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Convergence of Recursive Least Squares Based Input/Output System Identification with Model Order Mismatch

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***Abstract*— Discrete-time input/output models, also called infinite impulse response (IIR) models or autoregressive moving average (ARMA) models, are useful for online identification as they can be efficiently updated using recursive least squares (RLS) as new data is collected. Several works have studied the convergence of the input/output model coefficients identified using RLS under the assumption that the order of the identified model is the same as that of the true system. However, the case of model order mismatch is not as well addressed. This work begins by introducing the notion of *equivalence* of input/output models of different orders. Next, this work analyzes online identification of input/output models in the case where the order of the identified model is higher than that of the true system. It is shown that, given persistently exciting data, the higher-order identified model converges to the model equivalent to the true system that minimizes the regularization term of RLS.**

I. INTRODUCTION

Least squares based methods are fundamental and widely used in identification, signal processing, and control [1], [2]. One useful application is online identification of a linear model for adaptive control [3, ch. 3]. A particular model structure that lends itself to online identification is a discrete-time input/output model [4], also called an infinite-impulse-response (IIR) model [5] or autoregressive moving-average (ARMA) model [3, p. 32]. An advantage of discrete-time input/output models is they can be efficiently updated in real time as new data is collected [3, ch. 3.2]. Note that in the single-input-single-output (SISO) case, the coefficients of a discrete-time input/output model directly give the discrete-time transfer function. However, the multi-input-multi-output (MIMO) case is considerably more complex [4], [6] and converting between input/output models and other linear model structures is nontrivial [7]. As such, it is beneficial to directly study the online identification of MIMO input/output models without converting to another model structure.

This paper focuses on the online identification of input/output models using recursive least squares (RLS), which has been used in adaptive model predictive control [8] and retrospective cost adaptive control [9] with various applications [10]–[12]. While related works have discussed convergence of model coefficients when identifying input/output models using RLS [3, ch. 3.4], [13], these results assume the order of the true model is known. A natural question is whether similar guarantees can be made if the order of the identified model and true system do not match. This work will show that if the order of the identified model is higher than order of the true model, then the regressor of RLS is not persistently exciting, and hence standard convergence guarantees of RLS [3], [14], [15] do not apply. The main contribution of this work is developing new analysis of the case where the order of the identified input/output model is higher than that of the true input/output system.

This paper is organized as follows. Section II introduces discrete-time input/output models and a useful output transition equation. Next, Section III introduces the notion of *equivalence* of input/output models as models giving the same outputs under the same inputs and initial conditions. It is shown that a necessary and sufficient condition for equivalence can be written as a linear equation of the model coefficients. This section also introduces the notion of *reducibility* as the existence of a lower-order, equivalent input/output model. Finally, section IV discusses the online identification of input/output models using RLS. It is first shown that, in the case where the order of the identified model is the same as that of the true system, persistent excitation conditions guarantee global asymptotic stability of the coefficient estimation error. Next, in the case where the order of the identified model is higher than that of the true system, conditions are given under which the identified model converges to the higher-order model equivalent to the true system that minimizes the regularization term of RLS.

II. INPUT/OUTPUT MODELING

Let $k_0 \in \mathbb{N}$ be the initial time step and let $n \geq 0$ be the model order. Consider the input/output model where, for all $k \geq k_0$, $u_k \in \mathbb{R}^m$ is the input, $y_{k_0}, \ldots, y_{k_0+n-1} \in \mathbb{R}^p$ are the initial conditions and, for all $k \geq k_0$, the output $y_{k+n} \in \mathbb{R}^p$ is given by

$$y_{k+n} = -\sum_{i=1}^{n} F_i y_{k+n-i} + \sum_{i=0}^{n} G_i u_{k+n-i}, \tag{1}$$

where $F_1, \ldots, F_n \in \mathbb{R}^{p\times p}$, and $G_0, \ldots, G_n \in \mathbb{R}^{p\times m}$ are the input/output model coefficients. It follows that, for all $k \geq k_0$,

$$y_{k+n} = -\mathcal{F}_n \mathcal{Y}_{k,n} + \mathcal{G}_n \mathcal{U}_{k,n}, \tag{2}$$

where $\mathcal{F}_n \in \mathbb{R}^{p\times pn}$, $\mathcal{Y}_{k,n} \in \mathbb{R}^{pn}$, $\mathcal{G}_n \in \mathbb{R}^{p\times m(n+1)}$, and $\mathcal{U}_{k,n} \in \mathbb{R}^{m(n+1)}$ are defined as

$$\mathcal{F}_n \triangleq \begin{bmatrix} F_1 & \cdots & F_n \end{bmatrix}, \quad \mathcal{Y}_{k,n} \triangleq \begin{bmatrix} y_{k+n-1} \\ \vdots \\ y_k \end{bmatrix}, \tag{3}$$

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$$\mathcal{G}_n \triangleq \begin{bmatrix} G_0 & \cdots & G_n \end{bmatrix}, \quad \mathcal{U}_{k,n} \triangleq \begin{bmatrix} u_{k+n} \\ \vdots \\ u_k \end{bmatrix}. \tag{4}$$

Proposition 1 shows that, for all $k \geq k_0$ and $j \geq 0$, y_{k+n+j} can be written as a linear combination of the inputs $u_k, \ldots, u_{k+n+j-1}$ and the n outputs $y_k, \ldots, y_{k+n-1}$. We call this the output transition equation.

Proposition 1. *Let $k_0 \in \mathbb{N}$. For all $k \geq k_0$, let $u_k \in \mathbb{R}^m$, let $y_{k_0}, \ldots, y_{k_0+n-1} \in \mathbb{R}^p$, and, for all $k \geq k_0$, let $y_{k+n} \in \mathbb{R}^p$ be given by* (1)*. Then, for all $k \geq k_0$ and $j \geq 0$,*

$$y_{k+n+j} = -\mathcal{F}_{n,j}\mathcal{Y}_{k,n} + \mathcal{G}_{n,j}\mathcal{U}_{k,n+j}, \tag{5}$$

where $\mathcal{F}_{n,0} \triangleq \mathcal{F}_n$, $\mathcal{G}_{n,0} \triangleq \mathcal{G}_n$, and, for all $j \geq 1$, $\mathcal{F}_{n,j} \in \mathbb{R}^{p\times pn}$ and $\mathcal{G}_{n,j} \in \mathbb{R}^{p\times m(n+1+j)}$ are defined

$$\mathcal{F}_{n,j} \triangleq \mathcal{S}(\mathcal{F}_{n,0}, j) - \sum_{i=1}^{\min\{j,n\}} F_i \mathcal{F}_{n,j-i}, \tag{6}$$

$$\mathcal{G}_{n,j} \triangleq \begin{bmatrix} \mathcal{G}_{n,0} & 0_{p\times jm} \end{bmatrix} - \sum_{i=1}^{\min\{j,n\}} F_i \begin{bmatrix} 0_{p\times im} & \mathcal{G}_{n,j-i} \end{bmatrix}, \tag{7}$$

and where $\mathcal{S}(\mathcal{F}_{n,0}, j) \in \mathbb{R}^{p\times pn}$ is the j step shift of $\mathcal{F}_{n,0}$, defined as

$$\mathcal{S}(\mathcal{F}_{n,0}, j) \triangleq \begin{cases} \begin{bmatrix} F_{j+1} & \cdots & F_n & 0_{p\times jp} \end{bmatrix} & j \leq n-1, \\ 0_{p\times np} & j \geq n. \end{cases}$$

Proof. Proof follows by strong induction on $j \geq 0$. Note that for all $k \geq k_0$ and $j = 0$, (5) simplifies to (2). Next, let $k \geq k_0$, let $j \geq 1$ and suppose that, for all $\hat{j} \leq j-1$, (5) holds. Note that by (1), y_{k+n+j} can be written as $y_{k+n+j} = -\sum_{i=1}^{n} F_i y_{k+n+j-i} + \sum_{i=0}^{n} G_i u_{k+n+j-i}$. Next, for all $1 \leq i \leq \min\{j,n\}$, it follows that $0 \leq j-i \leq j-1$ and by inductive hypothesis, $y_{k+n+j-i}$ can be written as $y_{k+n+j-i} = -\mathcal{F}_{n,j-i}\mathcal{Y}_{k,n} + \mathcal{G}_{n,j-i}\mathcal{U}_{k,n+j-i} = -\mathcal{F}_{n,j-i}\mathcal{Y}_{k,n} + \begin{bmatrix} 0_{p\times im} & \mathcal{G}_{n,j-i} \end{bmatrix}\mathcal{U}_{k,n+j}$. Furthermore, note that $\sum_{i=0}^{n} G_i u_{k+n+j-i} = \begin{bmatrix} \mathcal{G}_{n,0} & 0_{p\times jm} \end{bmatrix}\mathcal{U}_{k,n+j}$. Hence, y_{k+n+j} can be written as $y_{k+n-j} = \sum_{i=1}^{\min\{j,n\}} F_i \left(\mathcal{F}_{n,j-i}\mathcal{Y}_{k,n} - \begin{bmatrix} 0_{p\times im} & \mathcal{G}_{n,j-i} \end{bmatrix}\mathcal{U}_{k,n+j}\right) - \sum_{i=\min\{j,n\}+1}^{n} F_i y_{k+n+j-i} + \begin{bmatrix} \mathcal{G}_{n,0} & 0_{p\times jm} \end{bmatrix}\mathcal{U}_{k,n+j}$. Noting that $\sum_{i=\min\{j,n\}+1}^{n} F_i y_{k+n+j-i} = \mathcal{S}(\mathcal{F}_{n,0}, j)\mathcal{Y}_{k,n}$ and combining terms yields (5). □

III. Equivalence and Reducibility of Input/Output Models

Consider an input/output model of order $\hat{n} \geq n$ where, for all $k \geq k_0$, $u_k \in \mathbb{R}^m$ is the input, $\hat{y}_{k_0}, \ldots, \hat{y}_{k_0+\hat{n}-1} \in \mathbb{R}^p$ are the initial conditions and, for all $k \geq k_0$, the output $\hat{y}_{k+\hat{n}} \in \mathbb{R}^p$ is given by

$$\hat{y}_{k+\hat{n}} = -\sum_{i=1}^{\hat{n}} \hat{F}_i \hat{y}_{k+\hat{n}-i} + \sum_{i=0}^{\hat{n}} \hat{G}_i u_{k+\hat{n}-i}, \tag{8}$$

where $\hat{F}_1, \ldots, \hat{F}_{\hat{n}} \in \mathbb{R}^{p\times p}$, and $\hat{G}_0, \ldots, \hat{G}_{\hat{n}} \in \mathbb{R}^{p\times m}$ are the input/output model coefficients. Definition 1 defines models (1) and (8) as *equivalent* if they give the same outputs under the same inputs and initial conditions.

Definition 1. *Let $k_0 \in \mathbb{N}$. Consider input/output model* (1) *with order n and and input/output Model* (8) *with order $\hat{n} \geq n$. For all $k \geq k_0$, let $u_k \in \mathbb{R}^m$ be arbitrary. For all $k_0 \leq k \leq k_0+n-1$, let $y_k \in \mathbb{R}^p$ be arbitrary and, for all $k \geq k_0+n$, let $y_k \in \mathbb{R}^p$ be given by* (1)*. Next, for all $k_0 \leq k \leq k_0+\hat{n}-1$, let $\hat{y}_k = y_k$ and, for all $k \geq k_0+\hat{n}$, let $\hat{y}_k \in \mathbb{R}^p$ be given by* (8)*. Models* (1) *and* (8) *are **equivalent** if, for all $k \geq k_0+\hat{n}$, $\hat{y}_k = y_k$.*

Proposition 2 shows that two input/output models of the same order are equivalent if and only if they have the same model coefficients.

Proposition 2. *Consider input/output models* (1) *and* (8) *with the same order $\hat{n} = n$. Models* (1) *and* (8) *are equivalent if and only if $\hat{G}_0 = G_0$ and, for all $1 \leq i \leq n$, $\hat{F}_i = F_i$ and $\hat{G}_i = G_i$.*

Proof. Proof of necessity follow immediately. To prove sufficiency, let $k_0 \in \mathbb{N}$. First consider that there exists $0 \leq i \leq n$ such that $\hat{G}_i \neq G_i$. Then, there exists $u \in \mathbb{R}^m$ such that $\hat{G}_i u \neq G_i u$. Next, consider $y_{k_0} = \cdots = y_{k_0+n-1} = 0$, for all $k \geq k_0$ such that $k \neq k_0+n-i$, $u_k = 0$, and $u_{k_0+n-i} = u$. Then, $\hat{y}_{k_0+n} = \hat{G}_i u$, $y_{k_0+n} = G_i u$, and $\hat{y}_{k_0+n} \neq y_{k_0+n}$.

Next, consider that there exists $1 \leq i \leq n$ such that $\hat{F}_i \neq F_i$. Similarly, there exists $y \in \mathbb{R}^p$ such that $\hat{F}_i y \neq F_i y$. Hence, if, for all $k_0 \leq k \leq k_0+n-1$, $k \neq k_0+n-i$, $y_k = 0$, $y_{k_0+n-i} = y$, and, for all $k \geq k_0$, $u_k = 0$, then $\hat{y}_{k_0+n} = \hat{F}_i y$, $y_{k_0+n} = F_i y$, and $\hat{y}_{k_0+n} \neq y_{k_0+n}$. Thus, (1) and (8) are not equivalent and sufficiency is proven. □

We next consider the equivalence of input/output models of different orders. Note that, for all $k \geq k_0$,

$$\hat{y}_{k+\hat{n}} = -\hat{\mathcal{F}}_{\hat{n}}\hat{\mathcal{Y}}_{k,\hat{n}} + \hat{\mathcal{G}}_{\hat{n}}\mathcal{U}_{k,\hat{n}}, \tag{9}$$

where $\mathcal{U}_{k,\hat{n}} \in \mathbb{R}^{m(\hat{n}+1)}$ is defined in (4) and $\hat{\mathcal{F}}_{\hat{n}} \in \mathbb{R}^{p\times p\hat{n}}$, $\hat{\mathcal{Y}}_{k,\hat{n}} \in \mathbb{R}^{p\hat{n}}$, and $\hat{\mathcal{G}}_{\hat{n}} \in \mathbb{R}^{p\times m(\hat{n}+1)}$ are defined as

$$\hat{\mathcal{F}}_{\hat{n}} \triangleq \begin{bmatrix} \hat{F}_1 & \cdots & \hat{F}_{\hat{n}} \end{bmatrix}, \quad \hat{\mathcal{Y}}_{k,\hat{n}} \triangleq \begin{bmatrix} \hat{y}_{k+\hat{n}-1} \\ \vdots \\ \hat{y}_k \end{bmatrix}, \tag{10}$$

$$\hat{\mathcal{G}}_{\hat{n}} \triangleq \begin{bmatrix} \hat{G}_0 & \cdots & \hat{G}_{\hat{n}} \end{bmatrix}. \tag{11}$$

Theorem 1 gives necessary and sufficient conditions for the equivalence of input/output models of different orders. We begin with two useful Lemmas.

Lemma 1. *Let $k_0 \in \mathbb{N}$. Consider input/output model* (1) *with order n and and input/output Model* (8) *with order $\hat{n} > n$. For all $k \geq k_0$, let $u_k \in \mathbb{R}^m$, for all $k_0 \leq k \leq k_0+n-1$, let $y_k \in \mathbb{R}^p$, and, for all $k \geq k_0+n$, let $y_k \in \mathbb{R}^p$ be given by* (1)*. Let $k^* \geq k_0$ and, for all $k_0 \leq k \leq k^*+\hat{n}-1$, let $\hat{y}_k = y_k$. Finally, let $\hat{y}_{k^*+\hat{n}} \in \mathbb{R}^p$ be given by* (8)*. Then, $\hat{y}_{k^*+\hat{n}}$ can be expressed as*

$$\hat{y}_{k^*+\hat{n}} = -\hat{\mathcal{F}}_{n,\hat{n}-n}\mathcal{Y}_{k^*,n} + \hat{\mathcal{G}}_{n,\hat{n}-n}\mathcal{U}_{k^*,\hat{n}}, \tag{12}$$

where $\hat{\mathcal{F}}_{n,\hat{n}-n} \in \mathbb{R}^{p\times pn}$ *and* $\hat{\mathcal{G}}_{n,\hat{n}-n} \in \mathbb{R}^{p\times m(\hat{n}+1)}$ *are defined*

$$\hat{\mathcal{F}}_{n,\hat{n}-n} \triangleq \begin{bmatrix}\hat{F}_{\hat{n}-n+1} & \cdots & \hat{F}_{\hat{n}}\end{bmatrix} - \sum_{i=1}^{\hat{n}-n} \hat{F}_i \mathcal{F}_{n,\hat{n}-n-i}, \quad (13)$$

$$\hat{\mathcal{G}}_{n,\hat{n}-n} \triangleq \hat{\mathcal{G}}_{\hat{n}} - \sum_{i=1}^{\hat{n}-n} \hat{F}_i \begin{bmatrix}0_{p\times im} & \mathcal{G}_{n,\hat{n}-n-i}\end{bmatrix}. \quad (14)$$

Proof. Let $k_0 \in \mathbb{N}$ and let $k^* \geq k_0$. By (8), it follows that $\hat{y}_{k^*+\hat{n}} = -\sum_{i=1}^{\hat{n}-n}(\hat{F}_i y_{k^*+\hat{n}-i}) - \sum_{i=\hat{n}-n+1}^{\hat{n}}(\hat{F}_i y_{k+\hat{n}-i}) + \hat{\mathcal{G}}_{\hat{n}}\mathcal{U}_{k^*,\hat{n}}$. Note that the second term can be written as $\sum_{i=\hat{n}-n+1}^{\hat{n}} \hat{F}_i y_{k^*+\hat{n}-i} = \begin{bmatrix}\hat{F}_{\hat{n}-n+1} & \cdots & \hat{F}_{\hat{n}}\end{bmatrix}\mathcal{Y}_{k^*,n}$. Next, for all $1 \leq i \leq \hat{n}-n$, note that $\hat{n}-n-i \geq 0$ and Proposition 1 (with $k = k^*$ and $j = \hat{n}-n-i$) implies that $y_{k^*+\hat{n}-i} = -\mathcal{F}_{n,\hat{n}-n-i}\mathcal{Y}_{k^*,n} + \mathcal{G}_{n,\hat{n}-n-i}\mathcal{U}_{k^*,\hat{n}-i}$. Substituting these two equations into the expression for $\hat{y}_{k^*+\hat{n}}$, it follows that $\hat{y}_{k^*+\hat{n}} = -\begin{bmatrix}\hat{F}_{\hat{n}-n+1} & \cdots & \hat{F}_{\hat{n}}\end{bmatrix}\mathcal{Y}_{k^*,n} + \hat{\mathcal{G}}_{\hat{n}}\mathcal{U}_{k^*,\hat{n}} - \sum_{i=1}^{\hat{n}-n}\hat{F}_i(-\mathcal{F}_{n,\hat{n}-n-i}\mathcal{Y}_{k^*,n} + \mathcal{G}_{n,\hat{n}-n-i}\mathcal{U}_{k^*,\hat{n}-i})$.

Finally, note that, for all $1 \leq i \leq \hat{n}-n$, $\mathcal{G}_{n,\hat{n}-n-i}\mathcal{U}_{k^*,\hat{n}-i} = \begin{bmatrix}0_{p\times im} & \mathcal{G}_{n,\hat{n}-n-i}\end{bmatrix}\mathcal{U}_{k^*,\hat{n}}$. Combining the previous two equations and simplifying yields (12). □

Lemma 2. *For all* $\hat{n} > n$*,*

$$\begin{bmatrix}-\hat{\mathcal{F}}_{n,\hat{n}-n} & \hat{\mathcal{G}}_{n,\hat{n}-n}\end{bmatrix} = \begin{bmatrix}-\hat{\mathcal{F}}_{\hat{n}} & \hat{\mathcal{G}}_{\hat{n}}\end{bmatrix} M_{n,\hat{n}}, \quad (15)$$

where $M_{n,\hat{n}} \in \mathbb{R}^{p\hat{n}+m(\hat{n}+1)\times pn+m(\hat{n}+1)}$ *is defined*

$$M_{n,\hat{n}} \triangleq \begin{bmatrix} M^1_{n,\hat{n}} & M^2_{n,\hat{n}} \\ I_{pn} & 0_{pn\times m(\hat{n}+1)} \\ 0_{m(\hat{n}+1)\times pn} & I_{m(\hat{n}+1)} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix}M^1_{n,\hat{n}} & M^2_{n,\hat{n}}\end{bmatrix} \\ I_{pn+m(\hat{n}+1)} \end{bmatrix}, \quad (16)$$

where $M^1_{n,\hat{n}} \in \mathbb{R}^{p(\hat{n}-n)\times pn}$ *and* $M^2_{n,\hat{n}} \in \mathbb{R}^{p(\hat{n}-n)\times m(\hat{n}+1)}$ *are defined*

$$M^1_{n,\hat{n}} \triangleq \begin{bmatrix} -\mathcal{F}_{n,\hat{n}-n-1} \\ -\mathcal{F}_{n,\hat{n}-n-2} \\ \vdots \\ -\mathcal{F}_{n,1} \\ -\mathcal{F}_{n,0} \end{bmatrix}, M^2_{n,\hat{n}} \triangleq \begin{bmatrix} \begin{bmatrix}0_{p\times m} & \mathcal{G}_{n,\hat{n}-n-1}\end{bmatrix} \\ \begin{bmatrix}0_{p\times 2m} & \mathcal{G}_{n,\hat{n}-n-2}\end{bmatrix} \\ \vdots \\ \begin{bmatrix}0_{p\times(\hat{n}-n-1)m} & \mathcal{G}_{n,1}\end{bmatrix} \\ \begin{bmatrix}0_{p\times(\hat{n}-n)m} & \mathcal{G}_{n,0}\end{bmatrix} \end{bmatrix} \quad (17)$$

Proof. First, note that it follows from (13) that $\hat{\mathcal{F}}_{n,\hat{n}-n} = \begin{bmatrix}\hat{F}_{\hat{n}-n+1} & \cdots & \hat{F}_{\hat{n}}\end{bmatrix} - \begin{bmatrix}\hat{F}_1 & \cdots & \hat{F}_{\hat{n}-n}\end{bmatrix} M^1_{n,\hat{n}}$. Next, it follows from (14) that $\hat{\mathcal{G}}_{n,\hat{n}-n} = \hat{\mathcal{G}}_{\hat{n}} - \begin{bmatrix}\hat{F}_1 & \cdots & \hat{F}_{\hat{n}-n}\end{bmatrix} M^2_{n,\hat{n}}$. Combining these two expressions yields (15). □

Theorem 1. *Consider input/output model* (1) *with order* n *and input/output model* (8) *with order* $\hat{n} > n$*. Models* (1) *and* (8) *are equivalent if and only if*

$$\hat{\mathcal{F}}_{n,\hat{n}-n} = \mathcal{F}_{n,\hat{n}-n}, \quad \hat{\mathcal{G}}_{n,\hat{n}-n} = \mathcal{G}_{n,\hat{n}-n}, \quad (18)$$

which holds if and only if

$$\begin{bmatrix}-\hat{\mathcal{F}}_{\hat{n}} & \hat{\mathcal{G}}_{\hat{n}}\end{bmatrix} M_{n,\hat{n}} = \begin{bmatrix}-\mathcal{F}_{n,\hat{n}-n} & \mathcal{G}_{n,\hat{n}-n}\end{bmatrix}. \quad (19)$$

Proof. It follows from Lemma 2 that (18) and (19) are equivalent. Hence, it suffices to show that Models (1) and (8) are equivalent if and only if (18) holds. To prove sufficiency, assume models (1) and (8) are equivalent and let $k_0 \in \mathbb{N}$. Then, it follows from Definition 1 that, for all $\mathcal{Y}_{k_0,n} \in \mathbb{R}^{pn}$ and $\mathcal{U}_{k_0,\hat{n}} \in \mathbb{R}^{m(\hat{n}+1)}$, $y_{k_0+\hat{n}} = \hat{y}_{k_0+\hat{n}}$ holds, where, for all $k_0 + n \leq k \leq k_0 + \hat{n}$, $y_k \in \mathbb{R}^p$ is given by (1), for all $k_0 \leq k \leq k_0 + \hat{n} - 1$, $\hat{y}_k = y_k$, and where $\hat{y}_{k+\hat{n}} \in \mathbb{R}^p$ is given by (8). It follows from Proposition 1 and Lemma 1 that $y_{k+\hat{n}} = -\mathcal{F}_{n,\hat{n}-n}\mathcal{Y}_{k,n} + \mathcal{G}_{n,\hat{n}-n}\mathcal{U}_{k,n}$, and $\hat{y}_{k+\hat{n}} = -\hat{\mathcal{F}}_{n,\hat{n}-n}\mathcal{Y}_{k,n} + \hat{\mathcal{G}}_{n,\hat{n}-n}\mathcal{U}_{k,n}$, Since $\mathcal{Y}_{k,n}$ and $\mathcal{U}_{k,\hat{n}}$ are chosen arbitrarily and $y_{k+\hat{n}} = \hat{y}_{k+\hat{n}}$, (18) follows.

To prove necessity, assume that (19) holds, let $k_0 \in \mathbb{N}$. For all $k \geq k_0$, let $u_k \in \mathbb{R}^m$ be arbitrary, for all $k_0 \leq k \leq k_0 + n - 1$, let $y_k \in \mathbb{R}^p$ be arbitrary and, for all $k \geq k_0 + n$, let $y_k \in \mathbb{R}^p$ be given by (1). Furthermore, for all $k_0 \leq k \leq k_0 + \hat{n} - 1$, let $\hat{y}_k = y_k$ and, for all $k \geq k_0 + \hat{n}$, let $\hat{y}_k \in \mathbb{R}^p$ be given by (8). We now show by strong induction that, for all $k^* \geq k_0$, $\hat{y}_{k^*+\hat{n}} = y_{k^*+\hat{n}}$. First, consider the base case $k^* = k_0$. It follows from (5) of Proposition 1 (with $k = k_0$ and $j = \hat{n}-n$) that $y_{k_0+\hat{n}} = -\mathcal{F}_{n,\hat{n}-n}\mathcal{Y}_{k_0,n} + \mathcal{G}_{n,\hat{n}-n}\mathcal{U}_{k_0,\hat{n}}$. Moreover, since by assumption, for all $k_0 \leq k \leq k_0 + \hat{n} - 1$, $\hat{y}_k = y_k$, it follows from (12) of Lemma 1 (with $k^* = k_0$) that $\hat{y}_{k_0+\hat{n}} = -\hat{\mathcal{F}}_{n,\hat{n}-n}\mathcal{Y}_{k_0,n} + \hat{\mathcal{G}}_{n,\hat{n}-n}\mathcal{U}_{k_0,\hat{n}}$. Finally, (18) implies that $y_{k_0+\hat{n}} = \hat{y}_{k_0+\hat{n}}$.

Next, let $k^* \geq k_0 + 1$ and assume for inductive hypothesis that, for all $k_0 \leq k \leq k^* - 1$, $\hat{y}_{k+\hat{n}} = y_{k+\hat{n}}$. It follows from (5) of Proposition 1 (with $k = k^*$ and $j = \hat{n} - n$) that $y_{k^*+\hat{n}} = -\mathcal{F}_{n,\hat{n}-n}\mathcal{Y}_{k^*,n} + \mathcal{G}_{n,\hat{n}-n}\mathcal{U}_{k^*,\hat{n}}$. Moreover, it follows from inductive hypothesis and (12) of Lemma 1 that $\hat{y}_{k^*+\hat{n}} = -\hat{\mathcal{F}}_{n,\hat{n}-n}\mathcal{Y}_{k^*,n} + \hat{\mathcal{G}}_{n,\hat{n}-n}\mathcal{U}_{k^*,\hat{n}}$. Finally, (18) implies that $y_{k^*+\hat{n}} = \hat{y}_{k^*+\hat{n}}$. Thus, by strong induction, for all $k^* \geq k_0$, $\hat{y}_{k^*+\hat{n}} = y_{k^*+\hat{n}}$, and models (1) and (8) are equivalent. □

Definition 2 defines an input/output model as *reducible* if there exists an equivalent input/output model of lower-order, and *irreducible* otherwise. Theorem 2 provides necessary and sufficient conditions for reducibility of input/output models.

Definition 2. *Consider input/output model* (8) *with order* $\hat{n}$*. The input/output model* (8) *is* ***reducible*** *if there exists an input/output model* (1) *with order* $n \leq \hat{n} - 1$ *such that* (1) *and* (8) *are equivalent. Otherwise, input/output model* (8) *is* ***irreducible****.*

Theorem 2. *The input/output model* (8) *is reducible if and only if there exists* $F_1 \in \mathbb{R}^{p\times p}$ *such that*

$$(\hat{F}_1 - F_1)F_{\hat{n}-1} = \hat{F}_{\hat{n}}, \quad (20)$$

$$(\hat{F}_1 - F_1)G_{\hat{n}-1} = \hat{G}_{\hat{n}}, \quad (21)$$

where $G_0 \triangleq \hat{G}_0$ *and, for all* $2 \leq i \leq \hat{n}-1$ *and* $1 \leq j \leq \hat{n}-1$*,*

$$F_i \triangleq = \hat{F}_i - (\hat{F}_1 - F_1)F_{i-1}, \quad (22)$$

$$G_j \triangleq = \hat{G}_j - (\hat{F}_1 - F_1)G_{j-1}. \quad (23)$$

Moreover, if there exists $F_1 \in \mathbb{R}^{p\times p}$ *satisfying* (20) *and* (21)*, then input/output model* (1) *with coefficients* $F_1, \ldots, F_{\hat{n}-1}$ *and* $G_0, \ldots, G_{\hat{n}-1}$ *defined by* (22) *and* (23) *is equivalent to input/output model* (8)*.*

Proof. To begin, if input/output model (1) with order $n < \hat{n}-1$ is equivalent to input/output model (8), then there exists an input/output model with order $\hat{n}-1$ that is equivalent to (8), namely a model with the coefficients $F_1,\ldots,F_n$, $G_0,\ldots,G_n$, and $F_i = 0_{p\times p}$ and $G_i = 0_{p\times m}$ for all $n+1 \le i \le \hat{n}-1$. Hence, input/output model (8), with order $\hat{n}$ is reducible if and only if there exists an equivalent input/output model (1) with order $n = \hat{n}-1$.

Next, it follows from Theorem 1 that model (1) with order $n = \hat{n}-1$ and model (8) with order $\hat{n}$ are equivalent if and only if $\hat{\mathcal{F}}_{\hat{n}-1,1} = \mathcal{F}_{\hat{n}-1,1}$ and $\hat{\mathcal{G}}_{\hat{n}-1,1} = \mathcal{G}_{\hat{n}-1,1}$. Moreover, it follows from (6), (13), (7), and (14), respectively, that $\hat{\mathcal{F}}_{\hat{n}-1,1} = \begin{bmatrix}\hat{F}_2 & \hat{F}_3 & \cdots & \hat{F}_{\hat{n}}\end{bmatrix} - \hat{F}_1\mathcal{F}_{\hat{n}-1}$, $\mathcal{F}_{\hat{n}-1,1} = -F_1\mathcal{F}_{\hat{n}-1} + \begin{bmatrix}\hat{F}_2 & \hat{F}_3 & \cdots & \hat{F}_{\hat{n}-1} & 0_{p\times p}\end{bmatrix}$, $\hat{\mathcal{G}}_{\hat{n}-1,1} = \hat{\mathcal{G}}_{\hat{n}} - \hat{F}_1\begin{bmatrix}0_{p\times m} & \mathcal{G}_{\hat{n}-1}\end{bmatrix}$, and $\mathcal{G}_{\hat{n}-1,1} = -F_1\begin{bmatrix}0_{p\times m} & \mathcal{G}_{\hat{n}-1}\end{bmatrix} + \begin{bmatrix}\mathcal{G}_{\hat{n}-1} & 0_{p\times m}\end{bmatrix}$. Rearranging terms, it follows that $\hat{\mathcal{F}}_{\hat{n}-1,1} = \mathcal{F}_{\hat{n}-1,1}$ holds if and only if $(\hat{F}_1 - F_1)\mathcal{F}_{\hat{n}-1} = \begin{bmatrix}\hat{F}_2 - F_2 & \cdots & \hat{F}_{\hat{n}-1} - F_{\hat{n}-1} & \hat{F}_{\hat{n}}\end{bmatrix}$, and $\hat{\mathcal{G}}_{\hat{n}-1,1} = \mathcal{G}_{\hat{n}-1,1}$ holds if and only if $(\hat{F}_1 - F_1)\begin{bmatrix}0_{p\times m} & \mathcal{G}_{\hat{n}-1}\end{bmatrix} = \begin{bmatrix}\hat{G}_0 - G_0 & \cdots & \hat{G}_{\hat{n}-1} - G_{\hat{n}-1} & \hat{G}_{\hat{n}}\end{bmatrix}$. These two expression can be expanded as (20), (21), $\hat{G}_0 = G_0$, and, for all $2 \le i \le \hat{n}-1$ and $1 \le j \le \hat{n}-1$, (22) and (23). □

IV. Online Identification of Input/Output Models using Recursive Least Squares

Next, we discuss the online identification of input/output models using recursive least squares. We consider input/output model (1) of order n to be the true system. We define $\theta_{n,\mathrm{true}} \in \mathbb{R}^{p\times pn+m(n+1)}$ as

$$\theta_{n,\mathrm{true}} \triangleq \begin{bmatrix}F_1 & \cdots & F_n & G_0 & \cdots & G_n\end{bmatrix} = \begin{bmatrix}\mathcal{F}_n & \mathcal{G}_n\end{bmatrix}. \tag{24}$$

It follows from (2) that, for all $k \ge 0$, $y_k \in \mathbb{R}^p$ is given by

$$y_k = \theta_{n,\mathrm{true}}\phi_{n,k}, \tag{25}$$

where, for all $k \ge 0$ and $n \ge 0$, $\phi_{n,k} \in \mathbb{R}^{pn+m(n+1)}$ is defined as

$$\begin{aligned}\phi_{n,k} &\triangleq \begin{bmatrix}-y_{k-1}^{\mathrm{T}} & \cdots & -y_{k-n}^{\mathrm{T}} & u_k^{\mathrm{T}} & \cdots & u_{k-n}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} \\ &= \begin{bmatrix}\mathcal{Y}_{k-n,n}^{\mathrm{T}} & \mathcal{U}_{k-n,n}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}.\end{aligned} \tag{26}$$

The objective of online identification is to identify the coefficients an input/output model (8) of order $\hat{n}$ using measurements of the inputs u_k and outputs y_k generated from (1). This can be accomplished by minimizing the cost function $J_{\hat{n},k}\colon \mathbb{R}^{p\times p\hat{n}+m(\hat{n}+1)} \to \mathbb{R}$, defined as

$$\begin{aligned}J_{\hat{n},k}(\hat{\theta}_{\hat{n}}) = &\sum_{i=0}^{k} z_{\hat{n},i}^{\mathrm{T}}(\hat{\theta}_{\hat{n}})z_{\hat{n},i}(\hat{\theta}_{\hat{n}}) \\ &+ \mathrm{tr}\left[(\hat{\theta}_{\hat{n}} - \theta_{\hat{n},0})P_{\hat{n},0}^{-1}(\hat{\theta}_{\hat{n}} - \theta_{\hat{n},0})^{\mathrm{T}}\right],\end{aligned} \tag{27}$$

where $\hat{\theta}_{\hat{n}} \in \mathbb{R}^{p\times p\hat{n}+m(\hat{n}+1)}$ are the coefficients to be identified, defined as

$$\hat{\theta}_{\hat{n}} \triangleq \begin{bmatrix}\hat{F}_1 & \cdots & \hat{F}_{\hat{n}} & \hat{G}_0 & \cdots & \hat{G}_{\hat{n}}\end{bmatrix} = \begin{bmatrix}\hat{\mathcal{F}}_{\hat{n}} & \hat{\mathcal{G}}_{\hat{n}}\end{bmatrix}, \tag{28}$$

where the residual error function $z_{\hat{n},k}\colon \mathbb{R}^{p\times p\hat{n}+m(\hat{n}+1)} \to \mathbb{R}^p$ is defined as

$$z_{\hat{n},k}(\hat{\theta}_{\hat{n}}) \triangleq y_k - \hat{\theta}_{\hat{n}}\phi_{\hat{n},k}, \tag{29}$$

where $\phi_{\hat{n},k} \in \mathbb{R}^{p\hat{n}+m(\hat{n}+1)}$ is defined in (26), and where $\theta_{\hat{n},0} \in \mathbb{R}^{p\times p\hat{n}+m(\hat{n}+1)}$ is an initial guess of the coefficients and $\bar{P}_{\hat{n},0}^{-1} \in \mathbb{R}^{[p^2\hat{n}+pm(\hat{n}+1)]\times[p^2\hat{n}+pm(\hat{n}+1)]}$ is the positive-definite regularization matrix. The following algorithm from [8] uses recursive least squares to minimize $J_{\hat{n},k}$.

Proposition 3. *For all $k \ge -\hat{n}$, let $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$. Furthermore, let $\theta_{\hat{n},0} \in \mathbb{R}^{p\times p\hat{n}+m(\hat{n}+1)}$ and let $P_{\hat{n},0} \in \mathbb{R}^{[p\hat{n}+m(\hat{n}+1)]\times[p\hat{n}+m(\hat{n}+1)]}$ be positive definite. Then, for all $k \ge 0$, $J_{\hat{n},k}$, defined in* (27)*, has a unique global minimizer, denoted*

$$\theta_{\hat{n},k+1} \triangleq \underset{\hat{\theta}_{\hat{n}} \in \mathbb{R}^{p\times\hat{n}(m+p)+m}}{\arg\min} J_k(\hat{\theta}_{\hat{n}}). \tag{30}$$

which, for all $k \ge 1$, is given by

$$\theta_{\hat{n},k} = (\mathcal{Y}_{0,k}\Phi_{\hat{n},k}^{\mathrm{T}} + \theta_{\hat{n},0}P_{\hat{n},0}^{-1})(\Phi_{\hat{n},k}\Phi_{\hat{n},k}^{\mathrm{T}} + P_{\hat{n},0}^{-1})^{-1}, \tag{31}$$

and where $\Phi_{\hat{n},k} \in \mathbb{R}^{p\hat{n}+m(\hat{n}+1)\times k}$ is defined

$$\Phi_{\hat{n},k} \triangleq \begin{bmatrix}\phi_{\hat{n},k-1} & \cdots & \phi_{\hat{n},0}\end{bmatrix}. \tag{32}$$

Moreover, for all $k \ge 0$, $\bar{\theta}_{\hat{n},k+1}$ is given recursively as

$$P_{\hat{n},k+1} = P_{\hat{n},k} - \frac{P_{\hat{n},k}\phi_{\hat{n},k}\phi_{\hat{n},k}^{\mathrm{T}}P_{\hat{n},k}}{1 + \phi_{\hat{n},k}^{\mathrm{T}}P_{\hat{n},k}\phi_{\hat{n},k}}, \tag{33}$$

$$\theta_{\hat{n},k+1} = \theta_{\hat{n},k} + (y_k - \theta_{\hat{n},k}\phi_{\hat{n},k})\phi_{\hat{n},k}^{\mathrm{T}}P_{\hat{n},k+1}. \tag{34}$$

Proof. See [8] and [16]. □

In practice, the recursive formulation (34) is used to update the coefficient identification in real time as new measurements are obtained. We've included the batch formulation (31) to aid in subsequent analysis. It will also be beneficial for analysis to note that using matrix inversion lemma, for all $k \ge 0$, (33) can be rewritten as

$$P_{\hat{n},k+1}^{-1} = P_{\hat{n},k}^{-1} + \phi_{\hat{n},k}\phi_{\hat{n},k}^{\mathrm{T}}. \tag{35}$$

A. Convergence with Correct Model Order

We begin by considering the case $\hat{n} = n$, where the correct model order is known. In this case, the identified coefficients $\theta_{n,k}$ and true model coefficients $\theta_{n,\mathrm{true}}$ are the same dimension, and it is natural to define the estimation error $\tilde{\theta}_{n,k} \in \mathbb{R}^{p\times pn+m(n+1)}$ as

$$\tilde{\theta}_{n,k} \triangleq \theta_{n,k} - \theta_{n,\mathrm{true}}. \tag{36}$$

It then follows from (25), (34), and (35) that, for all $k \ge 0$, the estimation error dynamics can be written as

$$\tilde{\theta}_{n,k+1} = \tilde{\theta}_{n,k}P_{\hat{n},k}^{-1}P_{\hat{n},k+1}. \tag{37}$$

It then follows from (35) that, for all $k \ge 0$,

$$\tilde{\theta}_{n,k} = \tilde{\theta}_{n,0}P_{n,0}^{-1}P_{n,k} = \tilde{\theta}_{n,0}P_{n,0}^{-1}(\Phi_{n,k-1}\Phi_{n,k-1}^{\mathrm{T}} + P_{n,0}^{-1})^{-1}. \tag{38}$$

We now define the notions of weak persistent excitation and persistent excitation. Note that persistent excitation implies weak persistent excitation.

Definition 3. $(\phi_k)_{k=0}^{\infty} \subset \mathbb{R}^{p\times n}$ *is* ***weakly persistently exciting*** *if* $\lim_{k\to\infty} \boldsymbol{\lambda}_{\mathbf{min}}\left[\sum_{i=0}^{k}\phi_i^{\mathrm{T}}\phi_i\right] = \infty$. $(\phi_k)_{k=0}^{\infty} \subset \mathbb{R}^{p\times n}$

*is **persistently exciting** if* $C \triangleq \lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} \phi_i^{\rm T}\phi_i$ *exists and is positive definite.*

Theorem 3 shows that weak persistent excitation is necessary and sufficient conditions for the global asymptotic stability (GAS) of the error dynamics (37). For definition of GAS and further discussion on weak persistent excitation, see [14]. Moreover, Theorem 3 shows that persistent excitation implies that the convergence of $\tilde{\theta}_{n,k}$ to zero is asymptotically proportional to $1/k$.

Theorem 3. *Consider the assumptions and notation of Proposition 3.* (37) *is GAS if and only of* $(\phi_{n,k}^{\rm T})_{k=0}^{\infty}$ *is weakly persistently exciting. Moreover, if* $(\phi_{n,k}^{\rm T})_{k=0}^{\infty}$ *is persistently exciting, then*

$$\lim_{k\to\infty} k\tilde{\theta}_{n,k} = \tilde{\theta}_{n,0}P_{n,0}^{-1}C_n^{-1}, \tag{39}$$

where $C_n \triangleq \lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1}\phi_{n,k-1}\phi_{n,k-1}^{\rm T}$.

Proof. Note that, for all $1 \le i \le p$,

$$(\tilde{\theta}_{\hat{n},k+1}^{i})^{\rm T} = P_{\hat{n},k+1}P_{\hat{n},k}^{-1}(\tilde{\theta}_{\hat{n},k}^{i})^{\rm T}, \tag{40}$$

where $\tilde{\theta}_{\hat{n},k}^{i} \in \mathbb{R}^{1\times pn+m(n+1)}$ is the $i^{\rm th}$ row of $\tilde{\theta}_{\hat{n},k}$. It follows from Theorem 3 of [14] that (40) is GAS if and only of $(\phi_{n,k}^{\rm T})_{k=0}^{\infty}$ is weakly persistently exciting. Hence, (37) is GAS if and only of $(\phi_{n,k}^{\rm T})_{k=0}^{\infty}$ is weakly persistently exciting. Next, if $(\phi_{n,k}^{\rm T})_{k=0}^{\infty}$ is persistently exciting, it follows from (38) that $\lim_{k\to\infty} k\tilde{\theta}_{n,k} = \lim_{k\to\infty}\tilde{\theta}_{n,0}P_{n,0}^{-1}(\frac{1}{k}\Phi_{n,k-1}\Phi_{n,k-1}^{\rm T} + \frac{1}{k}P_{n,0}^{-1})^{-1} = \tilde{\theta}_{n,0}P_{n,0}^{-1}C_n^{-1}$. □

Proposition 4 show that for $(\phi_{n,k}^{\rm T})_{k=0}^{\infty}$ to be weakly persistently exciting, it is necessary that (1) be irreducible.

Proposition 4. *If* (1) *is reducible, then* $(\phi_{n,k}^{\rm T})_{k=0}^{\infty}$ *is not weakly persistently exciting.*

Proof. If (1) is reducible, then there exists an equivalent input/output model of order $n_{\rm r} < n$. Lemma 3 implies that $\phi_{n,k} = M_{n_{\rm r},n}\begin{bmatrix}\mathcal{Y}_{k-n,n_{\rm r}}\\ \mathcal{U}_{k-n,n}\end{bmatrix}$. Hence, for all $N \ge 0$, $\operatorname{rank}(\sum_{k=0}^{N}\phi_{n,k}\phi_{n,k}^{\rm T}) \le pn_{\rm r} + m(n+1)$. but $\sum_{k=0}^{N}\phi_{n,k}\phi_{n,k}^{\rm T} \in \mathbb{R}^{pn+m(n+1)\times pn+m(n+1)}$. Therefore, for all $N \ge 0$, $\boldsymbol{\lambda}_{\mathbf{min}}\left[\sum_{k=0}^{N}\phi_{n,k}\phi_{n,k}^{\rm T}\right] = 0$. □

Note that if (1) is reducible, then there exists an equivalent input/output model of a lower-order which is irreducible. In other words, (1) being reducible can be viewed as the order of the identified model being higher than the order of the true model. This case is addressed in the following subsection.

B. Convergence with Higher Model Order

Next, we address the case $\hat{n} > n$, where the identified model order is higher than the model order of the true system. To begin, note that a model of order $\hat{n}$ which is equivalent to (1) is given by the coefficients[1]

$$\theta_{\hat{n},\rm true} = \begin{bmatrix}\mathcal{F}_n & 0_{p\times p(\hat{n}-n)} & \mathcal{G}_n & 0_{p\times m(\hat{n}-n)}\end{bmatrix}. \tag{41}$$

In particular, it holds that, for all $k \ge 0$,

$$y_k = \theta_{\hat{n},\rm true}\phi_{\hat{n},k}. \tag{42}$$

Since $\hat{n} > n$, it follows from Proposition 4 that $(\phi_{\hat{n},k}^{\rm T})_{k=0}^{\infty}$ is not weakly persistently exciting. However, it is possible that $(\phi_{n,\hat{n},k}^{\rm T})_{k=0}^{\infty}$ is weakly persistently exciting, where, for all $k \ge 0$, $\phi_{n,\hat{n},k} \in \mathbb{R}^{pn+m(\hat{n}+1)}$ is defined

$$\begin{aligned}\phi_{n,\hat{n},k} &\triangleq \begin{bmatrix}-y_{k-\hat{n}+n-1}^{\rm T} & \cdots & -y_{k-\hat{n}}^{\rm T} & u_k & \cdots & u_{k-\hat{n}}^{\rm T}\end{bmatrix}^{\rm T}\\ &= \begin{bmatrix}\mathcal{Y}_{k-\hat{n},n}^{\rm T} & \mathcal{U}_{k-\hat{n},\hat{n}}^{\rm T}\end{bmatrix}^{\rm T}.\end{aligned} \tag{43}$$

By similar reasoning to Proposition 4, for $(\phi_{n,\hat{n},k}^{\rm T})_{k=0}^{\infty}$ to be weakly persistently exciting, it is necessary that (1) be irreducible. Lemma 3 shows that $\phi_{\hat{n},k} = M_{n,\hat{n}}\phi_{n,\hat{n},k}$ where $M_{n,\hat{n}}$ is defined in (16).

Lemma 3. *For all* $k \ge 0$ *and* $\hat{n} > n$,

$$\phi_{\hat{n},k} = M_{n,\hat{n}}\phi_{n,\hat{n},k}. \tag{44}$$

Proof. Note that $\phi_{\hat{n},k}$ can be written as $\phi_{\hat{n},k} = \begin{bmatrix}\mathcal{Y}_{k-\hat{n},\hat{n}}^{\rm T} & \mathcal{Y}_{k-\hat{n},\hat{n}}^{\rm T}\end{bmatrix}^{\rm T} = \begin{bmatrix}\mathcal{Y}_{k-\hat{n}+n,\hat{n}-n}^{\rm T} & \mathcal{Y}_{k-\hat{n},n}^{\rm T} & \mathcal{U}_{k-\hat{n},\hat{n}}^{\rm T}\end{bmatrix}^{\rm T}$ For all $1 \le i \le \hat{n}-n$, note that $k-i = (k-\hat{n}) + n + (\hat{n}-n-i)$ and it follows from Proposition 1 that $y_{k-i} = -\mathcal{F}_{n,\hat{n}-n-i}\mathcal{Y}_{k-\hat{n},n} + \mathcal{G}_{n,\hat{n}-n-i}\mathcal{U}_{k-\hat{n},\hat{n}-i} = -\mathcal{F}_{n,\hat{n}-n-i}\mathcal{Y}_{k-\hat{n},n} + \begin{bmatrix}0_{p\times im} & \mathcal{G}_{n,\hat{n}-n-i}\end{bmatrix}\mathcal{U}_{k-\hat{n},\hat{n}-i}$. Hence, $\mathcal{Y}_{k-\hat{n}+n,\hat{n}-n} = \begin{bmatrix}M_{n,\hat{n}}^{1} & M_{n,\hat{n}}^{2}\end{bmatrix}\begin{bmatrix}\mathcal{Y}_{k-\hat{n},n}\\ \mathcal{U}_{k-\hat{n},\hat{n}}\end{bmatrix}$, and (44) follows from (16). □

Proposition 5 shows that input/output model (8) with coefficients $\hat{\theta}_{\hat{n}}$ is equivalent to (1) if and only if (45) holds. Next, Proposition 6 give an explicit formulation for the equivalent model of order $\hat{n}$ which minimizes the regularization term $\operatorname{tr}\left[(\hat{\theta}_{\hat{n}} - \theta_{\hat{n},0})P_{\hat{n},0}^{-1}(\hat{\theta}_{\hat{n}} - \theta_{\hat{n},0})^{\rm T}\right]$ of the of cost function $J_{\hat{n},k}$, defined in (27).

Proposition 5. *Input/Output models* (1) *and* (8) *are equivalent if and only if*

$$(\hat{\theta}_{\hat{n}} - \theta_{\hat{n},\rm true})M_{n,\hat{n}} = 0_{p\times pn+m(\hat{n}+1)}. \tag{45}$$

Proof. Note that an input/output model of order $\hat{n}$ with coefficients $\theta_{\hat{n},\rm true}$ is trivially equivalent to (1). Hence, by Theorem 1, $\theta_{\hat{n},\rm true}M_{n,\hat{n}} = \begin{bmatrix}-\mathcal{F}_{n,\hat{n}-n} & \mathcal{G}_{n,\hat{n}-n}\end{bmatrix}$. Then, it also follows from Theorem 1 that (8) is equivalent to (1) if and only of $\hat{\theta}_{\hat{n}}M_{n,\hat{n}} = \begin{bmatrix}-\mathcal{F}_{n,\hat{n}-n} & \mathcal{G}_{n,\hat{n}-n}\end{bmatrix} = \theta_{\hat{n},\rm true}M_{n,\hat{n}}$. □

Proposition 6. *The constrained optimization problem*

$$\min_{\hat{\theta}_{\hat{n}}\in\mathbb{R}^{p\times\hat{n}(m+p)+m}} \operatorname{tr}\left[(\hat{\theta}_{\hat{n}} - \theta_{\hat{n},0})P_{\hat{n},0}^{-1}(\hat{\theta}_{\hat{n}} - \theta_{\hat{n},0})^{\rm T}\right], \tag{46}$$

$$\text{such that } (\hat{\theta}_{\hat{n}} - \theta_{\hat{n},\rm true})M_{n,\hat{n}} = 0_{p\times pn+m(\hat{n}+1)},$$

has the unique solution

$$\theta_{\hat{n}}^{*} \triangleq \theta_{\hat{n},0} + (\theta_{\hat{n},\rm true} - \theta_{\hat{n},0})H_{\hat{n}}, \tag{47}$$

where hat matrix $H_{\hat{n}} \in \mathbb{R}^{p\hat{n}+m(\hat{n}+1)\times p\hat{n}+m(\hat{n}+1)}$ *is defined*

$$H_{\hat{n}} \triangleq M_{n,\hat{n}}(M_{n,\hat{n}}^{\rm T}P_{\hat{n},0}M_{n,\hat{n}})^{-1}M_{n,\hat{n}}^{\rm T}P_{\hat{n},0}. \tag{48}$$

[1]The trivial equivalent model (41) is chosen because it can be written in terms of only the true model coefficients.

Proof. Note that $\text{tr}\left[(\hat{\theta}_{\hat{n}} - \theta_{\hat{n},0})P_{\hat{n},0}^{-1}(\hat{\theta}_{\hat{n}} - \theta_{\hat{n},0})^{\rm T}\right] = \sum_{i=1}^{p}(\hat{\theta}_{\hat{n},i} - \theta_{\hat{n},0,i})P_{\hat{n},0}^{-1}(\hat{\theta}_{\hat{n},i} - \theta_{\hat{n},0,i})^{\rm T}$, where, for all $1 \le i \le p$, $\hat{\theta}_{\hat{n},i} \in \mathbb{R}^{1\times\hat{n}(m+p)+m}$ and $\theta_{\hat{n},0,i} \in \mathbb{R}^{1\times\hat{n}(m+p)+m}$ are the $i^{\rm th}$ row of $\hat{\theta}_{\hat{n}}$ and $\theta_{\hat{n},0}$, respectively. It then follows that (46) can be written as p separate optimization problems given by, for all $1 \le i \le p$, $\min_{\hat{\theta}_{\hat{n},i}\in\mathbb{R}^{1\times\hat{n}(m+p)+m}}(\hat{\theta}_{\hat{n},i}-\theta_{\hat{n},0,i})P_{\hat{n},0}^{-1}(\hat{\theta}_{\hat{n},i}-\theta_{\hat{n},0,i})^{\rm T}$, such that $M_{n,\hat{n}}^{\rm T}\hat{\theta}_{\hat{n},i}^{\rm T} = M_{n,\hat{n}}^{\rm T}\theta_{\hat{n},{\rm true},i}^{\rm T}$, where, for all $1 \le i \le p$, $\theta_{\hat{n},{\rm true},i} \in \mathbb{R}^{1\times\hat{n}(m+p)+m}$ is the $i^{\rm th}$ row of $\theta_{\hat{n},{\rm true},i}$. It then follows from equality constrained convex quadratic minimization (e.g. see section 10.1.1 of [17]) that, for all $1 \le i \le p$, this optimization problem has a unique solution $\theta_{\hat{n},i}^*$, given by

$$M_{n,\hat{n}}^{\rm T}{\theta_{\hat{n},i}^*}^{\rm T} = M_{n,\hat{n}}^{\rm T}\theta_{\hat{n},{\rm true},i}^{\rm T}, \quad (49)$$

$${\theta_{\hat{n},i}^*}^{\rm T} + P_{\hat{n},0}M_{n,\hat{n}}{\nu_i^*}^{\rm T} = \theta_{\hat{n},0,i}^{\rm T}. \quad (50)$$

Substituting (50) into (49), it follows that $M_{n,\hat{n}}^{\rm T}(\theta_{\hat{n},0,i}^{\rm T} - P_{\hat{n},0}M_{n,\hat{n}}{\nu_i^*}^{\rm T}) = M_{n,\hat{n}}^{\rm T}\theta_{\hat{n},{\rm true},i}^{\rm T}$. Hence, ${\nu_i^*}^{\rm T}$ is given as ${\nu_i^*}^{\rm T} = (M_{n,\hat{n}}^{\rm T}P_{\hat{n},0}M_{n,\hat{n}})^{-1}M_{n,\hat{n}}^{\rm T}(\theta_{\hat{n},0,i}^{\rm T} - \theta_{\hat{n},{\rm true},i}^{\rm T})$. Substituting this equation into (50), it follows that ${\theta_{\hat{n},i}^*}^{\rm T} = \theta_{\hat{n},0,i}^{\rm T} - P_{\hat{n},0}M_{n,\hat{n}}(M_{n,\hat{n}}^{\rm T}P_{\hat{n},0}M_{n,\hat{n}})^{-1}M_{n,\hat{n}}^{\rm T}(\theta_{\hat{n},0,i}^{\rm T} - \theta_{\hat{n},{\rm true},i}^{\rm T})$. Finally, taking the transpose and noting that $\theta_{\hat{n},i}^*$ is the $i^{\rm th}$ row of $\theta_{\hat{n}}^*$ yields (47). □

Finally, Theorem 4 show that if $(\phi_{n,\hat{n},k}^{\rm T})_{k=0}^{\infty}$ is weakly persistently exciting, then $\theta_{\hat{n},k}$ converges to $\theta_{\hat{n}}^*$. Moreover, if $(\phi_{n,\hat{n},k}^{\rm T})_{k=0}^{\infty}$ is persistently exciting, then the convergence of $\theta_{\hat{n},k} - \theta_{\hat{n}}^*$ to zero is asymptotically proportional to $1/k$.

Theorem 4. *Consider the assumptions and notation of Proposition 3. If $(\phi_{n,\hat{n},k}^{\rm T})_{k=0}^{\infty}$ is weakly persistently exciting, then*

$$\lim_{k\to\infty}\theta_{\hat{n},k} = \theta_{\hat{n}}^*, \quad (51)$$

where $\theta_{\hat{n}}^*$ is defined in (47). If, additionally, $(\phi_{n,\hat{n},k}^{\rm T})_{k=0}^{\infty}$ is persistently exciting, then

$$\lim_{k\to\infty} k(\theta_{\hat{n},k} - \theta_{\hat{n}}^*) = (\theta_{\hat{n},0} - \theta_{\hat{n},{\rm true}})M_{n,\hat{n}}W_{n,\hat{n}}^{-1}C_{n,\hat{n}}^{-1}W_{n,\hat{n}}^{-1}M_{n,\hat{n}}^{\rm T}P_{\hat{n},0}, \quad (52)$$

where $C_{n,\hat{n}} \triangleq \lim_{k\to\infty}\frac{1}{k}\sum_{k=0}^{k-1}\phi_{n,\hat{n},i}\phi_{n,\hat{n},i}^{\rm T}$ and $W_{n,\hat{n}} \triangleq M_{n,\hat{n}}^{\rm T}P_{\hat{n},0}M_{n,\hat{n}}$.

Proof. For brevity, denote $P_0 \triangleq P_{\hat{n},0}$, $\theta^* \triangleq \theta_{\hat{n}}^*$, $\theta_0 \triangleq \theta_{\hat{n},0}$, $\theta \triangleq \theta_{\hat{n},{\rm true}}$, $M \triangleq M_{n,\hat{n}}$, $H = H_{\hat{n}}$, $W \triangleq W_{n,\hat{n}}$, and, for all $k \ge 0$, $\theta_k \triangleq \theta_{\hat{n},k}$, $\Phi_k \triangleq \Phi_{\hat{n},k}$ and $\bar{\Phi}_k \triangleq \begin{bmatrix}\phi_{n,\hat{n},k-1} & \cdots & \phi_{n,\hat{n},0}\end{bmatrix}$. It follows from (31), (42), and Lemma 3 that, for all $k \ge 0$, $\theta_{k+1} = (\theta\Phi_k\Phi_k^{\rm T} + \theta_0P_0^{-1})(\Phi_k\Phi_k^{\rm T} + P_0^{-1})^{-1} = \theta + (\theta_0 - \theta)P_0^{-1}(\Phi_k\Phi_k^{\rm T} + P_0^{-1})^{-1} = \theta + (\theta_0 - \theta)P_0^{-1}(M\bar{\Phi}_k\bar{\Phi}_k^{\rm T}M^{\rm T} + P_0^{-1})^{-1}$. Subtracting both sides by θ^* and substituting (47) yields $\theta_{k+1} - \theta^* = (\theta_0 - \theta)\big[-I + P_0^{-1}(M\bar{\Phi}_k\bar{\Phi}_k^{\rm T}M^{\rm T} + P_0^{-1})^{-1} + H\big]$.

Since $(\phi_{n,\hat{n},k}^{\rm T})_{k=0}^{\infty}$ is weakly persistently exciting, there exists N such that, for all $k \ge N$, $\bar{\Phi}_k\bar{\Phi}_k^{\rm T}$ is nonsingular. Then, it follows from matrix inversion lemma that, for all $k \ge N$, $(M\bar{\Phi}_k\bar{\Phi}_k^{\rm T}M^{\rm T} + P^{-1})^{-1} = P_0 - P_0M[(\bar{\Phi}_k\bar{\Phi}_k^{\rm T})^{-1} + W]^{-1}M^{\rm T}P_0$, and hence $-I + P_0^{-1}(M\bar{\Phi}_k\bar{\Phi}_k^{\rm T}M^{\rm T} + P_0^{-1})^{-1} = -M[(\bar{\Phi}_k\bar{\Phi}_k^{\rm T})^{-1} + W]^{-1}M^{\rm T}P_0$. Substituting this and $H = M(M^{\rm T}P_0M)^{-1}M^{\rm T}P_0$ into the expression for $\theta_{k+1} - \theta^*$ implies that, for all $k \ge N$, $\theta_{k+1} - \theta^* = (\theta_0 - \theta)M\left[-[(\bar{\Phi}_k\bar{\Phi}_k^{\rm T})^{-1} + W]^{-1} + W^{-1}\right]M^{\rm T}P_0$. Again applying matrix inversion lemma, it follows that $-[(\bar{\Phi}_k\bar{\Phi}_k^{\rm T})^{-1} + W]^{-1} + W^{-1} = W^{-1}[\bar{\Phi}_k\bar{\Phi}_k^{\rm T} - W^{-1}]^{-1}W^{-1}$ which yields $\theta_{k+1} - \theta^* = (\theta_0 - \theta)MW^{-1}[\bar{\Phi}_k\bar{\Phi}_k^{\rm T} - W^{-1}]^{-1}W^{-1}M^{\rm T}P_0$. Since $(\phi_{n,\hat{n},k}^{\rm T})_{k=0}^{\infty}$ is weakly persistently exciting, it follows that $\lim_{k\to\infty}\boldsymbol{\lambda}_{\bf min}\left(\bar{\Phi}_k\bar{\Phi}_k^{\rm T}\right) = \infty$ and $\lim_{k\to\infty}\theta_{k+1} - \theta^* = 0$ and (51) follows. If additionally, $(\phi_{n,\hat{n},k}^{\rm T})_{k=0}^{\infty}$ is persistently exciting, then $\lim_{k\to\infty}k(\theta_{k+1} - \theta^*) = (\theta_0 - \theta)MW^{-1}[\lim_{k\to\infty}\frac{1}{k}\bar{\Phi}_k\bar{\Phi}_k^{\rm T} - \frac{1}{k}W^{-1}]^{-1}W^{-1}M^{\rm T}P_0$. Since $\lim_{k\to\infty}\frac{1}{k}\bar{\Phi}_k\bar{\Phi}_k^{\rm T} - \frac{1}{k}W^{-1} = C_{n,\hat{n}}$, (52) follows. □

V. Conclusions

This work shows that when using RLS to identify an input/output model of a higher-order than that of the true input/output system, the identified coefficients still converge to predictable values, given weakly persistently exciting data. In particular, we obtain the natural result that the higher-order identified model converges to the model equivalent to the true system that minimizes the regularization term of RLS.

References

[1] K. J. Astrom and B. Wittenmark, *Adaptive Control*, 2nd ed. USA: Addison-Wesley Longman Publishing Co., Inc., 1994.

[2] L. Ljung and T. Söderström, *Theory and practice of recursive identification*. MIT press, 1983.

[3] G. C. Goodwin and K. S. Sin, *Adaptive filtering prediction and control*. Courier Corporation, 2014.

[4] J. C. Willems, "From time series to linear system—part i. finite dimensional linear time invariant systems," *Automatica*, vol. 22, no. 5, pp. 561–580, 1986.

[5] E. Cuevas *et al.*, "A comparison of evolutionary computation techniques for iir model identification," *Journal of Applied Math.*, 2014.

[6] J. C. Willems and J. W. Polderman, *Introduction to mathematical systems theory*. Springer Science & Business Media, 1997, vol. 26.

[7] L. Shieh and Y. Tsay, "Transformations of a class of multivariable control systems to block companion forms," *IEEE Transactions on Automatic Control*, vol. 27, no. 1, pp. 199–203, 1982.

[8] T. W. Nguyen *et al.*, "Predictive cost adaptive control: A numerical investigation of persistency, consistency, and exigency," *IEEE Control Systems Magazine*, vol. 41, no. 6, pp. 64–96, 2021.

[9] S. A. U. Islam *et al.*, "Data-driven retrospective cost adaptive control for flight control applications," *Journal of Guidance, Control, and Dynamics*, vol. 44, no. 10, pp. 1732–1758, 2021.

[10] B. Y. Lai, S. A. U. Islam, S. Nivison, and D. S. Bernstein, "Data-driven retrospective cost adaptive control of a quadrotor uav," in *AIAA SCITECH 2023 Forum*, 2023, p. 0694.

[11] N. Mohseni *et al.*, "Adaptive force control for small celestial body sampling," in *Proc. Amer. Contr. Conf.* IEEE, 2022, pp. 2515–2520.

[12] A. Farahmandi and B. Reitz, "Predictive cost adaptive control of a planar missile with unmodeled aerodynamics," in *AIAA SCITECH 2024 Forum*, 2024, p. 2218.

[13] F. Ding, P. X. Liu, and G. Liu, "Multiinnovation least-squares identification for system modeling," *IEEE Transactions on Systems, Man, and Cybernetics, Part B*, vol. 40, no. 3, pp. 767–778, 2009.

[14] A. L. Bruce, A. Goel, and D. S. Bernstein, "Necessary and sufficient regressor conditions for the global asymptotic stability of recursive least squares," *Systems & Control Letters*, vol. 157, p. 105005, 2021.

[15] B. Lai, S. A. U. Islam, and D. S. Bernstein, "Regularization-induced bias and consistency in recursive least squares," in *2021 American Control Conference (ACC)*. IEEE, 2021, pp. 3987–3992.

[16] S. A. U. Islam and D. Bernstein, "Recursive least squares for real-time implementation," *IEEE Ctrl. Sys. Mag.*, vol. 39, no. 3, pp. 82–85, 2019.

[17] S. P. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.