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On Nash Equilibria for Decentralized Symbolic Control of Interconnected Finite State Systems

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Abstract— **In this paper we consider a pair of interconnected nondeterministic and metric finite state systems and address a decentralized symbolic control problem where controllers are designed for enforcing local specifications expressed in terms of regular languages, up to a desired accuracy. The considered control architecture is decentralized, i.e. each controller can only communicate with the corresponding plant. Since plant systems are interconnected, the part of the specification that can be enforced on one system depends on the one on the other system. We show how this dependency can be nicely formalized in terms of equilibria and in particular, of Nash equilibria. When controlled plants are at a Nash equilibrium, deviation of each plant from its control strategy may correspond to a loss in terms of the part of specification enforced. Algorithms are proposed which converge, when an equilibrium exists, to Nash equilibria.**

keywords: Interconnected finite state systems, supervisory control, regular languages, games, Nash equilibria.

I. Introduction

Decentralized control offers a compositional approach to controllers' synthesis that is effective in taming complexity of large–scale real–world systems. Several decentralized control techniques have been proposed in the literature which include decentralized stabilization and regulation [1], [2], decentralized robust stabilization optimization and reliability design [3], [4], [5], decentralized adaptive control [6], consensus and formation control problems in multi–agent systems [7], [8], [9], applications to mobile robotics [10], [11], decentralized supervisory control of discrete–event systems (DES) [12], [13]. More recently, [14], [15] proposed decentralized control algorithms based on discrete abstractions, for enforcing global regular language specifications on networks of discrete-time nonlinear systems. Work [16] addresses the case of discrete abstractions for nonlinear systems with disturbances. In this paper we consider a system that is given by the interconnection of a pair of nondeterministic and metric finite state systems and address a decentralized symbolic control problem where two controllers, each one sharing information only with corresponding plant system, are designed to enforce up to a desired accuracy, a desired local regular language specification. As discussed in e.g. [17], [18], regular languages are useful for modeling specifications of interest in cyber–physical systems. Since plants are interconnected, the part of the specification that can be enforced on one system depends on the one on the other system and hence, control strategies designed are conflicting, in general. In this paper we resolve this issue with a game theoretic approach. We propose a notion of equilibrium between specifications so that, at equilibrium, a control strategy of each player exists for enforcing a part of the local specification, given the control strategy of the other player, and vice versa. We then propose a stronger notion of equilibrium, called Nash equilibrium, where a strategy for each player maximizes the part of the corresponding local specification enforced, given the maximizing strategy of the other player, and vice versa. As a consequence, at a Nash equilibrium, unilateral deviation of each plant from its control strategy may correspond to a loss in terms of the part of specification enforced. We propose algorithms that converge, when an equilibrium exists, to Nash equilibria. An approach based on game theory, as in this paper, was also proposed in [19], [20] for designing controllers for enforcing doubly robustly invariant equilibria on discrete–time linear systems. Since each plant of the interconnected system we consider is nondeterministic and metric, and symbolic control problems require specifications to be met up to a desired accuracy, the results we propose are also applicable to the interconnection of purely continuous or hybrid systems that can be approximated by nondeterministic metric finite state systems. Literature in this regard is rather rich, see e.g. [21], [17], [22] and the references therein. For example, [14] proposes discrete abstractions for approximating networks of discrete-time nonlinear systems with arbitrary topological interconnection. A comparison with our previous work [15] follows. While here we consider nondeterministic systems and local specifications, [15] considers deterministic systems and global specifications. Enforcing global specifications is in general harder than enforcing local specifications. In fact, in [15], local controllers are needed to agree in advance on which word of the global specification to enforce. In our problem setting instead, there is no need for the controllers to adhere to some specific word, each controller takes care of enforcing a local specification of its plant. As a consequence, the overall specification enforced on the interconnected system may be composed of collection of words rather than a single word as in the case of [15].

This paper is organized as follows. Section II introduces notation and preliminary definitions, Section III interconnected systems and Section IV the notions of equilibria. Section V proposes algorithms converging to Nash equilibria. Section VI offers some concluding remarks. Most of the proofs of

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the results presented is omitted for lack of space.

II. Notation and preliminary definitions

Symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the set of non–negative integer, real, positive real, and non–negative real numbers, respectively. Given $a, b \in \mathbb{N}$ we set $[a; b] = \{x \in \mathbb{N} \,|\, a \leq x \leq b\}$. Symbol $\wedge$ denotes the logical conjunction and $\varnothing$ the empty set. Given a finite set X, 2^X denotes the power set of X, that is the collection of all subsets of X. A set X is a partially ordered set (poset) if there exists a relation on it satisfying the following properties for all $x_1, x_2, x_3 \in X$: (reflexivity) $x_1 \preceq x_1$; (anti-symmetry) if $x_1 \preceq x_2$ and $x_2 \preceq x_1$ then $x_1 = x_2$; and (transitivity) if $x_1 \preceq x_2$ and $x_2 \preceq x_3$ then $x_1 \preceq x_3$. We recall from e.g. [13] some notions on formal language theory. Let Y be a finite set representing the alphabet. A word over Y of length l is a finite sequence $y_1\, y_2 \ldots y_l$ of symbols in Y. The concatenation of two words $y_1\, y_2 \ldots y_l$ and $y_{l+1}\, y_{l+2} \ldots y_{l'}$ is the word $y_1\, y_2 \ldots y_l\, y_{l+1}\, y_{l+2}, \ldots, y_{l'}$. The empty word is denoted by ε. The symbol Y^* denotes the Kleene closure of Y, that is the collection of all words over Y including ε. Similarly, given a word y over Y, the symbol $\{y\}^*$ denotes the Kleene closure of word y, that is the collection of all words, including the empty word, obtained by concatenating y with itself, an arbitrary but finite number of times. A language L over Y is a subset of Y^*. We now recall the notion of transition system:

Definition 1: A transition system is a tuple $T = (X, X_0, U, \longrightarrow, X_m, Y, H)$, consisting of a set of states X, a set of initial states $X_0 \subseteq X$, a set of inputs U, a transition relation $\longrightarrow \subseteq X \times U \times X$, a set of marked states $X_m \subseteq X$, a set of outputs Y and an output function $H : X \to Y$.

A transition $(x, u, x') \in \longrightarrow$ of T is denoted by $x \xrightarrow{u} x'$. The evolution of transition systems is captured by the notions of state, input and output runs. Given a sequence of transitions of T

$$x(0) \xrightarrow{u(0)} x(1) \xrightarrow{u(1)} \ldots x(l-1) \xrightarrow{u(l-1)} x(l) \qquad (1)$$

with $x(0) \in X_0$, the sequences

$$\begin{aligned} r_X &: x(0)\, x(1) \ldots x(l), \\ r_U &: u(0)\, u(1) \ldots u(l-1), && (2) \\ r_Y &: H(x(0))\, H(x(1)) \ldots H(x(l)), && (3) \end{aligned}$$

are called a *state run*, an *input run* and an *output run* of T, respectively. Length of r_X and r_Y is $l+1$ while length of r_U is l. Transition system T is said to be *empty* if the set of initial states is empty, and *symbolic/finite* if X and U are finite sets. The *input language* (resp. *output language*) of T, denoted $\mathcal{L}^u(T)$ (resp. $\mathcal{L}^y(T)$), is the collection of all its input runs (resp. output runs). The *marked input language* (resp. *marked output language*) of T, denoted as $\mathcal{L}_m^u(T)$ (resp. $\mathcal{L}_m^y(T)$), is the collection of all input runs r_U in (2) (resp. output runs r_Y in (3)) such that the corresponding transitions sequence in (1) is with ending state $x_l \in X_m$. A language L over a finite set U is said *regular* if there exists a symbolic transition system T with input set U such that $L = \mathcal{L}_m^u(T)$. We also recall some unary operations on transition systems naturally adapted from the ones given for discrete event systems [13]. A transition system $T' = (X', X_0', U', \longrightarrow', X_m', Y', H')$ is said to be a *sub-transition system* of $T = (X, X_0, U, \longrightarrow, X_m, Y, H)$, denoted $T' \sqsubseteq T$, if $X' \subseteq X$, $X_0' \subseteq X_0$, $U' \subseteq U$, $\longrightarrow' \subseteq \longrightarrow$, $X_m' \subseteq X_m$, $Y' \subseteq Y$ and $H'(x) = H(x)$ for all $x \in X'$. The accessible part of T, denoted $\mathrm{Ac}(T)$, is the unique maximal[1] sub-transition system T' of T such that for any state x' of T' there exists a state run of T' ending in x'. By definition, if T is nonempty, $\mathrm{Ac}(T)$ is accessible; T is accessible if $\mathrm{Ac}(T) = T$. The co-accessible part of T, denoted $\mathrm{Coac}(T)$, is the unique maximal[1] sub-transition system T' of T such that for any state $x' \in X'$ there exists a transition sequence of T' starting from x' and ending in a marked state of T'. T is co-accessible if $\mathrm{Coac}(T) = T$. The trim of T, denoted $\mathrm{Trim}(T)$, is defined as $\mathrm{Trim}(T) = \mathrm{Coac}(\mathrm{Ac}(T)) = \mathrm{Ac}(\mathrm{Coac}(T))$. By definition, $\mathrm{Trim}(T)$, if not empty, is accessible and co-accessible.

III. Interconnected finite state systems

In this paper we consider two interconnected finite state systems described by the difference inclusions:

$$P_i : \begin{cases} x_i(t+1) \in F_i(x_i(t), x_{3-i}(t), u_i(t)), \\ x_i(0) \in X_{i,0}, x_{3-i}(0) \in X_{3-i,0}, \\ x_i(t) \in X_i, x_{3-i}(t) \in X_{3-i}, u_i(t) \in U_i, t \in \mathbb{N}, \end{cases} \qquad (4)$$

with $i = 1, 2$, where $x_i(t)$ is state, and $x_{3-i}(t)$ and $u_i(t)$ are the inputs at time $t \in \mathbb{N}$, respectively; X_i, $X_{i,0} \subseteq X_i$ and U_i are the finite sets of states, initial states and inputs, respectively. Input $x_{3-i}(t)$ of P_i is state of P_{3-i}. We suppose that X_i is metric with metric $\mathbf{d}_i : X_i \times X_i \to \mathbb{R}_0^+$. Map $F_i : X_i \times X_{3-i} \times U_i \to 2^{X_i}$ is the state transition (possibly partial) map. A state trajectory of P_i is a finite sequence of states $x_i(0)\, x_i(1) \ldots$ satisfying (4) for some finite sequence of inputs $x_{3-i}(0)\, x_{3-i}(1) \ldots$ and $u_i(0)\, u_i(1) \ldots$. It is easy to see that P_1 and P_2 are nondeterministic, in general. In this paper we consider a decentralized control architecture so that a controller C_i is associated with plant P_i, and C_i has only information about the state of P_i. Controllers C_1 and C_2 are dynamic and described by:

$$C_i : \begin{cases} z_i(t+1) \in G_i(z_i(t), x_i(t)), \\ u_i(t) \in h_i(z_i(t)), \\ z_i(0) \in Z_{i,0}, \\ z_i(t) \in Z_i, u_i(t) \in U_i, t \in \mathbb{N}, \end{cases} \qquad (5)$$

with $i = 1, 2$, where $z_i(t)$ and $u_i(t)$ are the state and the output at time $t \in \mathbb{N}$, respectively; Z_i, $Z_{i,0} \subseteq Z_i$ and U_i are the finite sets of states, initial states and outputs, respectively. Map $G_i : Z_i \times X_i \to 2^{Z_i}$ is the state transition (possibly partial) map and $h_i : Z_i \to 2^{U_i}$ is the output map and is total. Note that C_i is nondeterministic because of G_i and h_i that are maps and not functions. This modeling choice

[1]Here, maximality is with respect to (w.r.t.) the pre–order naturally induced by the binary operator $\sqsubseteq$.

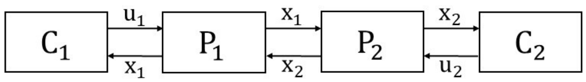

Fig. 1. Interconnected finite state system Σ.

is rather common in symbolic control problems [17], [21] and corresponds to the so–called maximal controller used when computing controlled–invariant sets [23]. For $i = 1, 2$, we denote by $P_i^{C_i}$ the difference inclusion describing the interconnection of P_i and C_i, resulting in:

$$P_i^{C_i} : \begin{cases} (x_i(t+1), z_i(t+1)) \in \phi_i(x_i(t), z_i(t), x_{3-i}(t)), \\ (x_i(0), z_i(0)) \in X_{i,0} \times Z_{i,0}, \\ (x_i(t), z_i(t)) \in X_i \times Z_i, \\ x_{3-i}(0) \in X_{3-i,0}, \\ x_{3-i}(t) \in X_{3-i}, t \in \mathbb{N}, \end{cases} \tag{6}$$

where:

$$\begin{array}{l} \phi_i(x_i, z_i, x_{3-i}) = \\ \bigcup_{u_i \in h_i(z_i(t))} (F_i(x_i, x_{3-i}, u_i), G_i(z_i, x_i)), \end{array} \tag{7}$$

for all $(x_i, z_i, x_{3-i}) \in X_i \times Z_i \times X_{3-i}$. A state trajectory of $P_i^{C_i}$ is a finite sequence of states $(x_i(0), z_i(0))\,(x_i(1), z_i(1))\ldots$ satisfying (6) for some finite sequence of states $x_{3-i}(0)\,x_{3-i}(1)\ldots$ of P_{3-i}. The interconnection among systems P_1, P_2, C_1 and C_2 is denoted by Σ, depicted in Fig. 1 and obtained by coupling (6) for $i = 1, 2$, thus obtaining

$$\Sigma : \begin{cases} (x_1(t+1), z_1(t+1)) \in \phi_1(x_1(t), z_1(t), x_2(t)), \\ (x_2(t+1), z_2(t+1)) \in \phi_2(x_2(t), z_2(t), x_1(t)), \\ (x_1(0), x_2(0), z_1(0), z_2(0)) \in \\ \qquad X_{1,0} \times X_{2,0} \times Z_{1,0} \times Z_{2,0}, \\ (x_1(t), x_2(t), z_1(t), z_2(t)) \in \\ \qquad X_1 \times X_2 \times Z_1 \times Z_2, t \in \mathbb{N}. \end{cases} \tag{8}$$

A state trajectory of Σ of length $l + 1$ is a finite sequence of states $(x_1(0), x_2(0), z_1(0), z_2(0))$ $(x_1(1), x_2(1), z_1(1), z_2(1)) \ldots (x_1(l), x_2(l), z_1(l), z_2(l))$ satisfying (8).

IV. Games and equilibria

In this section we introduce the notions of games and equilibria. We consider a game with two players P_i. Each player P_i is given with a specification Q_i defined as a regular language over alphabet X_i, i.e. $Q_i \subseteq (X_i)^*$. A strategy for player P_i is given by a pair $S_i = (C_i, \mathcal{R}_i)$ where C_i is the controller associated with P_i, as in (5), and $\mathcal{R}_i \subseteq X_{i,0} \times Z_{i,0}$ is the relation of initial states between P_i and C_i. The symbol $\mathcal{S}_i$ denotes the collection of strategies for player P_i. The goal of each player i is to enforce specification Q_i on plant P_i by selecting suitable strategy $S_i \in \mathcal{S}_i$. Reward function of each player is formalized by the notion of part of specification enforced, as follows.

Definition 2: For $i = 1, 2$, the part of the specification Q_i enforced by strategy $S_i = (C_i, \mathcal{R}_i) \in \mathcal{S}_i$, given strategy $S_{3-i} = (C_{3-i}, \mathcal{R}_{3-i}) \in \mathcal{S}_{3-i}$, denoted by $\mathcal{Q}_i(S_i, S_{3-i})$, is the collection of words $q_0\, q_1 \ldots q_l \in Q_i$ for which there exists a state trajectory $(x_1(.), x_2(.), z_1(.), z_2(.))$ of Σ with length $l + 1$, and initial state $(x_1(0), x_2(0), z_1(0), z_2(0))$ such that $(x_i(0), z_i(0)) \in \mathcal{R}_i$ and $(x_{3-i}(0), z_{3-i}(0)) \in \mathcal{R}_{3-i}$, such that the following inequality holds:

$$\mathbf{d}_i(x_i(t), q_t) \leq \theta_i, \forall t \in [0; l]. \tag{9}$$

Inputs		X_1			
X_2	U_1	1	3	5	7
2	a	3	-	-	-
2	b	-	5	5,7	-
2	c	-	-	7	-
4	a	3	-	-	-
4	b	-	5	5,7	-
4	c	-	-	7	-
6	a	3	-	-	-
6	b	-	1	5	-
6	c	-	-	7	-
8	a	3	-	-	-
8	b	-	1	5	-
8	c	-	-	7	-

TABLE I

Transition map F_1.

Inputs		X_2			
X_1	U_2	2	4	6	8
1	d	4	-	-	-
1	e	-	6	6,8	-
1	f	-	-	8	-
3	d	4	-	-	-
3	e	-	6	6,8	-
3	f	-	-	8	-
5	d	4	-	-	-
5	e	-	2	6	-
5	f	-	-	8	-
7	d	4	-	-	-
7	e	-	2	6	-
7	f	-	-	8	-

TABLE II

Transition map F_2.

Note that there may exist pairs $(S_i, S_{3-i}) \in \mathcal{S}_i \times \mathcal{S}_{3-i}$ for which $\mathcal{Q}_i(S_i, S_{3-i}) = \varnothing$. The collection of all $\mathcal{Q}_i(S_i, S_{3-i})$ with S_i ranging in $\mathcal{S}_i$ is a poset when equipped with $\subseteq$ and admits the maximum element Q_i. Equilibria in our game are then formalized in the following

Definition 3: The pair $(S_1, S_2) \in \mathcal{S}_1 \times \mathcal{S}_2$ is an equilibrium if $\mathcal{Q}_1(S_1, S_2) \neq \varnothing$ and $\mathcal{Q}_2(S_2, S_1) \neq \varnothing$. We denote by $\mathcal{E}$ the collection of all equilibria in Σ.

The set $\mathcal{E}$ may be empty, in general. If $\mathcal{E} \neq \varnothing$, the pair $(\mathcal{E}, \subseteq)$ is a poset. We can now introduce a special class of equilibria that we call Nash equilibria.

Definition 4: An equilibrium $(S_1, S_2) \in \mathcal{E}$ is Nash if

(i) $\mathcal{Q}_1(S_1', S_2) \subseteq \mathcal{Q}_1(S_1, S_2)$, $\forall S_1' \in \mathcal{S}_1$;
(ii) $\mathcal{Q}_2(S_2', S_1) \subseteq \mathcal{Q}_2(S_2, S_1)$, $\forall S_2' \in \mathcal{S}_2$.

The intuition behind this definition is that whenever player i deviates from a Nash equilibrium by selecting a different strategy, the corresponding part of the specification Q_i enforced on P_i may reduce. It corresponds *mutatis mutandis* to the classical notion of Nash equilibrium but applied to symbolic decentralized control of interconnected finite state systems as the following remark points out.

Remark 1: In standard static two–players games, one is given collection of strategies in sets V_1 and V_2 and a pair of reward functions $J_1 : V_1 \times V_2 \to \mathbb{R}$ and $J_2 : V_2 \times V_1 \to \mathbb{R}$, and a Nash equilibrium for this game is given by a pair $(v_1, v_2) \in V_1 \times V_2$ for which $J_1(v_1', v_2) \leq J_1(v_1, v_2)$ for all $v_1' \in V_1$ and $J_2(v_2', v_1) \leq J_2(v_2, v_1)$ for all $v_2' \in V_2$. In our framework, reward functions J_i correspond to $\mathcal{Q}_i$ ranging in the poset 2^{Q_i} rather than in the totally ordered set $\mathbb{R}$ as in the case of J_i.

We conclude this section with a simple example.

Example 1: Consider interconnected systems P_1 and P_2 given by $X_1 = X_{1,0} = \{1, 3, 5, 7\}$, $U_1 = \{a, b, c\}$,

$X_2 = X_{2,0} = \{2,4,6,8\}$, $U_2 = \{d,e,f\}$ and transition partial maps F_1 and F_2 given in Tables I and II, respectively. Tables read as follows: for example, $F_1(1,2,a) = \{3\}$ and $F_1(3,2,a)$ is not defined. Consider a pair of specifications $Q_1 = \{1\,1,\, 1\,3,\, 3\,5,\, 5\,5,\, 5\,7\}$ and $Q_2 = \{2\,2,\, 2\,4,\, 4\,6,\, 6\,6,\, 6\,8\}$ and choose accuracies $\theta_1 = \theta_2 = 0$. Since length of words involved in Q_1 and Q_2 is 2, it is readily seen that in this case, controllers can be chosen as static and of the form $C_i : X_i \to 2^{U_i}$, and $\mathcal{R}_i$ as subsets of $X_{i,0}$, instead of relations. We get:

- Define relations $\mathcal{R}_1 = \{1\}$ and $\mathcal{R}_2 = \{2\}$, and controllers $C_1(1) = \{a\}$ and $C_2(2) = \{d\}$. Define $S_i = (C_i, \mathcal{R}_i) \in \mathcal{S}_i$, $i = 1,2$. Since we get $\mathcal{Q}_1(S_1) = \{1\,3\}$ and $\mathcal{Q}_2(S_2) = \{2\,4\}$, then $(S_1, S_2) \in \mathcal{E}$.
- Define relations $\mathcal{R}_1 = \{1,5\}$, $\mathcal{R}_2 = \{2,6\}$, and controllers C_1 and C_2 such that $C_1(1) = \{a\}$, $C_1(5) = \{b,c\}$, $C_2(2) = \{d\}$, and $C_2(6) = \{e,f\}$. Define $S_i = (C_i, \mathcal{R}_i) \in \mathcal{S}_i$, $i = 1,2$. Since we get $\mathcal{Q}_1(S_1) = \{1\,3,\, 5\,5,\, 5\,7\}$ and $\mathcal{Q}_2(S_2) = \{2\,4,\, 6\,6,\, 6\,8\}$, then $(S_1, S_2) \in \mathcal{E}$. Moreover, this equilibrium is also Nash. Indeed, other strategies available for P_i are given by $S'_i = (C'_i, \mathcal{R}_i)$ with C'_i specified by $C'_1(1) = C_1(1)$, $C'_1(5) \in \{\{c\},\{b\}\}$, $C'_2(2) = C_2(2)$, and $C'_2(6) \in \{\{e\},\{f\}\}$. In all cases, conditions of Definition 4 are satisfied. In particular, for $C'_1(5) = \{c\}$ we get $\mathcal{Q}_1(S'_1) = \{1\,3,\, 5\,7\}$ and for $C'_2(6) = \{f\}$ we get $\mathcal{Q}_2(S'_2) = \{2\,4,\, 6\,8\}$. As a by-product, it is readily seen that pair $(S'_1, S'_2) \in \mathcal{E}$ and is not Nash.

V. Results

In this section, we start by introducing the following local problem, consisting in designing a strategy for P_i given a strategy for P_{3-i}. Then, the solution to this problem is used to compute a Nash equilibrium for the interconnected system (see Theorem 3).

Problem 1: (local symbolic control problem) Consider plant P_i, accuracy $\theta_i \in \mathbb{R}^+_0$ and specification Q_i. Let strategy $S_{3-i} = (C_{3-i}, \mathcal{R}_{3-i}) \in \mathcal{S}_{3-i}$ for P_{3-i} be given. Find a strategy $S_i = (C_i, \mathcal{R}_i) \in \mathcal{S}_i$ for P_i, such that specification Q_i is satisfied by Σ up to accuracy θ_i, i.e. for any state trajectory $(x_1(.), x_2(.), z_1(.), z_2(.))$ of Σ with initial state $(x_1(0), x_2(0), z_1(0), z_2(0))$ such that $(x_i(0), z_i(0)) \in \mathcal{R}_i$ and $(x_{3-i}(0), z_{3-i}(0)) \in \mathcal{R}_{3-i}$ and with length $l+1$, there exists a word $q_0\, q_1 \ldots q_l \in Q_i$, such that inequality (9) holds.

Note that in the problem above, accuracy θ_i may be set to 0, which corresponds to require specifications to be enforced exactly; this is of interest when considering non-metric finite state systems. As stressed in the introduction, introducing metric of X and requiring specification to be met approximately allows problem above to be also applicable to purely continuous or hybrid systems admitting discrete abstractions of the form of plants P_i, see e.g. [17], [21].
By using the results established in [15], it is possible to associate a transition system T_{Q_i} to specification Q_i, called dual transition system, specified by

$$T_{Q_i} = (X_{q,i}, X_{q,0,i}, U_{q,i}, \underset{q,i}{\longrightarrow}, X_{q,m,i}, Y_{q,i}, H_{q,i}) \quad (10)$$

with $Y_{q,i} = X_i$, and satifying the following properties:

- T_{Q_i} is symbolic, trim and metric with metric $\mathbf{d}_i$;
- the marked output language $\mathcal{L}^y_m(T_{Q_i})$ of T_{Q_i} coincides with Q_i.

In the sequel, we denote a state of $X_{q,i}$ by $x_{q,i}$ and a transition of T_{Q_i} by $x_{q,i} \underset{q,i}{\longrightarrow} x^+_{q,i}$. Solution to Problem 1 is based on Algorithm 1. It takes as input, plant P_i, specification Q_i, strategy $S_{3-i} = (C_{3-i}, \mathcal{R}_{3-i})$. It gives as output, strategy $S_i = (C_i, \mathcal{R}_i)$. In lines 7, transition system T^c is introduced which is initialized in line 6. At the end, pair $(C_i, \mathcal{R}_i)$ will be derived from T^c. In line 8, relation $\mathcal{R}_i$ is initialized to the empty set. In lines 11-15 for each state $(x_i, x_{q,i}, (x_{3-i}, z_{3-i})) \in X^c$ we look for a control input u_i that can enforce specification Q_i. More precisely, input $u_i \in U_i$ is required to enjoy the following property: for any next state $x^+_i \in F_i(x_i, x_{3-i}, u_i)$ there exists a transition $x_{q,i} \underset{q,i}{\longrightarrow} x^+_{q,i}$ such that $\mathbf{d}_i(x^+_i, H_q(x^+_{q,i})) \leq \theta_i$ (line 10). If such an input u_i exists, set of states X^c and state transition $\underset{c}{\longrightarrow}$ are updated in lines 12-13. In line 16 trim of T^c is computed so that not accessible and not co-accessible states are eliminated. In line 17 output of the algorithm is given, the entities of which are detailed in lines 18 and 19. The following results hold:

Proposition 1: Algorithm 1 terminates in a finite number of steps.

Theorem 1: Output of Algorithm 1 solves Problem 1 w.r.t. strategy $S_{3-i} = (C_{3-i}, \mathcal{R}_{3-i})$.

The following result is concerned with maximality of strategies.

Theorem 2: Let $S_i = (C_i, \mathcal{R}_i)$ be the output of Algorithm 1 with input P_i, specification Q_i, and strategy $S_{3-i} = (C_{3-i}, \mathcal{R}_{3-i})$. Then, $\mathcal{Q}_i(S'_i) \subseteq \mathcal{Q}_i(S_i)$, for any $S'_i = (C'_i, \mathcal{R}'_i) \in \mathcal{S}_i$.

We now use Algorithm 1 to compute Nash equilibria. To this purpose, we will make use of a function, denoted $\mathbb{A}$, associated to Algorithm 1, so that

$$S_i = \mathbb{A}(P_i, Q_i, S_{3-i}), \quad (11)$$

if inputs and outputs of Algorithm 1 are (P_i, Q_i, S_{3-i}) and S_i, respectively. The following result provides a sufficient condition for the existence of equilibria in terms of fixed points associated with operator $\mathbb{A}$.

Proposition 2: If (11) holds jointly with $i = 1$ and $i = 2$, then

$$(\mathcal{Q}_1(S_1, S_2), \mathcal{Q}_2(S_2, S_1)) \in \mathcal{E}. \quad (12)$$

The converse implication of the result above is not true because by Theorem 2, $\mathcal{Q}_i(S_i, S_{3-i})$ in (12) is the maximal part of the specification Q_i which can be enforced on P_i, condition that is not needed in Definition 3 for equilibria. We now provide algorithms for the computation of equilibria. In general, equilibria for Σ do not exist. However, it is easy to enforce trivial equilibria by appropriately modifying plants P_i and specifications Q_i, as follows. For $i = 1,2$, given P_i

input;
Plant P_i;
Specification Q_i;
Strategy $S_{3-i} = (C_{3-i}, \mathcal{R}_{3-i})$;
init;
$X^c = X^c_0 := \{(x_i, x_{q,i}, (x_{3-i}, z_{3-i})) \in X_{i,0} \times X_{q,i,0} \times \mathcal{R}_{3-i} \,|\, \mathbf{d}_i(x_i, H_q(x_{q,i})) \leq \theta_i\}$; $U^c := U_i$; $\xrightarrow[c]{} := \varnothing$;
$X^c_m := \{(x_i, x_{q,i}, (x_{3-i}, z_{3-i})) \in X^c \,|\, x_{q,i} \in X_{q,i,m}\}$; $Y^c := Y_q$; $H^c(x_i, x_{q,i}, (x_{3-i}, z_{3-i})) := H_q(x_{q,i})$, $\forall (x_i, x_{q,i}, (x_{3-i}, z_{3-i})) \in X^c$;
$T^c := (X^c, X^c_0, U^c, \xrightarrow[c]{}, X^c_m, Y^c, H^c)$;
$\mathcal{R}_i := \varnothing$;
foreach $(x_i, x_{q,i}, (x_{3-i}, z_{3-i})) \in X^c$ **do**
  if $\exists u_i \in U_i$ *s.t.* $[\, \forall x^+_i \in F_i(x_i, x_{3-i}, u_i), \exists\, x_{q,i} \xrightarrow[q,i]{} x^+_{q,i}$ *s.t.* $\mathbf{d}_i(x^+_i, H_q(x^+_{q,i})) \leq \theta_i \,]$
  then
    $X^c := X^c \cup \left(\bigcup_{(x^+_{3-i}, z^+_{3-i}) \in \phi_{3-i}(x_{3-i}, z_{3-i}, x_i)} \{(x^+_i, x^+_{q,i}, (x^+_{3-i}, z^+_{3-i}))\} \right)$;
    $\xrightarrow[c]{} := \xrightarrow[c]{} \cup \{((x_i, x_{q,i}, (x_{3-i}, z_{3-i})), u_i, (x^+_i, x^+_{q,i}, (x^+_{3-i}, z^+_{3-i})))\}$;
  end
end
$T^c := \mathrm{Trim}(T^c)$;
output Strategy $S_i = (C_i, \mathcal{R}_i)$ specified by
controller C_i defined by $Z_i := X^c$; $Z_{i,0} := X^c_0$;

$$G_i((x_i, x_{q,i}, (x_{3-i}, z_{3-i})), x_i) := \left\{ \begin{array}{l} (x^+_i, x^+_{q,i}, (x^+_{3-i}, z^+_{3-i})) \in X^c \,| \\ (x_i, x_{q,i}, (x_{3-i}, z_{3-i})) \xrightarrow[c]{u_i} (x^+_i, x^+_{q,i}, (x^+_{3-i}, z^+_{3-i})) \end{array} \right\};$$
$$h_i((x_i, x_{q,i}, (x_{3-i}, z_{3-i}))) := \left\{ u_i \in U_i \,|\, (x_i, x_{q,i}, (x_{3-i}, z_{3-i})) \xrightarrow[c]{u_i} (x^+_i, x^+_{q,i}, (x^+_{3-i}, z^+_{3-i})) \right\};$$

relation of initial states defined by $\mathcal{R}_i := \{(x'_i, (x_i, x_{q,i}, (x_{3-i}, z_{3-i}))) \in X_i \times X^c_0 \,|\, x'_i = x_i\}$.

Algorithm 1. Design of strategy S_i.

as in (4), define

$$\mathbb{P}_i : \left\{ \begin{array}{l} x_i(t+1) \in \mathbb{F}_i(x_i(t), x_{3-i}(t), u_i(t)), \\ x_i(0) \in \mathbb{X}_{i,0}, x_{3-i}(0) \in \mathbb{X}_{3-i,0} \\ x_i(t) \in \mathbb{X}_i, x_{3-i}(t) \in \mathbb{X}_{3-i}, \\ u_i(t) \in U_i, t \in \mathbb{N}, \end{array} \right. \tag{13}$$

where $\mathbb{X}_i = X_i \cup \{x^d_i\}$, with $x^d_i \notin X_i$ a dummy state; $\mathbb{X}_{i,0} = X_{i,0} \cup \{x^d_i\}$; $\mathbb{X}_{3-i} = X_{3-i} \cup \{x^d_{3-i}\}$ with $x^d_{3-i} \notin X_{3-i}$ a dummy state; $\mathbb{F}_i(x_i, x_{3-i}, u_i) = F_i(x_i, x_{3-i}, u_i)$ for all $(x_i, x_{3-i}, u_i) \in X_i \times X_{3-i} \times U_i$ and $\mathbb{F}_i(x^d_i, x_{3-i}, u_i) = \{x^d_i\}$ for all $(x_{3-i}, u_i) \in X_{3-i} \times U_i$. We equip $\mathbb{X}_i$ with a metric $\mathbf{d}^{ext}_i$ such that $\mathbf{d}^{ext}_i(x_i, x'_i) = \mathbf{d}_i(x_i, x'_i)$ for all $x_i, x'_i \in \mathbb{X}_i$. Given Q_i, define

$$\mathbb{Q}_i = Q_i \cup \{x^d_i\}^* \{x^d_i\}, \tag{14}$$

where we recall $*$ denotes the Kleene closure.

Definition 5: For $i = 1, 2$, let

$$S^{\min}_i = (C^{\min}_i, \mathcal{R}^{\min}_i) \tag{15}$$

with $C^{\min}_i$ in the form of C_i in (5) and specified by $Z^{\min}_i = Z^{\min}_{i,0} = \{x^d_i\}$, $G^{\min}_i(x^d_i, x^d_i) = \{x^d_i\}$, $h^{\min}_i(x^d_i) = U_i$, and let

$$\mathcal{R}^{\min}_i = \{(x^d_i, x^d_{3-i})\}. \tag{16}$$

Proposition 3: Pair $(\mathcal{Q}_1(S^{\min}_1), \mathcal{Q}_2(S^{\min}_2))$ is an equilibrium for plants $\mathbb{P}_1$ and $\mathbb{P}_2$.

Proof: Consider any state trajectory $\mathbf{x}(.) = (x_1(.), x_2(.), z_1(.), z_2(.))$ of Σ with initial state $\mathbf{x}(0) = (x_1(0), x_2(0), z_1(0), z_2(0))$ such that $(x_1(0), z_1(0)) \in \mathcal{R}^{\min}_1$ and $(x_2(0), z_2(0)) \in \mathcal{R}^{\min}_2$, and controllers $C^{\min}_1$ and $C^{\min}_2$. By definition of $\mathbb{P}_i$ in (13) and of $C^{\min}_i$, we get $x_i(t) = x^d_i$ for $i = 1, 2$ and all $t \in \mathbb{N}$. As a consequence, by picking word $q_0\, q_1 \ldots q_l \in \{x^d_i\}^* \{x^d_i\} \subseteq \mathbb{Q}_i$, inequality (9) is satisfied (for $\theta_i = 0$). Hence, $S^{\min}_i \in \mathcal{S}_i$, which concludes the proof. ■

In the sequel we will work with plants $\mathbb{P}_i$ in (13), specifications $\mathbb{Q}_i$ as in (14) and metric $\mathbf{d}^{ext}_i$. Some remarks in this regard are reported at the end of this section.
For the next developments we need to specialize to this framework, the notion of maximal permissive controller, often used in e.g. symbolic control.

Definition 6: For $i = 1, 2$, the maximal permissive controller for plant $\mathbb{P}_i$, denoted $C^{\max}_i$, is the controller of the form (5) with $Z_i = Z_{i,0} = \{z^d_i\}$, z^d_i is a dummy state, $G_i(z^d_i, x_i) = \{z^d_i\}$ and $h_i(z^d_i) = U_i$.

We now have all the ingredients to define a sequence of controllers and relations which is used later on to compute Nash equilibria. For $i = 1, 2$ and $k \in \mathbb{N}$, define the following recursive equations:

$$\begin{array}{ll} C_{3-i}(0) := C_{3-i}^{\max}; & \\ \mathcal{R}_{3-i}(0) := X_{3-i} \times \{z_{3-i}^d\}; & \\ S_{3-i}(0) := (C_{3-i}(0), \mathcal{R}_{3-i}(0)); & (17) \\ S_i(k+1) := \mathbb{A}(\mathbb{P}_i, \mathbb{Q}_i, S_{3-i}(k)); & \\ S_{3-i}(k+1) := \mathbb{A}(\mathbb{P}_{3-i}, \mathbb{Q}_{3-i}, S_i(k)). & \end{array}$$

Sequence above alternates updating of strategies $S_i(k)$ with those of strategies $S_{3-i}(k)$. Convergence of the sequences $S_i(k)$, $k \in \mathbb{N}$, $i = 1, 2$, is proven in the following

Proposition 4: There exists $K \in \mathbb{N}$ such that $S_i(k) = S_i(k+1)$ for $i = 1, 2$ and for all $k \geq K$.

Let $K \in \mathbb{N}$ be such that $S_i(k) = S_i(k+1)$ for $i = 1, 2$ and for all $k \geq K$. We can now give the main result of this paper.

Theorem 3: $(\mathcal{Q}_1(S_1(K), S_2(K)), \mathcal{Q}_2(S_2(K), S_1(K)) \in \mathcal{E}$ and is Nash.

Proof: By Proposition 4 and (17), we get

$$\begin{array}{l} S_i(K) = \mathbb{A}(\mathbb{P}_i, \mathbb{Q}_i, S_{3-i}(K)); \\ S_{3-i}(K) = \mathbb{A}(\mathbb{P}_{3-i}, \mathbb{Q}_{3-i}, S_i(K)), \end{array}$$

which, by Proposition 2, implies the statement. ■

Note that convergent strategies of recursive equations in (17) depend on index $i = 1, 2$ by which (17) is initialized and hence, corresponding Nash equilibria may be different. We derived results for plants $\mathbb{P}_i$ in (13) and specifications $\mathbb{Q}_i$ in (14). This guarantees convergence of proposed algorithms. Let $S_i(k) = (C_i(k), \mathcal{R}_i(k))$, $k \in \mathbb{N}$. Note that state trajectories of controlled systems $P_i^{C_i(K)}$ and $\mathbb{P}_i^{C_i(K)}$ coincide when initialized to $\mathcal{R}_i(K) \backslash \mathcal{R}_i^{\min}$ with $\mathcal{R}_i^{\min}$ as in (16). As a consequence, if one is interested in deriving strategies for enforcing Q_i on P_i, and not $\mathbb{Q}_i$ on $\mathbb{P}_i$, it suffices to initialize P_i with $\mathcal{R}_i(K) \backslash \mathcal{R}_i^{\min}$ and apply the controller $C_i(K)$. We conclude this section by coming back to

Example 1: (continued.) Since we know that equilibria exist, we can work directly with P_i and Q_i, instead of $\mathbb{P}_i$ and $\mathbb{Q}_i$. We now apply recursive equations in (17) to P_i and Q_i starting from $i = 1$. For static controllers $C_i(k)$ evaluated in state x_i we use here notation $C_i(k; x_i)$. For $k = 0$ we obtain $\mathcal{R}_2(0) = \{2, 4, 6, 8\}$. For $k = 1$: $C_1(k)$ is specified by $C_1(k; 1) = \{a\}$ and $C_1(k; 5) = \{b, c\}$, and $\mathcal{R}_1(k) = \{1, 5\}$; $C_2(k)$ is specified by $C_2(k; 2) = \{d\}$ and $C_2(k; 6) = \{e, f\}$, and $\mathcal{R}_2(k) = \{2, 6\}$. For $k = 2$ we get $C_1(2) = C_1(1)$ and $\mathcal{R}_1(2) = \mathcal{R}_1(1)$; hence, fixed point is found and $K = 2$. For $i = 1, 2$, we get $S_i(K) \in \mathcal{S}_i$, $\mathcal{Q}_1(S_1(K), S_2(K)) = \{1\,3,\, 5\,5,\, 5\,7\}$ and $\mathcal{Q}_2(S_2(K), S_1(K)) = \{2\,4,\, 6\,6,\, 6\,8\}$. By Theorem 4, pair $(\mathcal{Q}_1(S_1(K), S_2(K)), \mathcal{Q}_2(S_2(K), S_1(K))$ is a Nash equilibrium. By reversing order of $i = 1$ and $i = 2$ in the recursive equations (17) we get the same result. However, this is a consequence of the symmetry in dynamics of the plants P_1 and P_2. Note that these findings are consistent with what discussed for this example on page 4.

VI. Conclusions

In this paper we proposed notions of equilibria and Nash equilibria to study decentralized symbolic control problems of interconnected nondeterministic and metric finite state systems with regular language specifications. Algorithms were designed which converge, when an equilibrium exists, to Nash equilibria.

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