretical assumptions, and set the stepsizes $\beta = 0.1$ and $\gamma = 0.1$. The evolutions of $\|\mathbf{x}_k - \mathbf{x}^*\|/\|\mathbf{x}^*\|$ generated by Algorithm 1 and the proposed forward-backward (FB) algorithm in [16] are provided in Figure 2 under different stepsizes α and τ, respectively, where two algorithms are performed with the same initial conditions and implemented by Matlab R2020a running on a laptop equipped with Intel(R) Core(TM) i7-1065G7 CPU @ 1.30GHz. It can be seen from Figure 2 that our algorithm performs a faster convergence, and larger feasible stepsizes also lead to the faster convergence.

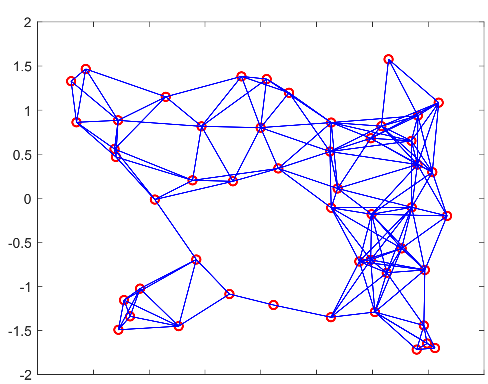

Fig. 1: Random communication graph with 50 nodes.

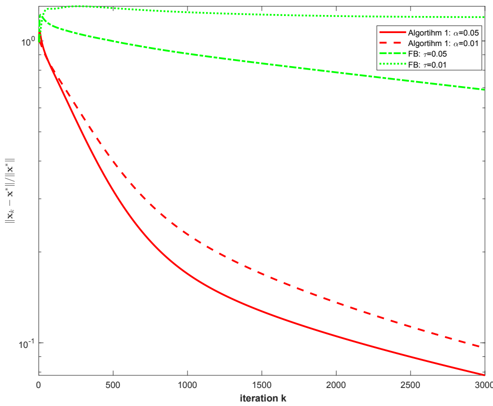

Fig. 2: Distance from the variational GNE for Algorithm 1 and the FB algorithm in [16] with different α and τ.

V. Conclusion

In this paper, the GNE seeking problem for continuous games with coupled affine inequality constraints was solved by designing a novel primal-dual algorithm. The linear last-iterate convergence of the designed algorithm was also rigorously analyzed and the bounds of feasible stepsizes were provided. Future work of interest may be on the design of linearly convergent algorithms for continuous games with compact strategy set constraints and coupled nonlinear inequality constraints.

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