2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Trackability Compensation for Iterative Learning Control: A Data-Based Approach

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***Abstract*— This paper aims at proposing a data-based trackability compensation strategy for iterative learning control systems to enhance their tracking performances when confronted with untrackable references. By designing and leveraging offline input-output test principles, an alternative data-based representation is constructed, based on which a data-based trackability criterion is developed. In scenarios where the reference outputs are untrackable, by interconnecting the original system with an auxiliary system, the trackability set of the interconnected system is modified. Consequently, the originally untrackable references become trackable for the interconnected system, and the perfect tracking preformances of iterative learning control can be guaranteed.**

I. Introduction

Iterative learning control (ILC), as a powerful intelligent control method, has been widely applied in various industrial fields such as robotics [1], chemical processes [2], and high-speed trains [3]. ILC is especially applicable for systems that undergo repetitive operations within some specific time duration, and its implementation relies on the learning mechanism from past experiences [4]. To be specific, by employing the knowledge of tracking errors and inputs from past iterations, ILC recursively modifies the control inputs throughout the iteration axis and ultimately achieves accurate tracking of the desired reference over the entire time duration [5]. There has been an abundant research history of ILC, for which we refer readers to a comprehensive survey for more explanations [6].

Convergence analysis is indispensable for the synthesis of ILC, essential for characterizing the tracking performance of the designed iterative learning controller. In the most of the existing literatures of ILC, the mainstream convergence analysis frameworks are building upon the Banach fixed point theorem or the composed energy functions [7]. Nevertheless, both of these analytical frameworks depend on a conservative assumption on the existence of the desired input with respect to the desired reference. Unfortunately, the existence of such desired input is strictly dependent on the model knowledge, raising doubts on the rationality of this assumption. Once this assumption fails to hold, ILC can solely achieve conservative tracking performances. To address this essential issue, some recent results have introduced the *trackability* and established its connection with the solvability of linear algebraic equations [8]. Benifiting from these efforts, one can efficiently verify the trackability of some reference, thereby determining whether perfect tracking performance can be achieved [9].

This work was supported by the National Natural Science Foundation of China under Grant U2333215 and Grant 62273018. *(Corresponding author: Deyuan Meng.)*

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Inspired by the results mentioned earlier, a further question arises: whether there exists certain trackability compensation strategy to guarantee that ILC can achieve perfect tracking of those references that were originally untrackable. This question motivates the proposal of a data-based trackability compensation strategy via interconnection in this paper. Due to the absence of model knowledge, the offline input-output tests are designed, from which an alternative data-based I-O representation is constructed. By leveraging this data-based I-O representation, the trackability of certain references and the trackability set of original ILC systems can be identified. For untrackable references, by interconnecting the original ILC system with the specifically designed auxiliary system, the trackability set of the interconnected system may be compensated. Through the proposed trackability compensation strategy, the originally untrackable references are expected to lie within the compensated trackability set. Subsequently, we can proceed with the synthesis and analysis of ILC for the interconnected system, and the perfect tracking performances for the originally untrackable references can be achieved.

The reminder of this paper is organized as follows. The preliminaries on ILC and the data-based trackability compensation problems via interconnection are formulated in Section II. In Section III-A, the offline I-O tests are designed, from which a data-based I-O representation is constructed. The trackability compensation strategy is demonstrated in Section III-B. Afterward, for those originally untrackable references, a modified data-based ILC framework is presented in Section III-C. An illustrative example is provided in Section IV. Finally, Section V summarizes the contributions in this paper.

Notation: Let $\mathbb{Z}_N = \{0,1,\cdots,N\}$ and $\mathbb{Z}_+ = \{0,1,2,\cdots\}$. For a matrix A, its transpose and Moore-Penrose inverse are denoted as A^{T} and $A^{\dagger}$, respectively. The null and identity matrices with appropriate dimensions are denoted as 0 and I. For arbitrary vectors a and b, $\operatorname{col}(a,b)$ refers to $\left[a^{\mathrm{T}}\ b^{\mathrm{T}}\right]^{\mathrm{T}}$.

II. Preliminaries and Problem Statement

A. Iterative Learning Control and Trackability

Consider a class of repetitive systems whose dynamics can be described by the state space representation as

$$\begin{aligned} x_k(t+1) &= A_s(t)x_k(t) + B_s(t)u_k(t) \\ y_k(t) &= C_s(t)x_k(t) + D_s(t)u_k(t) \end{aligned} \quad (1)$$

where $t \in \mathbb{Z}_T$ and $k \in \mathbb{Z}_+$ refers to the time and iteration axes, respectively. The input and output in k-th iteration are denoted by $u_k(t) \in \mathbb{R}^{n_u}$ and $y_k(t) \in \mathbb{R}^{n_y}$, respectively. Owing to the existence of non-minimal representation, the internal state with unknown dimension is denoted as $x_k(t) \in \mathbb{R}^{\bullet}$. Correspondingly, $(A_s(t), B_s(t), C_s(t), D_s(t))$ represent the unknown model matrices with appropriate dimensions. For a reference output $y_d(t) \in \mathbb{R}^{n_y}$, ILC focuses on addressing the output tracking issues over a specific time duration $t \in \mathbb{Z}_T$, for which the *lifting technique* is widely leveraged to introduce the supervectors as follows [6]:

$$\begin{aligned} U_k &= \left[u_k^{\mathrm{T}}(0),\ u_k^{\mathrm{T}}(1),\ \cdots,\ u_k^{\mathrm{T}}(T)\right]^{\mathrm{T}} \in \mathbb{R}^{n_u T}, \\ Y_k &= \left[y_k^{\mathrm{T}}(0),\ y_k^{\mathrm{T}}(1),\ \cdots,\ y_k^{\mathrm{T}}(T)\right]^{\mathrm{T}} \in \mathbb{R}^{n_y T}, \\ Y_d &= \left[y_d^{\mathrm{T}}(0),\ y_d^{\mathrm{T}}(1),\ \cdots,\ y_d^{\mathrm{T}}(T)\right]^{\mathrm{T}} \in \mathbb{R}^{n_y T}. \end{aligned}$$

Based on these defined supervectors, the input-output relationship of (1) over the time duration $\mathbb{Z}_T$ is represented as

$$Y_k = GU_k + Lx_k(0),\ \forall k \in \mathbb{Z}_+ \tag{2}$$

where the input-output transfer matrix G and initial state-output transfer matrix L can be steadily constructed through the model knowledge $(A_s(t), B_s(t), C_s(t), D_s(t))$ [10]. To deal with the tracking issues over $t \in \mathbb{Z}_T$, ILC recursively modifies the input along the iteration axis in the form of

$$U_{k+1} = U_k + \Delta U_{k+1},\ \forall k \in \mathbb{Z}_+ \tag{3}$$

where ΔU_{k+1} is usually designed by exploiting the tracking error information from past iterations. Given the repetitive characteristics of the ILC system (1), it is required that the system starts from an identical initial state in each iteration, and the following assumption is commonly employed.

Assumption 1. The initial state of the ILC system (1) is iteration-invariant, i.e., $x_k(0) = x_0,\ \forall k \in \mathbb{Z}_+$.

Of note is that Assumption 1 does not require the specific value of x_0, and it is only employed to constrain the system (1) to commence from some identical initially stored energy at the beginning of each iteration. With the ILC system (1) and Assumption 1, we further introduce the definitions of *trackability* and *trackability set*.

Definition 1. For the ILC system (1) with $x_k(0) = x_0,\ \forall k \in \mathbb{Z}_+$, a reference output Y_d is said to be trackable if there exists some input U_d such that (U_d, Y_d) fullfills (2). Moreover, the trackability set of ILC system (1) is defined as

$$\boldsymbol{Y}_{\text{track}} = \{Y_d \,|\, \exists U_d \text{ such that } (U_d, Y_d) \text{ fullfills } (2)\}.$$

Remark 1. Of note is that the definitions of trackability and trackability set depend on the initial state x_0. The necessity of introducing the notion of trackability lies in the fact that if a reference output is trackable, then there must exist some controllers in the form of (3), ensuring the achievement of *perfect tracking* performance

$$\lim_{k\to\infty} Y_k = Y_d. \tag{4}$$

Additionally, for any untrackable reference output in $\mathbb{R}^{n_y T} - \boldsymbol{Y}_{\text{track}}$, perfect tracking (4) can not be achieved, resulting in a loss of tracking performance. For a untrackable reference output, proposing strategies to compensate for its trackability is an effective method to enhance the tracking performance.

B. Data-Based Trackability Compensation Problems

Model information is indispensable for ILC in the controller design and convergence analysis. Moreover, as emphasized in Remark 1, once a reference output is untrackable, ILC can only achieve conservative tracking performances. In response to these two issues, the data-based trackability compensation problems are formulated as follows:

Problem Statement. For the unknown ILC system (1) under Assumption 1, let $Y_d^c \in \boldsymbol{Y}_{\text{track}}^c := \mathbb{R}^{n_y T} - \boldsymbol{Y}_{\text{track}}$ represent some untrackable reference output. This paper focuses on solving the following two problems:

1) Develop an interconnection-based trackability compensation strategy by leveraging the sampled data from ILC system (1). By interconnecting the ILC system (1) with some specifically designed system or controller, the trackability set of the interconnected system is modified to $\boldsymbol{Y}_{\text{track}}^{\text{inter}}$, ensuring that $Y_d^c \in \boldsymbol{Y}_{\text{track}}^{\text{inter}}$.
2) For the interconnected system and originally untrackable reference Y_d^c, design a data-based iterative learning controller in the form of (3), ensuring that the perfect tracking performance (4) can be achieved.

III. Data-Based Trackability Compensation

For the scenarios where the model knowledge of the ILC system (1) is inaccessible, this section attempts to develop a data-based trackability compensation strategy, for which we first construct an alternative I-O representation utilizing data.

A. Input-Output Representation Based on Sampled Data

As emphasized in Section II-A, the state space model (1) is somewhat unreliable in model-free scenarios owing to the presence of non-minimal representation [11]. Therefore, an I-O representation is preferred. To collect a sufficient amount of I-O data, offline I-O tests need to be conduct. To address the output tracking problems over the time duration $\mathbb{Z}_T$, at least $n_u T + 1$ times I-O tests are required. In each I-O test, the initial state is fixed as x_0 which may be unknown. In the i-th I-O test, the system (1) is applied with the input $U^i \in \mathbb{R}^{n_u T}$ over the entire time duration $\mathbb{Z}_T$, and corresponding response $Y^i \in \mathbb{R}^{n_y T}$ is obtained. The inputs and outputs are collected into data matrices as

$$\begin{aligned} \boldsymbol{U}_{\text{Test}} &= \left[U^1,\ U^2,\ \cdots,\ U^{n_u T+1}\right] \in \mathbb{R}^{n_u T \times (n_u T+1)}, \\ \boldsymbol{Y}_{\text{Test}} &= \left[Y^1,\ Y^2,\ \cdots,\ Y^{n_u T+1}\right] \in \mathbb{R}^{n_y T \times (n_u T+1)}. \end{aligned}$$

To guarantee the sufficiency of the sampled data, additional constraints need to be imposed on the test inputs as follows:

Offline Test Principles. In the offline I-O tests, the following two test principles are executed:

1) In the first I-O test, the input is designed as
$$U^1 = \left[0_{n_u}^{\mathrm{T}},\ 0_{n_u}^{\mathrm{T}},\ \cdots,\ 0_{n_u}^{\mathrm{T}}\right] \in \mathbb{R}^{n_u T}; \tag{5}$$
2) In later $n_u T$ tests, the inputs are designed to satisfy
$$\operatorname{rank}\left(\left[U^2,\ U^3,\ \cdots,\ U^{n_u T+1}\right]\right) = n_u T. \tag{6}$$

Let the vector consisting of the test I-O pair be denoted as $w^i = \mathrm{col}\left(U^i, Y^i\right)$. Because of the identical initial state x_0, it is convincing that each w^i satisfies (2). From the offline test principle (6), the inputs $\left\{U^2,\ U^3,\ \cdots,\ U^{n_u T+1}\right\}$ form a set of bases of $\mathbb{R}^{n_u T}$. Without the loss of generality, the sampled data are supposed to satisfy the following assumption.

Assumption 2. The offline sampled data $(\boldsymbol{U}_{\text{Test}}, \boldsymbol{Y}_{\text{Test}})$ satisfy the offline test principles (5) and (6).

With the application of the offline sampled data, a data-based I-O representation can be constructed, which will be detailedly interpreted in Lemma 1.

Lemma 1. For the unknown system (1) under Assumption 1, let Assumption 2 hold. Then (U_k, Y_k) satisfies (2) if and only if there exists some $g_k \in \mathbb{R}^{n_u T+1}$ such that

$$\begin{bmatrix} 1^{\mathrm{T}}_{n_u T+1} \\ \boldsymbol{U}_{\text{Test}} \\ \boldsymbol{Y}_{\text{Test}} \end{bmatrix} g_k = \begin{bmatrix} 1 \\ U_k \\ Y_k \end{bmatrix}. \tag{7}$$

Proof. Sufficiency: Under the Assumptions 1 and 2, each pair of the offline I-O data $\left(U^i, Y^i\right),\ i \in \mathbb{Z}_{n_u T+1} \backslash \{0\}$ satisfies the following non-homogeneous linear algebraic equation

$$\left[-G,\ I_{n_y T}\right] w^i = L x_0. \tag{8}$$

From this LAE, an insightful conclusion is that the affine combinition of any two pairs of offline I-O data, denoted by

$$\alpha w^i + (1-\alpha) w^j,\ \forall \alpha \in \mathbb{R},\ \forall i, j \in \mathbb{Z}_{n_u T+1} \backslash \{0\}$$

still satisfies the same equation (8), or equivalently, fullfills (2). From the first equation of (7), it can be readily concluded that $\sum_{i=1}^{n_u T+1} g_{k,i} = 1$, where $g_{k,i}$ represents the i-th element of g_k. Consequently, under the constraints of the first equation, $\left[\boldsymbol{U}^{\mathrm{T}}_{\text{Test}},\ \boldsymbol{Y}^{\mathrm{T}}_{\text{Test}}\right]^{\mathrm{T}} g_k$ exactly represents the affine combination of w_i for all $i \in \mathbb{Z}_{n_u T+1} \backslash \{0\}$. Therefore, (U_k, Y_k) satisfies (2).

Necessity: From the offline test principles (5) and (6), the test inputs $\left\{U^2, U^3, \cdots, U^{n_u T+1}\right\}$ form a set of bases of $\mathbb{R}^{n_u T}$. This fact indicates that, for any input U_k, there must exist a series of real number $g_{k,i} \in \mathbb{R},\ k \in \mathbb{Z}_{n_u T+1} \backslash \{0\}$ such that

$$U_k = \sum_{i=2}^{n_u T+1} g_{k,i} U^i + g_{k,1} U^1. \tag{9}$$

Since (U_k, Y_k) satisfies (2), by designing $g_{k,1} = 1 - \sum_{i=2}^{n_u T+1}$ and leveraging (9), the corresponding output can be likewise expressed as

$$\begin{aligned} Y_k &= \sum_{i=2}^{n_u T+1} g_{k,i} Y^i + g_{k,1} Y^1 - \sum_{i=1}^{n_u T+1} g_{k,i} L x_0 + L x_0 \\ &= \sum_{i=2}^{n_u T+1} g_{k,i} Y^i + g_{k,1} Y^1. \end{aligned} \tag{10}$$

By simultaneously taking into account (9) and (10), for any (U_k, Y_k) satisfying (2), there always exist some vector $g_k = \left[g_{k,1},\ g_{k,2},\ \cdots,\ g_{k,n_u T+1}\right]^{\mathrm{T}}$ such that (7) holds. ■

Remark 2. By employing the I-O data collected under the offline test principles (5) and (6), Lemma 1 has established a data-based I-O representation to equivalently characterize the dynamics of the original ILC system (1). In the absence of model knowledge, (7) can serve as an alternative data-based model. Following the established representation (7), a data-based criterion for trackability can be further developed.

Remark 3. From Lemma 1, another inspirative result is that the set involving all possible I-O pairs, which is defined as

$$\mathscr{B}_{x_0} = \{w = \mathrm{col}(U, Y) \,|\, (U, Y) \text{ satisfies } (2)\}$$

constitutes an affine set. Additionally, this affine set can be decomposed into the sum of subspace and offset components, which can be constructed from the sampled data as

$$\begin{aligned} &\mathscr{B}_{x_0} = \mathscr{W} + w^1, \\ &\mathscr{W} = \mathrm{span}(H), \\ &H = \left[w^2 - w^1, w^3 - w^1, \cdots, w^{n_u T+1} - w^1\right]. \end{aligned} \tag{11}$$

It is quite obvious that the subspace $\mathscr{W}$ is exactly the kernel space of the matrix $\left[-G,\ I\right]$, and the columns of H constitute a set of bases of $\mathscr{W}$.

Corollary 1. For the unknown system (1) under Assumption 1, let the sampled data $(\boldsymbol{U}_{\text{Test}}, \boldsymbol{Y}_{\text{Test}})$ satisfy Assumption 2. A reference output Y_d is trackable if and only if there exists a vector $\tilde{g} \in \mathbb{R}^{n_u T+1}$ such that

$$\begin{bmatrix} 1^{\mathrm{T}}_{n_u T+1} \\ \boldsymbol{Y}_{\text{Test}} \end{bmatrix} \tilde{g} = \begin{bmatrix} 1 \\ Y_d \end{bmatrix}. \tag{12}$$

Proof. Sufficiency: Due to the existence of the vector $\tilde{g}$, we adopt the control input $U_d = \boldsymbol{U}_{\text{Test}} \tilde{g}$ and can readily deduce

$$\begin{bmatrix} 1^{\mathrm{T}}_{n_u T+1} \\ \boldsymbol{U}_{\text{Test}} \\ \boldsymbol{Y}_{\text{Test}} \end{bmatrix} \tilde{g} = \begin{bmatrix} 1 \\ \boldsymbol{U}_{\text{Test}} \tilde{g} \\ Y_d \end{bmatrix}.$$

With the application of Definition 1 and Lemma 1, it can be concluded that Y_d is trackable.

Necessity: From Definition 1, the output Y_d is trackable if and only if there exists some U_d such that (U_d, Y_d) satisfies (2) or (7). By leveraging Lemma 1, there must exist some $g_k \in \mathbb{R}^{n_u T+1}$ such that (7) holds. Therefore, (12) holds by simply choosing $\tilde{g} = g_k$. ■

Remark 4. By exploiting the data-based I-O representation presented in Lemma 1, Corollary 1 further develops an data-based trackability criterion. Once a reference output Y_d is trackable, then adopting the conventional data-based ILC can achieve the perfect tracking performances [12]. Otherwise, for the untrackable reference output Y^c_d, a trackability compensation strategy and modified ILC need to be developed.

B. Trackability Compensation via Interconnection

To further enhance the tracking performances of ILC when confronted with untrackable reference outputs $Y^c_d \in \boldsymbol{Y}^c_{\text{track}}$, we attempt to interconnect the ILC system (1) with another system or controller, thus modifying the trackability set of the interconnected system, which is denoted by $\boldsymbol{Y}^{\text{inter}}_{\text{track}}$. Through such compensation, it is expected that $Y^c_d \in \boldsymbol{Y}^{\text{inter}}_{\text{track}}$. Such a trackability compensation strategy can be interpreted from a geometric perspective, as shown in Fig. 1. From Corollary 1, the trackability set $\boldsymbol{Y}_{\text{track}}$ constitutes an affine hyperplane in Euclidean space. For an untrackable reference Y^c_d, it must lie outside of this affine hyperplane. Nevertheless, through the trackability compensation strategy, the trackability set is modified to another affine hyperplane $\boldsymbol{Y}_{\text{track}}^{\text{inter}}$ ensuring $Y_d^c \in \boldsymbol{Y}_{\text{track}}^{\text{inter}}$, such that ILC can achieve perfect tracking.

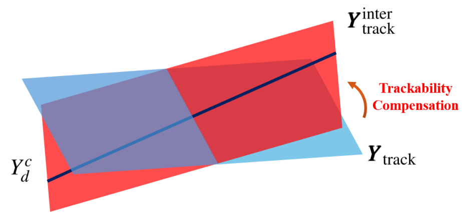

Fig. 1. A geometric interpretation for trackability compensation strategy.

Consider an auxiliary system whose input-output relationship over the time duration $\mathbb{Z}_T$ can be described as

$$\Sigma: \; \overline{Y}_k = G_\Sigma \overline{U}_k, \; \forall k \in \mathbb{Z}_+ \tag{13}$$

where the dimensions of $\overline{U}_k$ and $\overline{Y}_k$, denoted by $n_{\overline{u}}T$ and $n_{\overline{y}}T$, depend on the specific interconnection structure. We consider two fundamental interconnection structures: feedforward and feedback, as shown in Figs. 2 and 3. Despite the differences

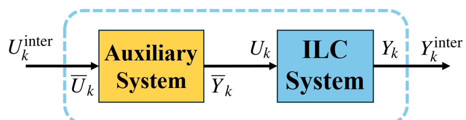

Fig. 2. Feedforward interconnection structure.

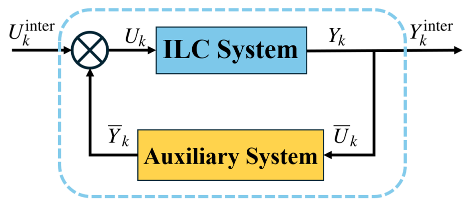

Fig. 3. Feedback interconnection structure.

in the interconnection structures, we may attempt to establish a unified framework to further analysis the input-output relationship of the interconnected system. Let $w_k = \text{col}(U_k, Y_k)$, $\overline{w}_k = \text{col}(\overline{U}_k, \overline{Y}_k)$, and $w_k^{\text{inter}} = \text{col}\left(U_k^{\text{inter}}, Y_k^{\text{inter}}\right)$ denote the I-O pairs of the ILC, auxiliary, and interconnected systems, respectively. Without loss of generality, we assume that $n_{\overline{u}} = n_u$ in the feedforward interconnection, hence maintaining the dimensions of the inputs and outputs regardless of the structures of interconnection. Then, there must be

$$w_k^{\text{inter}} = K_1 w_k + K_2 \overline{w}_k \tag{14}$$

subject to addition structral constraint

$$S_1 w_k + S_2 \overline{w}_k = 0. \tag{15}$$

For different interconnection structures, S_1, S_2, K_1, and K_2 possess different meanings as follows:

1) In the feedforward interconnection:

$$\begin{aligned} K_1 &= \text{blkdiag}\left(0_{n_uT\times n_uT}, I_{n_yT}\right), \; S_1 = \left[I_{n_uT} \;\; 0_{n_uT\times n_yT}\right], \\ K_2 &= \text{blkdiag}\left(I_{n_uT}, 0_{n_yT\times n_uT}\right), \; S_2 = \left[0_{n_uT\times n_uT} \;\; -I_{n_uT}\right]; \end{aligned} \tag{16}$$

2) In the feedback interconnection:

$$\begin{aligned} K_1 &= \text{blkdiag}\left(I_{n_uT}, I_{n_yT}\right), \; S_1 = \left[0_{n_yT\times n_uT} \;\; I_{n_yT}\right], \\ K_2 &= \begin{bmatrix} 0_{n_uT\times n_yT} & -I_{n_uT} \\ 0_{n_yT\times n_yT} & 0_{n_yT\times n_uT} \end{bmatrix}, \; S_2 = \left[-I_{n_yT} \;\; 0_{n_yT\times n_uT}\right]. \end{aligned} \tag{17}$$

Other more complex interconnection structures can always be decomposed into combinations of feedforward and feedback interconnections, and corresponding interconnection parameter matrices can be derived from (16) and (17). Afterward, the data-based I-O representation of the interconnected system needs to be explored. Similar to Remark 3, let span(H_Σ), where H_Σ is of full column rank, represent the subspace $\ker\left(\left[-G_\Sigma, \; I\right]\right)$. Specifically, rank$(H_\Sigma) = n_uT$ in feedforward interconnection, and rank$(H_\Sigma) = n_yT$ for the feedback interconnection. Let $K_{1,1}$ and $K_{2,1}$ represent matrices composed of the first n_u rows of K_1 and K_2, repsectively. Correspondingly, the matrices composed of the last n_y rows are denoted by $K_{1,2}$ and $K_{2,2}$. Then the I-O relationship of the interconnected system is given through the following theorem.

Theorem 1. For the unknown ILC system (1) under Assumption 1, let sampled data $(\boldsymbol{U}_{\text{Test}}, \boldsymbol{Y}_{\text{Test}})$ satisfy Assumption 2. By interconnecting the systems (1) and (13) via some interconnection structure, $\left(U_k^{\text{inter}}, Y_k^{\text{inter}}\right)$ is an I-O pair of the interconnected system if and only if

$$\begin{bmatrix} U_k^{\text{inter}} \\ Y_k^{\text{inter}} \\ 0 \end{bmatrix} = \begin{bmatrix} K_{1,1}H & K_{2,1}H_\Sigma \\ K_{1,2}H & K_{2,2}H_\Sigma \\ S_1H & S_2H_\Sigma \end{bmatrix} \alpha_k + \begin{bmatrix} K_{1,1}w^1 \\ K_{1,2}w^1 \\ S_1w^1 \end{bmatrix} \tag{18}$$

holds for some vector α_k.

Proof. Inspired by Remark 3, for any $w_k = \text{col}(U_k, Y_k)$, there always exist some vector $\alpha_{k,1} \in \mathbb{R}^{n_uT}$ such that $w_k = H\alpha_{k,1} + w^1$. Likewise, any $\overline{w}_k$ that satisfies (13) can be expressed as $\overline{w}_k = H_\Sigma \alpha_{k,2}$. Therefore, (14) and (15) can be rewritten as

$$\begin{aligned} w_k^{\text{inter}} &= K_1 H\alpha_{k,1} + K_1 w^1 + K_2 H_\Sigma \alpha_{k,2}, \\ 0 &= S_1 H\alpha_{k,1} + S_1 w^1 + S_2 H_\Sigma \alpha_{k,2}. \end{aligned}$$

By designing a vector as $\alpha_k = \text{col}\left(\alpha_{k,1}, \alpha_{k,2}\right)$, it is concluded that $\left(U_k^{\text{inter}}, Y_k^{\text{inter}}\right)$ is an I-O pair of the interconnected system if and only if (18) holds. ■

Following the established I-O representation of the interconnected system, the trackability of Y_d^c for interconnected system can be immediately determined through Corollary 2.

Corollary 2. For the unknown ILC system (1) under Assumption 1, let the sampled data $(\boldsymbol{U}_{\text{Test}}, \boldsymbol{Y}_{\text{Test}})$ satisfy Assumption 2, and let Y_d^c be a untrackable reference output for (1). Through the interconnection presented in Theorem 1, $Y_d^c \in \boldsymbol{Y}_{\text{track}}^{\text{inter}}$ holds if and only if there exists some $\tilde{\alpha}$ such that

$$\begin{bmatrix} Y_d^c \\ 0 \end{bmatrix} = \begin{bmatrix} K_{1,2}H & K_{2,2}H_\Sigma \\ S_1H & S_2H_\Sigma \end{bmatrix} \tilde{\alpha} + \begin{bmatrix} K_{1,2}w^1 \\ S_1w^1 \end{bmatrix}. \tag{19}$$

Proof. A direct consequence of Theorem 1. ■

Remark 5. Even if the reference output output Y_d^c is untrackable for the original ILC system (1), Corollary 2 proposes a trackability compensation strategy to render Y_d^c is trackable for the interconnected system, i.e., $Y_d^c \in \boldsymbol{Y}_{\text{track}}^{\text{inter}}$. To ensure that

Y_d^c satisfies (19), the interconnection parameters K_1, K_2, S_1, S_2 and controller-related parameters H_Σ need to be designed. Based on the proposed trackability compensation strategy, ILC can achieve perfect tracking of Y_d^c.

C. Modified Data-Based ILC

After completing the trackability compensation, we further design the required data-based iterative learning controller for the interconnected system, such that the perfect tracking of Y_d^c can be achieved. Taking optimization-based ILC as an demonstration, the controller is designed as

$$U_{k+1}^{\text{inter}} = U_k^{\text{inter}} + \Delta U_{k+1}^{\text{inter}}$$

where $\Delta U_{k+1}^{\text{inter}}$ is derived through minimizing the designed cost function. Nevertheless, such optimization-based ILC strategies depend on precise model knowledge. To release the dependence on system model, we adopt the alternative data-based representation in (18), and the control updation can be obtained by solving the convex optimization as follows:

$$\begin{aligned} &\text{minimize} \quad J_{k+1} = \left\| Y_d^c - Y_{k+1}^{\text{inter}} \right\|_2^2 + \left\| \Delta U_{k+1}^{\text{inter}} \right\|_2^2 \\ &\text{subject to} \quad (18) \text{ holds.} \end{aligned} \tag{20}$$

To present the controller design more clearly, we define

$$\begin{bmatrix} M_1 & M_2 \end{bmatrix} = \begin{bmatrix} K_{1,1}H & K_{2,1}H_\Sigma \\ S_1 H & S_2 H_\Sigma \end{bmatrix}^\dagger$$

where the dimensions of M_1 and M_2 rely on the interconnection structure. Following this, the iterative learning controller is developed via the following theorem.

Theorem 2. For the unknown ILC system (1) under Assumption 1, let

1) The sampled data $(\boldsymbol{U}_{\text{Test}}, \boldsymbol{Y}_{\text{Test}})$ satisfy Assumption 2;
2) $Y_d^c \in \boldsymbol{Y}_{\text{track}}^c$ be a untrackable reference output for (1);
3) $Y_d^c \in \boldsymbol{Y}_{\text{track}}^{\text{inter}}$ be trackable for the interconnected system (18) via the trackability compensation in Theorem 1.

By solving the convex optimization (20), the required iterative learning controller is designed as

$$\begin{aligned} \Delta U_{k+1}^{\text{inter}} = &\left(I + M_1^{\mathrm{T}} \left[K_{1,2}H \;\; K_{2,2}H_\Sigma \right]^{\mathrm{T}} \left[K_{1,2}H \;\; K_{2,2}H_\Sigma \right] M_1 \right)^{-1} \\ &\times M_1^{\mathrm{T}} \left[K_{1,2}H \;\; K_{2,2}H_\Sigma \right]^{\mathrm{T}} E_k^{\text{inter}}. \end{aligned} \tag{21}$$

Proof. From (18), it is concluded that for any $\left(U_k^{\text{inter}}, Y_k^{\text{inter}}\right)$ and $\left(U_{k+1}^{\text{inter}}, Y_{k+1}^{\text{inter}}\right)$, there must exist α_k and α_{k+1} such that

$$\begin{bmatrix} -\Delta U_{k+1}^{\text{inter}} \\ Y_k^{\text{inter}} - Y_{k+1}^{\text{inter}} \\ 0 \end{bmatrix} = \begin{bmatrix} K_{1,1}H & K_{2,1}H_\Sigma \\ K_{1,2}H & K_{2,2}H_\Sigma \\ S_1 H & S_2 H_\Sigma \end{bmatrix} (\alpha_k - \alpha_{k+1}).$$

Consider the following homogeneous algebraic equation as

$$\begin{bmatrix} -\Delta U_{k+1}^{\text{inter}} \\ 0 \end{bmatrix} = \begin{bmatrix} K_{1,1}H & K_{2,1}H_\Sigma \\ S_1 H & S_2 H_\Sigma \end{bmatrix} (\alpha_k^* - \alpha_{k+1}^*).$$

This equation must be solvable since I-O pairs $\left(U_k^{\text{inter}}, Y_k^{\text{inter}}\right)$ and $\left(U_{k+1}^{\text{inter}}, Y_{k+1}^{\text{inter}}\right)$ satisfy (18), and its least square minimal norm solution can be expressed as

$$\alpha_k^* - \alpha_{k+1}^* = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} -\Delta U_{k+1}^{\text{inter}} \\ 0 \end{bmatrix}. \tag{22}$$

By exploiting (22), the difference in output along the iteration axis can be represented as

$$\begin{aligned} Y_k^{\text{inter}} - Y_{k+1}^{\text{inter}} &= \left[K_{1,2}H \quad K_{2,2}H_\Sigma \right] \left[M_1 \quad M_2 \right] \begin{bmatrix} -\Delta U_{k+1}^{\text{inter}} \\ 0 \end{bmatrix} \\ &= - \left[K_{1,2}H \quad K_{2,2}H_\Sigma \right] M_1 \Delta U_{k+1}^{\text{inter}}. \end{aligned}$$

Therefore, the cost function in (20) can be rewritten as

$$\begin{aligned} J_{k+1} &= \left\| Y_d^c - Y_{k+1}^{\text{inter}} \right\|_2^2 + \left\| \Delta U_{k+1}^{\text{inter}} \right\|_2^2 \\ &= \left\| E_k^{\text{inter}} - \left[K_{1,2}H \quad K_{2,2}H_\Sigma \right] M_1 \Delta U_{k+1}^{\text{inter}} \right\|_2^2 + \left\| \Delta U_{k+1}^{\text{inter}} \right\|_2^2. \end{aligned}$$

Hence, the cost function J_{k+1} is strictly convex with respect to $\Delta U_{k+1}^{\text{inter}}$, and its derivative to $\Delta U_{k+1}^{\text{inter}}$ is calculated as

$$\begin{aligned} \frac{\mathrm{d}J_{k+1}}{\mathrm{d}\Delta U_{k+1}^{\text{inter}}} = &2M_1^{\mathrm{T}} \left[K_{1,2}H \;\; K_{2,2}H_\Sigma \right]^{\mathrm{T}} \left[K_{1,2}H \;\; K_{2,2}H_\Sigma \right] M_1 \Delta U_{k+1}^{\text{inter}} \\ &+ 2\Delta U_{k+1}^{\text{inter}} - 2M_1^{\mathrm{T}} \left[K_{1,2}H \;\; K_{2,2}H_\Sigma \right]^{\mathrm{T}} E_k^{\text{inter}}. \end{aligned}$$

By setting this derivative to zero, the required iterative learning controller can be derived, and the proof is completed. ■

Remark 6. Once the cost function takes the form of (20) and the reference output Y_d^c is trackable for the interconnected system, the convergence properties of the optimization-based ILC have been extensively proven (see Proposition 1 in [13]), and we will not elaborate further on them. Therefore, by compensating the trackability of the original ILC system (1) and leveraging the controller in (21), ILC can achieve perfect tracking performances for $Y_d^c \in \boldsymbol{Y}_{\text{track}}^c$.

To summarize the main contributions of this paper, we organize the proposed trackability compensation strategy and the modified data-based ILC into Algorithm 1.

Algorithm 1 Trackability compensation and data-based ILC

Offline I-O Tests:

1: Apply the test inputs $\boldsymbol{U}_{\text{Test}}$ satisfying (5) and (6) to (1);
2: Collect the offline I-O data in $(\boldsymbol{U}_{\text{Test}}, \boldsymbol{Y}_{\text{Test}})$;
3: Construct the data-based I-O representation (7);

Trackability Compensation and Modified ILC:

4: Check the trackability of Y_d^c for the original ILC system (1) through (12)
   If Y_d^c is untrackable, **then** go to step 5;
   Else, quit and apply conventional ILC.
5: Choose interconnection parameters K_1, K_2, S_1, S_2 and auxiliary system parameter H_Σ such that (19) holds;
6: Solve the constrained optimization (20) and obtain (21).

IV. Simulation Examples

In this section, we provide simulation examples to illustrate the effectiveness of the proposed data-based trackability compensation strategy. Consider an ILC system in the form of (1) whose model matrices are presented as follows [14]:

$$A_s(t) = \begin{bmatrix} 1.607 + 0.05t & -0.6086 & -0.9282 \\ 1 & 0.05t & 0 \\ 0 & 1 & 0.05t \end{bmatrix},$$

$$B_s(t) \equiv \begin{bmatrix} 1.2390 \\ 0 \\ 1 \end{bmatrix}, \; C_s(t) \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}}, \; D_s(t) \equiv 0.$$

The initial states are fixed as $x_k(0) = \left[0.500,\ 0,\ 1\right]^{\mathrm{T}}$ for all $k \in \mathbb{Z}_+$. Here we present the model information solely for clearly elaborating the simulation settings, and it will not be leveraged in the trackability compensation and ILC synthesis. The control objectives are to track two reference outputs over the time duration $\mathbb{Z}_{24}$, and the reference outputs are given as

$$\begin{aligned} y_d^{c,1}(t) &= \sin\left(\frac{\pi}{6}t\right) \\ y_d^{c,2}(t) &= e^{-0.1t}\sin\left(\frac{\pi}{4}t\right) \end{aligned}, \ t \in \mathbb{Z}_{24}.$$

In order to collect sufficiently many I-O pairs, the offline I-O tests need to be executed for at least 26 times, where the test input $\boldsymbol{U}_{\mathrm{Test}}$ is designed as

$$U^1 = 0_{25},\ \left[U^2,\ U^3,\ \cdots,\ U^{26}\right] = I_{25}.$$

Corresponding test outputs are collected in the matrix $\boldsymbol{Y}_{\mathrm{Test}} = \left[Y^1,\ Y^2,\ \cdots,\ Y^{26}\right]$. With the designed test inputs, the offline test principles (5) and (6) are satisfied. By leveraging the data-based trackability criterion in (12), it directly follows that both $y_d^{c,1}$ and $y_d^{c,2}$ are untrackable, thus existing ILC can only achieve consevative tracking performances. To further enhance the tracking performances of ILC, the trackability compensation strategy is leveraged. By choosing the interconnection parameters as

$$K_1 = \begin{bmatrix} I_{25} & 0_{25\times 25} \\ 0_{25\times 25} & I_{25} \end{bmatrix},\ K_2 = \begin{bmatrix} 0_{25\times 25} & -I_{25} \\ 0_{25\times 25} & I_{25} \end{bmatrix}$$
$$S_1 = \begin{bmatrix} 0_{25\times 25} & I_{25} \end{bmatrix},\ S_2 = \begin{bmatrix} I_{25} & 0_{25\times 25} \end{bmatrix}$$

the interconnection structure is depicted as in Fig. 4. Through such an interconnection structure, by designing the appropriate controller parameters H_Σ, both $y_d^{c,1}$ and $y_d^{c,2}$ are trackable for the interconnected system. In order to demonstrate the

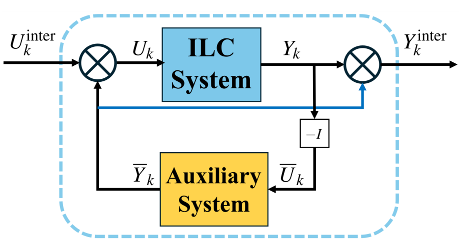

Fig. 4. Interconnection structure corresponding to the chosen parameters.

effectiveness of the modified ILC in enhancing tracking performances, we apply both conventional optimization-based ILC and modified ILC presented in (21) to the original ILC system and the interconnected system, respectively. After 1000 iterations, the tracking performances are depicted in Fig. 5. From Fig. 5, it can be observed that although $y_d^{c,1}(t)$ and $y_d^{c,2}$ are untrackable for the original ILC system (1), we can find the required controller parameters H_Σ by utilizing the interconnection structure presented in Fig. 4. As a result, by exploiting the modified data-based ILC in (21), the perfect tracking performances for $y_d^{c,1}$ and $y_d^{c,2}$ can be achieved.

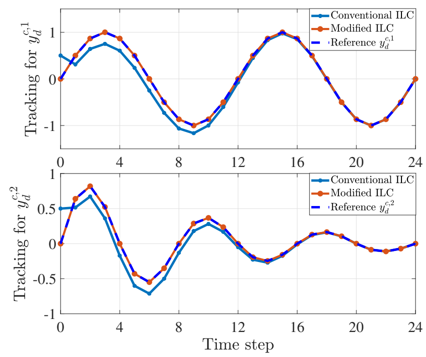

Fig. 5. Tracking performances of conventional and modified ILC.

V. Conclusions

This paper has developed a data-based trackability compensation strategy via interconnection to compensate for the trackability set of ILC systems. In the absence of model knowledge, the I-O representations of ILC systems have been constructed leveraging the sampled data, based on which a data-based trackability criterion has been developed. Afterward, by interconnecting the original ILC system with an auxiliary system, the trackability set of the interconnected system has been modified. As a result, even when confronted with untrackable references, the modified ILC has achieved perfect tracking performances.

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