2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Model Predictive Control of Two-level Open Quantum Systems

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Abstract— **This paper presents a robust control approach for a two-level open quantum system subject to bounded uncertainties through the application of model predictive control. We demonstrate two common cases: depolarizing decoherence and phase-damping decoherence in open quantum systems. We develop a model predictive control method using quantum measurements and provide a lower bound on the probability of obtaining the predicted state. Numerical results illustrate the effectiveness of this model predictive control approach in achieving stability and robustness in open quantum systems.**

I. INTRODUCTION

Quantum control theory is important in steering the state of quantum systems, which is essential in quantum computing and quantum information processing [1]–[4]. The field includes quantum optimal control, quantum learning control, and quantum feedback control [5]–[7]. Among these, robust performance remains important for the practical development of quantum technologies, employing strategies like feedback, learning control, and model predictive control (MPC) to maintain system stability in the presence of uncertainties [8]–[10]. This paper investigates the application of MPC in improving the robustness of quantum control systems.

MPC has become a key method for controlling systems with constraints and uncertainties. It is widely used in many areas [11], [12] and has also been applied in quantum systems [13]–[15]. In this paper, we use time-optimal model predictive control (TOMPC [16]) to steer a quantum system and the strategy involves positive operator-valued measures (POVM) [17] for quantum measurement. The measurements will lead the measured state to collapse into a desired state with a certain probability of success. Hence, developing an effective method to calculate the probability of success is a key task in the proposed MPC method.

In this paper, we focus on calculating the probability that an open quantum system follows the expected trajectory when subjected to decoherence and disturbances. We demonstrate that an open quantum system, subject to an uncertain Hamiltonian and decoherence, can evolve to the correct nominal state with a calculated lower bound on the probability of success. This work extends the robust TOMPC approach illustrated in [18] for closed two-level quantum systems to the context of two-level open quantum systems.

The structure of this paper is as follows. Section II discusses open quantum systems affected by depolarizing and phase-damping decoherence. In Section III, we derive a lower bound on the probability that the system state can be transferred to the correct terminal state under both depolarizing and phase-damping decoherence. In Section IV, numerical simulations demonstrate that employing our results facilitates TOMPC to steer quantum system states to desired terminal states. Finally, we provide concluding remarks in Section V.

This work was supported by the Australian Research Council under grant Project DP240101494 and Project FT220100656

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II. FORMULATION OF THE QUANTUM CONTROL PROBLEM

In open quantum systems, the state is described by a density matrix, ρ, which is a complex matrix satisfying the condition $\text{Tr}(\rho) = 1$, $\rho = \rho^\dagger$ and $\rho \succeq 0$. Specifically, when the system is closed, the quantum state is pure, and the density matrix can be represented as $\rho = |\psi\rangle\langle\psi|$ with a pure quantum state represented by $|\psi\rangle$, a unit vector in the complex Hilbert space. Compared to closed systems, open systems interact with their environment. The evolution of such mixed states ρ, under Markovian dynamics can be described by the Lindblad equation [19]

$$\dot{\rho} = -i[H(t), \rho] + \mathcal{D}(\rho). \tag{1}$$

Here, $[H(t), \rho] = H(t)\rho - \rho H(t)$ is the commutation operator and $H(t)$ denotes the total Hamiltonian $H(t) = H_0 + H_u$, where H_0 signifies the free Hamiltonian of the system, and H_u denotes the control Hamiltonian, which accounts for the external influence due to control signals. Specifically, for the control Hamiltonian $H_u = \sum_\mu u_\mu(t) H_\mu$, the input signal $u_\mu(t)$ is considered as a real-valued function, $u_\mu(t) \in \mathbb{R}$ and H_μ corresponds to the control Hamiltonian associated with the μ-th control signal. Additionally, $\mathcal{D}(\rho)$ corresponds to the dissipative dynamics induced by environmental interactions, defined by:

$$\mathcal{D}(\rho) = \sum_{j=1}^{3} \left(L_j \rho L_j^\dagger - \frac{1}{2} L_j^\dagger L_j \rho - \frac{1}{2} \rho L_j^\dagger L_j \right), \tag{2}$$

where L_j are the Lindblad operators, accounting for environmental influences such as decoherence and $\dagger$ means taking the Hermitian transpose.

Without decoherence, we can simplify (1) to the von Neumann equation, i.e., $\mathcal{D}(\rho) = 0$. Then, to discretize the process, we assume a constant sampling time T_s. Within each sampling interval $t \in [kT_s, (k+1)T_s]$ for $k = 0, 1, 2, \ldots$, the discretization of this continuous time model using the sampling period T_s results in the discrete-time evolution for the nominal system:

$$\rho_{1|k} = U(kT_s) \rho_k U^\dagger(kT_s), \tag{3}$$

which involves the unitary matrix $U(kT_s) = e^{-iT_s H(kT_s)}$.

When the system is affected by uncertainties, we introduce an uncertain Hamiltonian for a two-level system $H_\Delta = \vec{\Delta} \cdot \vec{\sigma} = \sum_{i=1}^{3} \Delta_i \sigma_i$, where $\sigma_{1,2,3}$ are the Pauli matrices, defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

and $\Delta_{1,2,3} \in \mathbb{R}$. We have

$$\rho_{k+1} = U_\Delta(kT_s)\rho_k U_\Delta^\dagger(kT_s), \tag{5}$$

where

$$U_\Delta(kT_s) = e^{-iT_s(H(kT_s)+H_\Delta)}. \tag{6}$$

In the following, we present the Lindblad equation for depolarizing decoherence and phase-damping decoherence [20]. The corresponding Lindblad operators for these types of decoherence are defined as follows.

1) **Depolarizing decoherence:** This case is characterized by Lindblad operators σ_1, σ_2, and σ_3, leading to the Lindblad equation:

$$\dot{\rho} = -i[H(t), \rho] + \gamma(t) \sum_{i=1}^{3} (\sigma_i \rho \sigma_i - \rho), \tag{7}$$

where we assume the intensity of decoherence is the time-varying uncertain parameter $\gamma(t)$.

2) **Phase-damping decoherence:** This case is governed by the Lindblad operator σ_3, yielding the Lindblad equation:

$$\dot{\rho} = -i[H(t), \rho] + \gamma(t)(\sigma_3 \rho \sigma_3 - \rho). \tag{8}$$

In this paper, we utilize TOMPC to steer the state of the quantum system and incorporate a POVM quantum measurement [17]. As shown in Fig. 1 (a), our objective is to steer the state of the quantum system from the initial state ρ_{ini} to the final state ρ_{fin} within the Bloch sphere. Initially, we follow the nominal evolution (3) and apply an MPC strategy to guide the system toward the final state. This generates a nominal trajectory illustrated by the blue line. However, the red line represents the actual evolution, which is influenced by decoherence and the uncertainties in the Hamiltonian.

Subsequently, we use a POVM to measure the state ρ_1 along the nominal trajectory. The probability p of attaining the correct nominal state is represented by the star point. Conversely, there is a probability $1-p$ of reaching the corresponding orthogonal state $\rho_{1,\perp}$ as shown with a triangle point. Then, based on the measurement outcome, either the star point as in Fig. 1 (b) or the triangle point as in Fig. 1 (c), we design a new input control based on the nominal evolution (3) (blue line) and measure the actual process (red line) again. This procedure is repeated until the final state is achieved. The detailed algorithm and the application of TOMPC are discussed in Section IV.

As the strategy illustrated in Fig. 1 indicates, we can steer the system state to the target state if it consistently follows

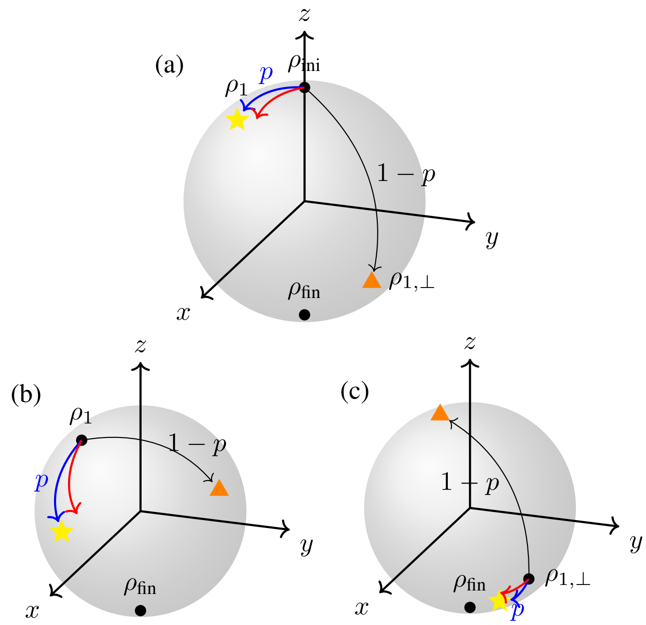

Fig. 1: Diagram representing the proposed control strategy: (a) The control is designed based on the initial state ρ_{ini}. After a POVM measurement, we have either the correct nominal state ρ_1 (the star point) or the corresponding orthogonal state $\rho_{1,\perp}$ (the triangle point). Depending on the measurement result, if we obtain the nominal state ρ_1, we design the new control as shown in (b). On the other hand, if we obtain the orthogonal state $\rho_{1,\perp}$, we follow the approach outlined in (c).

the nominal trajectory. Therefore, our primary concern is to establish a lower bound for p that ensures the system remains close to the nominal trajectory. Once this is achieved, we can assert the robustness of the system for reaching the target state [18]. In the following section, we discuss in detail how to calculate this probability for two-level open quantum systems. To facilitate the calculation of this lower bound on the probability, we employ two approaches: (i) The representation of the Lindblad equation using the Bloch vector framework [21]. (ii) The quantum trajectory method, which provides a stochastic unraveling of quantum dynamics [22].

III. The Minimum Success Probability

In the case of pure quantum systems, a lower bound on the probability of obtaining the correct nominal state has been provided in [18, Theorem 2]. Based on the principle of mathematical induction, we obtain Lemma 1 involving a generalization to measurements on l-step intervals and the incorporation of time-varying uncertainties in two-level systems.

Lemma 1: In a two-level quantum system (5), if the sampling time T_s satisfies the condition $l\bar{\Delta}T_s < \pi/2$, where $\bar{\Delta}$ denotes the bound on the uncertainties such that $|\vec{\Delta}| \leq \bar{\Delta}$, the probability of transferring to the correct nominal state

$\rho_{l|k}$ satisfies the lower bound

$$\mathrm{Tr}(\rho_{l|k}\rho_{k+l}) \geq \cos^2(l\bar{\Delta}T_s). \tag{9}$$

By splitting each step into infinitesimal steps and applying Lemma 1, it is possible to handle time-varying uncertainties $\vec{\Delta}(t)$.

In the following discussion, considering that the open quantum system is affected by decoherence and an uncertain Hamiltonian, we calculate a lower bound on the probability of achieving the correct nominal state.

A. Analysis of Depolarizing Decoherence

Theorem 1: In the two-level quantum system with depolarizing decoherence (7), the probability of transferring to the correct nominal state $\rho_{l|k}$ satisfies the lower bound

$$\mathrm{Tr}(\rho_{k+l}\rho_{l|k}) \geq \frac{1}{2} + \frac{1}{2}e^{-4\bar{\gamma}(lT_s)}, \tag{10}$$

where $\bar{\gamma}$ represents an upper bound on the Lindblad coefficients. i.e., for any time t, $\gamma(t) \leq \bar{\gamma}$. The state ρ_{k+1} evolves from the state ρ_k according to the Lindblad equation (7), over a fixed sampling period T_s.

Proof: To facilitate our analysis, the total Hamiltonian is defined as $H(t) = v_x(t)\sigma_1 + v_y(t)\sigma_2 + v_z(t)\sigma_3$. Subsequently, we can transform the Lindblad equation (7) into the Bloch vector representation [21]. Consequently, the time evolution of the corresponding Bloch vector of the quantum state ρ_k, denoted as $[x_k, y_k, z_k]^T$, is [20]

$$\begin{bmatrix} \dot{x}_k \\ \dot{y}_k \\ \dot{z}_k \end{bmatrix} = \begin{bmatrix} -4\gamma(t) & -v_z & v_y \\ v_z & -4\gamma(t) & -v_x \\ -v_y & v_x & -4\gamma(t) \end{bmatrix} \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix}. \tag{11}$$

In this case, the decoherence term is represented by $-4\gamma(t)\mathbb{I}$ and we define

$$\mathcal{L}_H(t) = \begin{bmatrix} 0 & -v_z(t) & v_y(t) \\ v_z(t) & 0 & -v_x(t) \\ -v_y(t) & v_x(t) & 0 \end{bmatrix}. \tag{12}$$

Since $-4\gamma(t)\mathbb{I}$ and $\mathcal{L}_H(t)$ commute, the state $[x_{k+l}, y_{k+l}, z_{k+l}]^T$ following the evolution (11), is given by

$$\begin{bmatrix} x_{k+l} \\ y_{k+l} \\ z_{k+l} \end{bmatrix} = \exp\left(-4\int_{kT_s}^{(k+l)T_s} \gamma(t)\,dt\right) \begin{bmatrix} x_{l|k} \\ y_{l|k} \\ z_{l|k} \end{bmatrix}, \tag{13}$$

where the nominal state $[x_{l|k}, y_{l|k}, z_{l|k}]^T$ is obtained from

$$\begin{bmatrix} x_{l|k} \\ y_{l|k} \\ z_{l|k} \end{bmatrix} = \exp\left(\int_{kT_s}^{(k+l)T_s} \mathcal{L}_H(t)\,dt\right) \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix}. \tag{14}$$

Therefore, the probability $\mathrm{Tr}(\rho_{k+l}\rho_{l|k})$ can be bounded as follows:

$$\begin{aligned} \mathrm{Tr}(\rho_{k+l}\rho_{l|k}) &= \frac{1}{2}(1 + x_{l|k}x_{k+1} + y_{l|k}y_{k+l} + z_{l|k}z_{k+l}) \\ &= \frac{1}{2} + \frac{1}{2}\exp\left(-4\int_{kT_s}^{(k+l)T_s} \gamma(t)dt\right) \\ &\geq \frac{1}{2} + \frac{1}{2}e^{-4\bar{\gamma}lT_s}, \end{aligned} \tag{15}$$

since for any $\gamma(t) \leq \bar{\gamma}$,

$$\exp\left(-4\int_{kT_s}^{(k+l)T_s} \gamma(t)dt\right) \geq e^{-4\bar{\gamma}(lT_s)}. \tag{16}$$

■

Corollary 1: If a two-level quantum system is subject to the depolarizing decoherence (7) and includes an uncertain Hamiltonian H_Δ, then the probability of transitioning to the correct nominal state $\rho_{1|k}$ satisfies the lower bound

$$\mathrm{Tr}(\rho_{k+l}\rho_{l|k}) \geq \frac{1}{2}\cos^2(l\bar{\Delta}T_s)(1 + e^{-4\bar{\gamma}(lT_s)}) \tag{17}$$

under the condition $\gamma(t) \leq \bar{\gamma}$ and $l\bar{\Delta}T_s < \pi/2$.

Proof: We assume that the system with uncertain Hamiltonian has evolved to the state ρ'_{k+l}. Its corresponding Bloch vector is $[x'_{k+l}, y'_{k+l}, z'_{k+l}]^T$, which can be obtained from

$$\begin{bmatrix} x_{l|k} \\ y_{l|k} \\ z_{l|k} \end{bmatrix} = e^{\int_{kT_s}^{(k+l)T_s} L'_H(t)dt} \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix}, \tag{18}$$

where

$$L'_H(t) = \begin{bmatrix} 0 & -v_z(t) - \Delta_z(t) & v_y(t) + \Delta_y(t) \\ v_z(t) + \Delta_z(t) & 0 & -v_x(t) - \Delta_x(t) \\ -v_y - \Delta_y(t)(t) & v_x(t) + \Delta_x(t) & 0 \end{bmatrix}. \tag{19}$$

Then, we introduce a pseudo measurement, a conceptual approach used to simplify the calculation, allowing us to decompose the probability as follows:

$$\mathrm{Tr}(\rho_{k+l}\rho_{l|k}) = \mathrm{Tr}(\rho_{k+l}\rho'_{k+l})\mathrm{Tr}(\rho'_{k+l}\rho_{l|k}). \tag{20}$$

Here, the pseudo measurement helps isolate the contributions of different components of the state. Specifically, terms involving the orthogonal state $\rho'_{\perp,k+l}$, defined as $\mathbb{I} - \rho'_{k+l}$, can be ignored. This is because the decoherence term, represented by the identity matrix, ensures that the state $\rho'_{\perp,k+l}$ is orthogonal to ρ_{k+l}. Then, using Theorem 1, we have

$$\mathrm{Tr}(\rho_{k+l}\rho'_{k+l}) \geq \frac{1}{2} + \frac{1}{2}e^{-4\bar{\gamma}lT_s}. \tag{21}$$

Using Lemma 1, this yields

$$\mathrm{Tr}(\rho'_{k+l}\rho_{l|k}) \geq \cos^2(l\bar{\Delta}T_s). \tag{22}$$

Therefore, the probability of obtaining the correct quantum state satisfies the lower bound

$$\mathrm{Tr}(\rho_{k+l}\rho_{l|k}) \geq \cos^2(l\bar{\Delta}T_s)(\frac{1}{2} + \frac{1}{2}e^{-4\bar{\gamma}lT_s}). \tag{23}$$

■

In the case of phase-damping decoherence, the decoherence term in the Bloch vector representation does not commute with the other terms, making it challenging to compute expectations within this framework. Therefore, an alternative approach using stochastic unraveling is considered to calculate a lower bound on the probability of obtaining the correct state [22].

B. Analysis of Phase-Damping Decoherence

Theorem 2: In the case of a two-level quantum system with phase-damping decoherence (8), the probability of transferring to the correct nominal state $\rho_{l|k}$ satisfies the lower bound

$$\text{Tr}(\rho_{k+l}\rho_{l|k}) \geq e^{-\bar{\gamma}(lT_s)} \tag{24}$$

under the condition $\gamma(t) \leq \bar{\gamma}$.

This proof follows [22]. When we consider the quantum state evolving according to the deterministic evolution [22],

$$\frac{d}{dt}|\tilde{\psi}_d(t)\rangle = -iH_{\text{eff}}|\tilde{\psi}_d(t)\rangle, \tag{25}$$

where $|\tilde{\psi}_d(t)\rangle$ denotes the unnormalized state vector and the effective Hamiltonian in phase-damping decoherence is

$$H_{\text{eff}} = H - \frac{\gamma(t)}{2}\mathbb{I}. \tag{26}$$

Since the contribution from the dissipative term is the identity, we can calculate the minimum probability based on deterministic evolution to obtain the lower bound in (24).

Corollary 2: If the two-level quantum system with phase-damping decoherence (8) also includes an uncertain Hamiltonian H_Δ, the probability of transferring to the correct nominal state $\rho_{l|k}$ satisfies the lower bound

$$\text{Tr}(\rho_{k+l}\rho_{l|k}) \geq \cos^2(l\bar{\Delta}T_s)e^{-\bar{\gamma}(lT_s)} \tag{27}$$

under the condition $\gamma(t) \leq \bar{\gamma}$ and $l\bar{\Delta}T_s < \pi/2$.

Following the approach used in Corollary 1, we introduce a pseudo-measurement, then apply Theorem 2 and Lemma 1 to obtain (27).

In conclusion, we have provided lower bounds on the probability $\text{Tr}(\rho_{l|k}\rho_{k+l})$ of transferring to the correct nominal state $\rho_{l|k}$ with an uncertain Hamiltonian and decoherence. The results are summarized in Table I.

TABLE I: The lower bounds on the probability of transferring to the correct nominal state.

Cases	Lower bound of $\text{Tr}(\rho_{l|k}\rho_{k+l})$
Closed system	$\cos^2(l\bar{\Delta}T_s)$
Depolarizing	$\frac{1}{2}\cos^2(l\bar{\Delta}T_s)(1+e^{-4\bar{\gamma}(lT_s)})$
Phase-damping	$\cos^2(l\bar{\Delta}T_s)e^{-\bar{\gamma}(lT_s)}$

IV. Numerical Results

In our simulation, we follow [18, Algorithm 1] to solve a quantum TOMPC, as outlined in Algorithm 1. Building on the study in [18], we further demonstrate the lower bound of the probability of remaining in the correct nominal state for an open quantum system. In an open quantum system, Algorithm 1 optimizes the optimal control problem defined in (28).

Algorithm 1 qTOMPC: TOMPC for quantum systems

Require: Initial state ρ_0 and target state ρ_f, prediction horizon $\bar{L}$, performance index $J_{\bar{L}}(\rho_k)$

1: **repeat** (for each step $k = 0, 1, \ldots, N-1$)
2: Calculate optimal control $\{u^\star_{0|k}, u^\star_{1|k}, \ldots, u^\star_{\bar{L}-1|k}\}$ by minimizing $J_{\bar{L}}(\rho_0)$ using the nominal model
3: Use $u^\star_{0|k}$ to get $\rho_{1|k}$ from the nominal model
4: Make a measurement with $M_{1|k} = \{\rho_{1|k}, \mathbb{I} - \rho_{1|k}\}$, and obtain the post-measurement state $\rho'_{1|k}$
5: **update** $\rho_k = \rho'_{1|k}$

$$\begin{aligned} \min_{u_0,\ldots,u_{L-1}} \quad & J_{\bar{L}}(\rho_k) = \sum_{l=0}^{\bar{L}-1} \theta^l \sqrt{1 - \text{Tr}\left(\bar{\rho}_{l|k}\rho_f\right)} \\ \text{s.t.} \quad & \bar{\rho}_{0|k} = \rho_k, \\ & \bar{\rho}_{l+1|k} = U_{l|k}\bar{\rho}_{l|k}U^\dagger_{l|k}, \\ & |u_{l|k,\mu}| \leq B, \\ & \bar{\rho}_{L|k} = \rho_f, \end{aligned} \tag{28}$$

where $U_{l|k} = e^{-iT_sH((k+l)T_s)}$ and the cost function $J_{\bar{L}}(\rho_k)$ includes the weight $\theta \in \mathbb{R}$ and measures the deviation of $\bar{\rho}$ from the desired target state ρ_f. Our study defines the trace distance between quantum states as described in [23]. The trace distance is equal to half the Euclidean norm of the difference between the corresponding Bloch vectors. Here, $u_{l|k}$ represents a multi-input control with each component $u_{l|k,\mu}$. The inequality constraints ensure that each input signal $u_{l|k,\mu}$ satisfies a specified magnitude bound $B \in \mathbb{R}$.

Upon finding the optimal value of the cost function $J_{\bar{L}}(\rho_k)$, the optimal control sequence over the prediction horizon $\bar{L}$ is given as $\{u^\star_{0|k}, u^\star_{1|k}, \ldots, u^\star_{\bar{L}-1|k}\}$. Following (3), a unitary matrix is calculated as $U^\star_{l|k} = e^{-iT_sH^\star((k+l))}$, where $H^\star((k+l) = H_0 + \sum_\mu u^\star_{l|k,\mu}H_\mu$. In the nominal system, the state $\rho_{l|k}$ at step $k+l$ is then expressed as

$$\rho_{l|k} = \prod_{j=0}^{l-1} U^\star_{l-1-j|k}\rho_k \prod_{j=0}^{l-1} U^{\star\dagger}_{j|k}. \tag{29}$$

When we have the corrected nominal state $\rho_{l|k}$, it is feasible to establish the POVM. By applying the theorems from Section III, we can ensure that the state evolves to this corrected nominal state with high probability. In the subsequent simulation, we demonstrate that quantum TOMPC effectively manages uncertainties to keep the state close to the target, even under depolarizing and phase-damping decoherence.

A. Simulation on open quantum systems

In our simulation of open quantum systems, we consider the dynamics outlined in (7) and (8). We select the free Hamiltonian $H_0 = \frac{1}{2}\sigma_3$ and incorporate two control signals, one aligned with σ_1 and the other with σ_2. In these systems, the coefficient $\gamma(t)$ is uniformly randomized within the range $[0.025, 0.075]$.

The simulation parameters are set as follows: the sampling time T_s is 0.4, and the total time duration is 2.8. We impose control constraints such that $|u_1(t)| \leq 1$ and $|u_2(t)| \leq 1$. Additionally, the prediction horizon $\bar{L}$ is 15, and the weight parameter θ is fixed at 1.9.

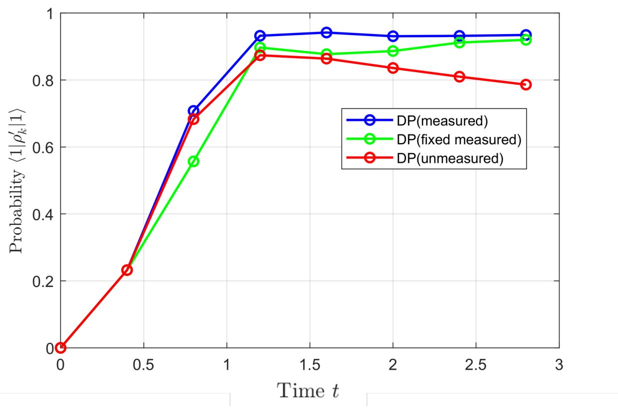

Fig. 2: Simulation results using Algorithm 1 for depolarizing decoherence, comparing scenarios with and without measurement. The label 'DP(measured)' refers to the state measured along the nominal trajectory. 'DP(fixed measured)' indicates measurement with a fixed POVM, as defined in (30). 'DP(unmeasured)' denotes the scenario where the state is not measured, showcasing the average of states in the simulation.

In addition to measuring the quantum system with the nominal state, we also implement a fixed measurement strategy to demonstrate a more practical approach. In this strategy, the quantum state is measured using a fixed set of POVMs, resulting in control inputs derived from the MPC that depend on the initial state. This strategy is more practical because it involves a fixed input signal and a trajectory that adheres to the selected POVMs. The predetermined set of POVMs is defined as $M_i = \{|\phi_i\rangle\langle\phi_i|, \mathbb{I} - |\phi_i\rangle\langle\phi_i|\}$, where four specific POVM elements are chosen. These fixed POVM elements are defined as follows:

$$\begin{aligned}
|\phi_1\rangle &= |0\rangle, \\
|\phi_2\rangle &= \cos(\frac{\pi}{8})|0\rangle + e^{-i\frac{\pi}{2}}\sin(\frac{\pi}{8})|1\rangle, \\
|\phi_3\rangle &= \cos(\frac{\pi}{6})|0\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\pi}{6})|1\rangle, \\
|\phi_4\rangle &= \cos(\frac{\pi}{4})|0\rangle + e^{i\frac{\pi}{4}}\sin(\frac{\pi}{4})|1\rangle.
\end{aligned} \tag{30}$$

In our simulations with fixed measurements, we consistently select the POVM that is nearest to the quantum state at each measurement step, determined by the nominal trajectory.

Tables II and III present the infidelity and error tracking metrics, denoted as E_{track}. Among them, we illustrate the scenario of depolarizing decoherence in Fig. 2.

In the presence of uncertainties, we assess the performance of the algorithm by calculating the accumulated tracking errors over the entire period N. This process is repeated 3000 times to obtain an average performance metric.

Here, the accumulated tracking error, denoted by E_{track}, quantifies the deviation from the nominal path and is defined as follows:

$$E_{\text{track}} = \sum_{t=0}^{N-1} 1 - \text{Tr}\left(\bar{\rho}_{1|k}\rho'_{k+1}\right), \tag{31}$$

where ρ'_{k+1} represents the post-measurement state.

In scenarios without measurement, $\bar{\rho}_{1|k}$ represents the evolution of the nominal system. On the other hand, the state ρ'_{k+1} is obtained by evolving the system with uncertainties.

TABLE II: Infidelity between final and target states for depolarizing and phase-damping decoherence.

Infidelity	Unmeasured	Fixed Measured	Measured
Depolarizing	0.2139	8.020×10^{-2}	6.572×10^{-2}
Phase-damping	0.03173	3.512×10^{-3}	1.172×10^{-4}

TABLE III: Average accumulated tracking error, E_{track}, for depolarizing and phase-damping decoherence.

E_{track}	Unmeasured	Fixed Measured	Measured
Depolarizing	7.078	0.4650	0.2567
Phase-damping	7.814	0.1930	3.767×10^{-2}

V. CONCLUSIONS AND FUTURE WORKS

In our study, we apply an MPC strategy to two-level open quantum systems. To facilitate this strategy with a quantum measurement, POVM, we demonstrated the minimum success probability when quantum states are affected by decoherence and an uncertain Hamiltonian. Notably, the fixed measurement strategy in our method represents a practical approach. It allows for controlling the quantum state using predetermined input signals and following a predefined set of POVMs, thereby simplifying the quantum control process step by step. We will explore scenarios involving more general N-level open quantum systems in future work,.

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