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Relations between modules associated to input-output nonlinear equations with delays and their realizations

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Abstract— **The relations between a control system with delays given by a nonlinear input-output equation and its realization are addressed. The algebraic formalism based on rings of polynomials over the rings associated with the considered systems and modules of differential one-forms is used to show the relations between submodules corresponding to the input-output equation and its realization.**

I. INTRODUCTION

Control systems can be described in different ways. One of the descriptions is given by an input-output (i/o) equation. In this case there is no state of the system, we have only the relation between inputs, outputs, and their derivatives at different delayed time instances. It is well known that some i/o equations can be described in the state space form called a realization. Therefore the other description is associated with control system given in the state space. We will consider both types of descriptions and explore the relations between certain algebraic structures related to them.

Studying the realization problem for systems with delays by using the algebraic formalism based on differential algebra and one-forms can be difficult because the algebraic objects characterizing the i/o equation and its realization are different. Most previous results simply ignore the issue. The possible reason for the latter is that if the order of the realization is equal to the order of the i/o equation then there exists a simple isomorphism between the corresponding algebraic structures. However, if one studies lower order realizations, then the map between the algebraic structures has more complicated characteristics. In [5] a kind of sketch is provided for addressing difficulties arising in case of unequal orders of system representations. This aspect will be handled in this paper in a full mathematical rigor. The purpose of the current paper is to describe the relationship between certain rings and modules associated with an i/o equation and its realization. Compared to [5] we use rings of analytic functions instead of fields of meromorphic functions. In our opinion rings are more useful for describing the relations between the algebraic structures associated to i/o equations and their realizations. The map ξ, which appears in the definition of realization, is not always injective. This happens if the dimension of realization is lower than the order of the input-output equation. Then the considered map ξ cannot be an isomorphism, so its kernel is not trivial. This map ξ may be extended to a map between the fields of fractions of rings $\mathcal{A}$ (associated to an i/o equation) and $\hat{\mathcal{A}}$ (associated to a realization) only if it is injective. If additionally this map is surjective, then it is an isomorphism. Obviously, it is better to have an isomorphism, because there is then one-to-one correspondence between the rings (and their fields of fractions), which is transferred to one-to-one correspondence between different modules. Unfortunately, if the dimension of realization is lower than the order of the input-output equation, then the map ξ cannot be an isomorphism on both levels: rings and modules. But then one gets a nice interpretation when we restrict the considered map ξ to the map from $\mathcal{H}_\infty$ to $\hat{\mathcal{H}}_\infty$. Then the isomorphism appears in a different context. Namely, the image $\operatorname{im} \xi|_{\mathcal{H}_\infty}$ is isomorphic to the quotient module $\mathcal{H}_\infty / \ker \xi|_{\mathcal{H}_\infty}$. Therefore in this paper we take into account a more general approach where the rings of left polynomials are over the ring while in [5] the rings of polynomials over the field were considered.

The presented relations can provide a framework for analyzing and designing control systems, in particular for constructing minimal realizations, i.e. for transforming a higher order differential equation relating the system outputs and inputs into a set of first order differential equations (the so-called state equations), which are observable and accessible. Besides helping to address minimal realization problem, the suggested relationship may be useful when addressing problems whose proofs require to move from one system representation (input-output equation for instance) to the other representation (state equations). Moreover, these relations can help in understanding system behaviour and determining the system properties as for instance accessibility, controllability or observability. Additionally, thanks to the presented relationship engineers can develop algorithms for computing different realizations for a retarded type time delay i/o equation.

The paper is organized as follows. In Section II we present the description of control systems with one input and one output and introduce the algebraic approach that allows to check whether the state-space system is an realization of the considered single-input single-output equation. In Section III we study observability and accessibility of considered systems. Section IV is devoted to presenting the realizability problem and the relation between modules associated with an input-output equation and its realization. Finally, an example to illustrate our results is given.

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II. CONTROL SYSTEMS WITH DELAYS

Let us recall from [1], [5] the methodology and basic notations used in the paper.

A. Description of control systems

Let us start with a single-input and single-output (SISO) nonlinear retarded time-delay system, described by the following input-output (i/o) equation

$$\begin{aligned} y^{(n)}(t) = F(y^{(n-1)}(t-jd), \ldots, y(t-jd), \\ u^{(s-1)}(t-jd), \ldots, u(t-jd); j = 0, 1 \ldots, p), \end{aligned} \quad (1)$$

where F is analytic in some open subset of $\mathbb{R}^{n(p+1)} \times \mathbb{R}^{s(p+1)}$, d is the delay that is assumed to be a non-negative real number, and p is a non-negative integer, corresponding to the maximum multiple of the delay d, which is present in (1). The variables $y^{(i)}(t-jd)$ and $u^{(i)}(t-jd)$ denote the ith time derivative of the output y and the input u at delayed time $t - jd$, respectively.

The control systems can be also described by

$$\begin{aligned} \dot{x}(t) =& f(x(t-jd); u(t-jd); j = 0, \ldots, q) \\ y(t) =& h(x(t-jd); j = 0, \ldots, q) \end{aligned} \quad (2)$$

for some $q \in \{0\} \cup \mathbb{N}$, where $x(t) = (x_1(t), \ldots, x_{\tilde{n}}(t))^T \in \mathcal{X} \subseteq \mathbb{R}^{\tilde{n}}$, $u(t) \in \mathcal{U} \subseteq \mathbb{R}$ and $y(t) \in \mathcal{Y} \subseteq \mathbb{R}$, where $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are open, and functions $f = (f_1, \ldots, f_{\tilde{n}})^T$ and h are analytic in their domains. Similarly as in (1), d is the delay that is assumed to be a non-negative real number, and q is a non-negative integer, corresponding to the maximum multiple of the delay d present in (2).

B. Rings and operators associated with control systems

Similarly as in [1] and [5], one can introduce rings and operators associated with systems. Then some modules defined over these rings are used to check systems' properties like observability or accessibility.

Denote by $\mathcal{A}$ the ring of analytic functions depending on a finite number of variables from the set $Y \cup U$, where

$$\begin{aligned} Y :=& \{y^{(i)}[j] : \; i = 0, \ldots, n-1, j \in \{0\} \cup \mathbb{N}\}, \\ U :=& \{u^{(i)}[j] : \; i, j \in \{0\} \cup \mathbb{N}\}. \end{aligned} \quad (3)$$

The variables $y^{(i)}[j]$ and $u^{(i)}[j]$ in (3) correspond to $y^{(i)}(t-jd)$ and $u^{(i)}(t-jd)$, though they are not seen as functions of time, but as independent variables. For the sake of simplicity, $y[j]$, $y^{(1)}[j]$ and $y^{(i)}[0]$ are also denoted as $y[j]$, $\dot{y}[j]$ and $y^{(i)}$, respectively. Similar simplified notations are also used for $u^{(i)}[j]$. Then, the i/o equation (1) can be rewritten as follows:

$$\begin{aligned} y^{(n)} = F(y^{(n-1)}[j], \ldots, y[j], \\ u^{(s-1)}[j], \ldots, u[j]; j = 0, 1 \ldots, p), \end{aligned} \quad (4)$$

where $F \in \mathcal{A}$.

Let $\hat{\mathcal{A}}$ be the corresponding ring of analytic functions in a finite number of variables from the set $\{x[j], u^{(i)}[j]; i, j \in \{0\} \cup \mathbb{N}\}$. The variables $x[j]$ and $u^{(i)}[j]$ correspond to $x(t-jd)$ and $u^{(i)}(t-jd)$. Again they are not seen as functions of time, but as independent variables. Then (2) can be rewritten as follows:

$$\begin{aligned} \dot{x} =& f(x[j]; u[j]; j = 0, \ldots, q) \\ y =& h(x[j]; j = 0, \ldots, q). \end{aligned} \quad (5)$$

For the systems (4) and (5) algebraic setting that allows to study for instance system's realizability or accessibility is described in [9], [5].

Similarly as in [5], on the ring $\mathcal{A}$ a time-derivative operator $\mathrm{d}/\mathrm{d}t : \mathcal{A} \to \mathcal{A}$ and a delay operator $D : \mathcal{A} \to \mathcal{A}$ are defined. Since the time derivative of $y^{(i)}(t-jd)$ is $y^{(i+1)}(t-jd)$ and its (one-step) time delay is $y^{(i)}(t-(j+1)d)$, it is natural to define the operators $\mathrm{d}/\mathrm{d}t$ and D such that $\mathrm{d}/\mathrm{d}t(y^{(i)}[j]) := y^{(i+1)}[j]$ and $D(y^{(i)}[j]) := y^{(i)}[j+1]$. However, because the set Y contains only time-derivatives up to the order $n-1$, then $\mathrm{d}/\mathrm{d}t(y^{(n-1)}[j]) := D^j F(\cdot)$. The operators $\mathrm{d}/\mathrm{d}t$ and D act in a similar manner on $u^{(i)}[j]$. Moreover,

$$\begin{aligned} &\mathrm{d}/\mathrm{d}t(G(y^{(i)}[j]; u^{(i)}[j]; i = 0, \ldots k; j = 0, \ldots, s)) := \\ &\sum_{i=0}^{k} \sum_{j=0}^{s} \left(\frac{\partial G}{\partial y^{(i)}[j]} \mathrm{d}/\mathrm{d}t(y^{(i)}[j]) + \frac{\partial G}{\partial u^{(i)}[j]} u^{(i+1)}[j]) \right) \end{aligned} \quad (6)$$

and

$$\begin{aligned} D(G(y^{(i)}[j]; u^{(i)}[j]; i = 0, \ldots, k; j = 0, \ldots, s)) := \\ G(y^{(i)}[j+1]; u^{(i)}[j+1]; i = 0, \ldots, k; j = 0, \ldots, s) \end{aligned} \quad (7)$$

for $G \in \mathcal{A}$.

In the case of $\hat{\mathcal{A}}$ one defines a time-derivative operator $\widehat{\mathrm{d}/\mathrm{d}t} : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ and the delay operator $\hat{D} : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ in a similar manner as above, namely $\widehat{\mathrm{d}/\mathrm{d}t}(x_i[j]) = f_i(\cdot)[j]$, $i = 1, \ldots, \tilde{n}$, $\hat{D}(x_i[j]) = x_i[j+1]$, $i = 1, \ldots, \tilde{n}$, and moreover,

$$\begin{aligned} \widehat{\mathrm{d}/\mathrm{d}t}(\hat{G}(x[j]; u^{(i)}[j]; & i = 0, \ldots k; j = 0, \ldots, s)) := \\ \sum_{j=0}^{s} \Bigg(& \sum_{i=1}^{\tilde{n}} \frac{\partial \hat{G}}{\partial x_i[j]} \widehat{\mathrm{d}/\mathrm{d}t}(x_i[j]) \\ & + \sum_{i=0}^{k} \frac{\partial \hat{G}}{\partial u^{(i)}[j]} u^{(i+1)}[j]) \Bigg), \end{aligned} \quad (8)$$

$$\begin{aligned} \hat{D}(\hat{G}(x[j]; u^{(i)}[j]; i = 0, \ldots, k; j = 0, \ldots, s)) := \\ \hat{G}(x[j+1]; u^{(i)}[j+1]; i = 0, \ldots, k; j = 0, \ldots, s) \end{aligned} \quad (9)$$

for and $\hat{G} \in \hat{\mathcal{A}}$.

C. Modules associated with control systems

Consider modules $\mathcal{E} := \mathrm{span}_{\mathcal{A}}\{\mathrm{d}\varphi : \varphi \in \mathcal{A}\}$ and $\hat{\mathcal{E}} := \mathrm{span}_{\hat{\mathcal{A}}}\{\mathrm{d}\varphi : \varphi \in \hat{\mathcal{A}}\}$ of one-forms, where $\mathrm{d} : \mathcal{A} \to \mathcal{E}$ and $\mathrm{d} : \hat{\mathcal{A}} \to \hat{\mathcal{E}}$ are defined as standard differentials of a function from the rings $\mathcal{A}$ and $\hat{\mathcal{A}}$. Next, the time-derivative operators $\mathrm{d}/\mathrm{d}t : \mathcal{A} \to \mathcal{A}$, $\widehat{\mathrm{d}/\mathrm{d}t} : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ and time-delay operators $D : \mathcal{A} \to \mathcal{A}$, $\hat{D} : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ are extended to the modules $\mathcal{E}$, $\hat{\mathcal{E}}$. Note that every element $\omega \in \mathcal{E}$ ($\hat{\omega} \in \hat{\mathcal{E}}$) can be represented as

$$\omega = \sum_{i=1}^{r} a_i \mathrm{d}\varphi_i \quad (\hat{\omega} = \sum_{i=1}^{r} \hat{a}_i \mathrm{d}\hat{\varphi}_i) \quad (10)$$

for some functions $a_i, \varphi_i \in \mathcal{A}$ ($\hat{a}_i, \hat{\varphi}_i \in \hat{\mathcal{A}}$) and integer r. Based on these representations, time-derivative operators $\mu : \mathcal{E} \to \mathcal{E}$, $\hat{\mu} : \hat{\mathcal{E}} \to \hat{\mathcal{E}}$ and time-delay operators $\delta : \mathcal{E} \to \mathcal{E}$, $\hat{\delta} : \hat{\mathcal{E}} \to \hat{\mathcal{E}}$ are defined on $\mathcal{E}$, $\hat{\mathcal{E}}$ as follows:

$$\mu\Big(\sum_{i=1}^{r} a_i \mathrm{d}\varphi_i\Big) := \sum_{i=1}^{r}\Big(\frac{\mathrm{d}}{\mathrm{d}t}(a_i)\mathrm{d}\varphi_i + a_i \mathrm{d}\big(\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_i)\big)\Big), \tag{11}$$

$$\hat{\mu}\Big(\sum_{i=1}^{r} \hat{a}_i \mathrm{d}\hat{\varphi}_i\Big) := \sum_{i=1}^{r}\Big(\frac{\hat{\mathrm{d}}}{\mathrm{d}t}(\hat{a}_i)\mathrm{d}\hat{\varphi}_i + \hat{a}_i \mathrm{d}\big(\frac{\hat{\mathrm{d}}}{\mathrm{d}t}(\hat{\varphi}_i)\big)\Big), \tag{12}$$

$$\delta\left(\sum_{i=1}^{r} a_i \mathrm{d}\varphi_i\right) := \sum_{i=1}^{r} D(a_i)\mathrm{d}(D(\varphi_i)), \tag{13}$$

$$\hat{\delta}\Big(\sum_{i=1}^{r} \hat{a}_i \mathrm{d}\hat{\varphi}_i\Big) := \sum_{i=1}^{r} \hat{D}(\hat{a}_i)\mathrm{d}(\hat{D}(\hat{\varphi}_i)). \tag{14}$$

The operators D and $\hat{D}$ are used to define left polynomial rings $\mathcal{A}[\vartheta]$ and $\hat{\mathcal{A}}[\vartheta]$, respectively. Addition is defined in $\mathcal{A}[\vartheta]$, $\hat{\mathcal{A}}[\vartheta]$ as usual, but for multiplication the following rules are used: $\vartheta\varphi = D(\varphi)\vartheta$ for $\varphi \in \mathcal{A}$ and $\vartheta\hat{\varphi} = \hat{D}(\hat{\varphi})\vartheta$ for $\hat{\varphi} \in \hat{\mathcal{A}}$. The polynomials in $\mathcal{A}[\vartheta]$ ($\hat{\mathcal{A}}[\vartheta]$) act as operators on $\mathcal{E}$ ($\hat{\mathcal{E}}$) by the rule $\vartheta\omega = \delta(\omega)$ for all $\omega \in \mathcal{E}$ ($\vartheta\hat{\omega} = \hat{\delta}(\hat{\omega})$ for all $\hat{\omega} \in \hat{\mathcal{E}}$). Now, the one-forms can be alternatively viewed as elements of the modules

$$\mathcal{N} := \mathrm{span}_{\mathcal{A}[\vartheta]}\{\mathrm{d}\varphi \mid \ \varphi \in \mathcal{A}\}, \tag{15}$$

$$\hat{\mathcal{N}} := \mathrm{span}_{\hat{\mathcal{A}}[\vartheta]}\{\mathrm{d}\varphi \mid \ \varphi \in \hat{\mathcal{A}}\}. \tag{16}$$

1) Properties of modules: Unlike a vector space, not every module has a basis. The modules, that do have bases, are called free modules. Similarly as in [9], $\mathcal{A}[\vartheta]$ and $\hat{\mathcal{A}}[\vartheta]$ satisfy the left Ore condition, i.e. for all $a[\vartheta], b[\vartheta] \in \mathcal{A}[\vartheta]$ there exist nonzero $a_1[\vartheta], b_1[\vartheta] \in \mathcal{A}[\vartheta]$ such that $a_1[\vartheta]b[\vartheta] = b_1[\vartheta]a[\vartheta]$ (similarly for the ring $\hat{\mathcal{A}}[\vartheta]$), and any two bases of a free module over such a ring have the same cardinality, which is called the *rank of the free module* and denoted as $\mathrm{rank}\,(\mathcal{F})$ for a free module $\mathcal{F}$. The definitions presented below will be given for the ring $\mathcal{A}[\vartheta]$ and the module $\mathcal{N}$, but they also hold for $\hat{\mathcal{A}}[\vartheta]$ and $\hat{\mathcal{N}}$.

Definition 1: [9] The *closure of a submodule $\mathcal{F}$ of $\mathcal{N}$*, denoted by $cl_{\mathcal{A}[\vartheta]}(\mathcal{F})$, is defined as $cl_{\mathcal{A}[\vartheta]}(\mathcal{F}) := \{\omega \in \mathcal{N} \mid \exists 0 \neq p \in \mathcal{A}[\vartheta],\ \mathrm{s.t.}\ p(\vartheta)\omega \in \mathcal{F}\}$. If the closure of the submodule $\mathcal{F}$ is equal to itself, then $\mathcal{F}$ is said to be *closed*.

A property of free submodules $\mathcal{F}$ is that the closure $cl_{\mathcal{A}[\vartheta]}(\mathcal{F})$ is the largest free submodule, containing $\mathcal{F}$, and having the same rank as $\mathcal{F}$, see [9]. One also has the following result.

Lemma 2: [1] A finitely generated closed submodule $\mathcal{F}$ of $\mathcal{N}$ is always free.

One is often interested in free modules, whose elements can be written as linear combination of k (where k is the rank of the free module) exact elements, i.e. $\mathrm{d}\varphi_i$, $\varphi_i \in \mathcal{A}$, $i = 1, \ldots, k$, over the ring $\mathcal{A}[\vartheta]$. Such modules are called *integrable*.

Definition 3: [6] A set of one-forms $\{\omega_1, \ldots, \omega_k\}$, linearly independent over $\mathcal{A}[\vartheta]$, is said to be *integrable* if there exist k independent functions $\{\varphi_1, \ldots, \varphi_k\}$, such that $\mathrm{span}_{\mathcal{A}[\vartheta]}\{\omega_1, \ldots, \omega_k\} = \mathrm{span}_{\mathcal{A}[\vartheta]}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_k\}$.

If the set of one-forms $\{\omega_1, \ldots, \omega_k\}$ is integrable, then the corresponding submodule $\mathrm{span}_{\mathcal{A}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$ is said to be *integrable*.

Note that the time-derivative operators $\mathrm{d}/\mathrm{d}t$ and $\hat{\mathrm{d}}/\mathrm{d}t$ can be extended to polynomials $p \in \mathcal{A}[\vartheta]$ and $\hat{p} \in \hat{\mathcal{A}}[\vartheta]$ naturally in the following manner

$$\frac{\mathrm{d}}{\mathrm{d}t}p(\vartheta) = \sum_{i=0}^{k} \frac{\mathrm{d}}{\mathrm{d}t}(p_i)\vartheta^i, \tag{17}$$

$$\frac{\hat{\mathrm{d}}}{\mathrm{d}t}\hat{p}(\vartheta) = \sum_{i=0}^{k} \frac{\hat{\mathrm{d}}}{\mathrm{d}t}(\hat{p}_i)\vartheta^i, \tag{18}$$

where $p(\vartheta) = \sum_{i=0}^{k} p_i\vartheta^i$, $p_i \in \mathcal{A}$, $\hat{p}(\vartheta) = \sum_{i=0}^{k} \hat{p}_i\vartheta^i$, $\hat{p}_i \in \hat{\mathcal{A}}$, $i = 0, \ldots, k$.

2) Submodules associated with control systems: In the modules $\mathcal{N}$ and $\hat{\mathcal{N}}$ one can define the following sequences of submodules:

$$\begin{aligned} \mathcal{H}_1 :=& \mathrm{span}_{\mathcal{A}[\vartheta]}\{\mathrm{d}y^{(n-1)}, \ldots, \mathrm{d}y, \mathrm{d}u^{(s-1)}, \ldots \mathrm{d}u\} \\ \mathcal{H}_{i+1} :=& \{\omega \in \mathcal{H}_i \mid \ \mu(\omega) \in \mathcal{H}_i\},\ i \geq 1 \end{aligned} \tag{19}$$

and

$$\begin{aligned} \hat{\mathcal{H}}_1 :=& \mathrm{span}_{\hat{\mathcal{A}}[\vartheta]}\{\mathrm{d}x_1, \ldots, \mathrm{d}x_{\tilde{n}}\} \\ \hat{\mathcal{H}}_{i+1} :=& \{\hat{\omega} \in \hat{\mathcal{H}}_i \mid \ \hat{\mu}(\hat{\omega}) \in \hat{\mathcal{H}}_i\},\ i \geq 1. \end{aligned} \tag{20}$$

associated with systems (4) and (5), respectively.

Sequences $(\mathcal{H}_i)_{i\in\mathbb{N}}$ and $(\hat{\mathcal{H}}_i)_{i\in\hat{\mathbb{N}}}$ are non-increasing and converge to a submodules $\mathcal{H}_\infty$ and $\hat{\mathcal{H}}_\infty$, respectively, i.e. there exist k and $\hat{k}$ such that $\mathcal{H}_k = \mathcal{H}_j =: \mathcal{H}_\infty$ for all $j > k$ and $\hat{\mathcal{H}}_{\hat{k}} = \hat{\mathcal{H}}_j =: \hat{\mathcal{H}}_\infty$ for all $j > \hat{k}$, see [9]. Similarly as in [9] and [1], the submodules $\mathcal{H}_i$ and $\hat{\mathcal{H}}_i$, $i \in \mathbb{N}$, are closed and free.

The properties of the submodule $\mathcal{H}_\infty$ are presented in [1]. Some of them are recalled in the following.

Lemma 4: [1] A one-form $\omega \in \mathcal{N}$ ($\hat{\omega} \in \hat{\mathcal{N}}$) belongs to $\mathcal{H}_\infty$ ($\hat{\mathcal{H}_\infty}$) if and only if there exists $k \in \mathbb{N}$ such that ω, $\mu(\omega)$, $\ldots$, $\mu^k(\omega)$ ($\hat{\omega}$, $\hat{\mu}(\hat{\omega})$, $\ldots$, $\hat{\mu}^k(\hat{\omega})$) are linearly dependent over $\mathcal{A}[\vartheta]$ ($\hat{\mathcal{A}}[\vartheta]$).

Since the submodules $\mathcal{H}_\infty$ and $\hat{\mathcal{H}}_\infty$ are the limits of nonincreasing sequences of submodules, they can be alternatively defined as follows:

$$\mathcal{H}_\infty := \{\omega \in \mathcal{H}_1 \mid \ \mu^k(\omega) \in \mathcal{H}_1,\ k \geq 0\} \tag{21}$$

and

$$\hat{\mathcal{H}}_\infty := \{\hat{\omega} \in \hat{\mathcal{H}}_1 \mid \ \hat{\mu}^k(\hat{\omega}) \in \hat{\mathcal{H}}_1,\ k \geq 0\}. \tag{22}$$

Taking into account (21) and (22) one gets

Proposition 5: The submodules $\mathcal{H}_\infty$ and $\hat{\mathcal{H}}_\infty$ are the biggest invariant submodules of $\mathcal{H}_1$ and $\hat{\mathcal{H}}_1$ with respect to μ and $\hat{\mu}$, respectively.

Theorem 6: [1] The submodule $\mathcal{H}_\infty$ ($\hat{\mathcal{H}_\infty}$) is always integrable.

III. PROPERTIES OF CONSIDERED CONTROL SYSTEMS

One of the properties of control systems in the form (2) (or equivalently, in (5)) is observability. Different notion of observability for time-delay systems of the form (5) are described in [3].

Definition 7: [3] System (5) is *weakly observable* if there exist polynomials $\hat{\alpha}_i \in \hat{\mathcal{A}}[\vartheta]$, $i = 1, \ldots, \tilde{n}$, such that

$$\hat{\alpha}_i(\vartheta)\mathrm{d}x_i \in \mathrm{span}_{\hat{\mathcal{A}}[\vartheta]}\{\mathrm{d}h^{(\tilde{n}-1)}, \ldots, \mathrm{d}h, \mathrm{d}u^{(\tilde{n}-2)}, \ldots, \mathrm{d}u\}, \tag{23}$$

where $i = 1, \ldots, \tilde{n}$. When $\hat{\alpha}_i$ can be chosen as $\hat{\alpha}_i(\vartheta) = 1$ for $i = 1, \ldots, \tilde{n}$, then system (5) is said to be *strongly observable*. If there exist $\tilde{N} \geq \tilde{n}$ such that

$$\mathrm{d}x_i \in \mathrm{span}_{\hat{\mathcal{A}}[\vartheta]}\{\mathrm{d}h^{(\tilde{N}-1)}, \ldots, \mathrm{d}h, \mathrm{d}u^{(\tilde{N}-2)}, \ldots, \mathrm{d}u\},$$

where $i = 1, \ldots, \tilde{n}$, then system (5) is said to be *regularly observable*.

Observe that (23) is equivalent to

$$\mathrm{d}x_i \in cl_{\hat{\mathcal{A}}[\vartheta]}\big(\mathrm{span}_{\hat{\mathcal{A}}[\vartheta]}\{\mathrm{d}h^{(\tilde{n}-1)}, \ldots, \mathrm{d}h, \mathrm{d}u^{(\tilde{n}-2)}, \ldots, \mathrm{d}u\}\big).$$

Directly form Definition 7 one gets:

Proposition 8: If system (5) is strongly observable, then it is regularly observable. Moreover, regular observability of (5) implies its weak observability.

Let g be an r-dimensional vector with entries $g_j \in \hat{\mathcal{A}}$. Then $\partial g/\partial x$ denotes the $r \times \tilde{n}$ matrix with entries

$$\left(\frac{\partial g}{\partial x}\right)_{j,i} = \sum_{\ell \geq 0} \frac{\partial g_j}{\partial x_i[\ell]}\vartheta^\ell \in \hat{\mathcal{A}}[\vartheta].$$

Let us define the *rank of a matrix* over $\hat{\mathcal{A}}[\vartheta]$ as the number of linearly independent rows.

Definition 9: The least nonnegative integer s such that

$$\mathrm{rank}_{\hat{\mathcal{A}}[\vartheta]} \frac{\partial\left(h, \ldots, h^{(s-1)}\right)}{\partial x} = \mathrm{rank}_{\hat{\mathcal{A}}[\vartheta]} \frac{\partial\left(h, \ldots, h^{(s)}\right)}{\partial x} \tag{24}$$

is called the *observability index* of (5).

Proposition 10: System (5) is weakly observable if and only if its observability index equals to $\tilde{n}$.

Proof: Let the observability index of (5) be s, i.e. the rows of the matrix $A := \begin{pmatrix} \frac{\partial h}{\partial x} \\ \vdots \\ \frac{\partial h^{(s-1)}}{\partial x} \end{pmatrix}$ are linearly independent over $\hat{\mathcal{A}}[\vartheta]$ and the rows of the matrix $\begin{pmatrix} \frac{\partial h}{\partial x} \\ \vdots \\ \frac{\partial h^{(i)}}{\partial x} \end{pmatrix}$ are linearly dependent over $\hat{\mathcal{A}}[\vartheta]$ for $i \geq s$, where $\frac{\partial h^{(i)}}{\partial x} = \left(\sum_{\ell=0}^q \frac{\partial h^{(i)}}{\partial x_1[\ell]}\vartheta^\ell \;\; \ldots \;\; \sum_{\ell=0}^q \frac{\partial h^{(i)}}{\partial x_{\tilde{n}}[\ell]}\vartheta^\ell\right) \in \left(\hat{\mathcal{A}}[\vartheta]\right)^{1\times\tilde{n}}$. Since $\mathrm{d}h^{(i)} = \frac{\partial h^{(i)}}{\partial x}\mathrm{d}x + \frac{\partial h^{(i)}}{\partial u}\mathrm{d}u + \frac{\partial h^{(i)}}{\partial u^{(1)}}\mathrm{d}u^{(1)} + \ldots + \frac{\partial h^{(i)}}{\partial u^{(i-1)}}\mathrm{d}u^{(i-1)}$,

$$A\mathrm{d}x = \begin{pmatrix} \mathrm{d}h \\ \mathrm{d}h^{(1)} \\ \vdots \\ \mathrm{d}h^{(s-1)} \end{pmatrix} - \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \frac{\partial h^{(1)}}{\partial u} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h^{(s-1)}}{\partial u} & \frac{\partial h^{(s-1)}}{\partial u^{(1)}} & \ldots & \frac{\partial h^{(s-1)}}{\partial u^{(s-2)}} \end{pmatrix} \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}u^{(1)} \\ \vdots \\ \mathrm{d}u^{(s-2)} \end{pmatrix}. \tag{25}$$

where $\frac{\partial h^{(i)}}{\partial u^{(j)}} = \sum_{\ell=0}^q \frac{\partial h^{(i)}}{\partial u^{(j)}[\ell]}\vartheta^\ell \in \hat{\mathcal{A}}[\vartheta]$, $j = 0, \ldots, s-2$. If $s = \tilde{n}$, then A is a $\tilde{n} \times \tilde{n}$ matrix with entries in the ring $\hat{\mathcal{A}}[\vartheta]$. Since the ring $\hat{\mathcal{A}}[\vartheta]$ satisfies the left Ore condition, the Gauss elimination can be applied (by adding a linear combination of rows, see [2]) and system (25) can be transformed to the form

$$\alpha_i(\vartheta)\mathrm{d}x_i = \sum_{j=0}^{s-1} \beta_j(\vartheta)\mathrm{d}h^{(j)} + \sum_{k=0}^{s-2} \gamma_k(\vartheta)\mathrm{d}u^{(k)} \tag{26}$$

and consequently, one gets the weak observability of (5).

For $s < \tilde{n}$, A is a $s \times \tilde{n}$ matrix with entries in the ring $\hat{\mathcal{A}}[\vartheta]$ and $\mathrm{rank}_{\hat{\mathcal{A}}[\vartheta]} A \leq s$. Then it is not possible to transform matrix A to a diagonal form and hence (25) cannot be transformed to the form (26). Consequently, (5) is not weakly observable. ■

Another property associated with the considered systems is their accessibility that can be defined by using the idea of autonomous one-forms.

Definition 11: Let $\omega \in \mathcal{N}$ and $\hat{\omega} \in \hat{\mathcal{N}}$. The one form ω ($\hat{\omega}$) is called the *autonomous* one-form of system (4) (system (5)) if there exist polynomials $\alpha_\ell \in \mathcal{A}[\vartheta]$ ($\hat{\alpha}_\ell \in \hat{\mathcal{A}}[\vartheta]$), $\ell = 0, \ldots, k$ such that the following relation holds

$$\sum_{\ell=0}^{k} \alpha_\ell(\vartheta)\mu^\ell(\omega) = 0 \quad \Big(\sum_{\ell=0}^{k} \hat{\alpha}_\ell(\vartheta)\hat{\mu}^\ell(\hat{\omega}) = 0\Big), \tag{27}$$

where $k \in \{0\} \cup \mathbb{N}$, and the one-forms ω, $\mu(\omega)$, …, $\mu^{k-1}(\omega)$ ($\hat{\omega}$, $\hat{\mu}(\hat{\omega})$, …, $\hat{\mu}^{k-1}(\hat{\omega})$) are linearly independent over $\mathcal{A}[\vartheta]$ ($\hat{\mathcal{A}}[\vartheta]$).

Definition 12: We say that system (1) is *accessible* if there is no nonzero autonomous one-form of (1). Similarly, system (2) is *accessible* if there is no nonzero autonomous one-form of (2). Otherwise the considered systems are said to be *non-accessible*.

Similarly as in [1] one can show that

Proposition 13: ω is an autonomous one-form of (1) ((2)) if and only if $\omega \in \mathcal{H}_\infty$ ($\hat{\mathcal{H}}_\infty$).

Corollary 14: System (1) ((2)) is accessible if and only if $\mathcal{H}_\infty = \{0\}$ ($\hat{\mathcal{H}}_\infty = \{0\}$).

IV. REALIZATIONS OF I/O EQUATION

In this section we study the realizability of (4) (or equivalently, (1)). Let $\xi : \mathcal{A} \to \hat{\mathcal{A}}$ be the map given by

$$\begin{aligned}&\xi(y^{(i)}[j]) := \mathrm{d}^i\hat{/}\mathrm{d}t^i(h(x[\ell+j]; \ell = 0,\dots,q));\\&\qquad\qquad i = 0,\dots,n-1, j \in \{0\}\cup\mathbb{N}\,,\\&\xi(u^{(k)}[j]) := u^{(k)}[j]; \quad k, j \in \{0\}\cup\mathbb{N}\,;\\&\xi(G(y^{(n-1)}[j],\dots,y[j],u^{(\kappa)}[j],\dots,u[j])) :=\\&\quad G(\xi(y^{(n-1)}[j]),\dots,\xi(y[j]),\xi(u^{(\kappa)}[j]),\\&\qquad\qquad \dots,\xi(u[j])), \quad j \in \{0\}\cup\mathbb{N}\,.\end{aligned} \tag{28}$$

Definition 15: System (5) is called a *realization of* (4) if the map ξ satisfies the condition

$$\begin{aligned}\xi(F(y^{(n-1)}[j],\dots,y[j],u^{(s-1)}[j],\dots,u[j];\\ j = 0,1\dots,p)) = h^{(n)}\end{aligned} \tag{29}$$

where $h^{(n)} := \mathrm{d}^n\hat{/}\mathrm{d}t^n(h(x[\ell]; \ell = 0,\dots,q))$.

Remark 16: Observe that if we replace $\mathcal{X}$ in realization (5) by a smaller open subset $\mathcal{X}'$, then we still get a realization of (4).

Using the introduced rings and their modules similarly as in [7], [5], one can formulate the conditions that guarantee the existence of realization (2) of i/o equation (1).

Theorem 17: [7] There exists a strongly observable realization (2) of dimension $\tilde{n} = n$ of i/o equation (1) if and only if $\mathcal{H}_{s+1}$ is integrable.

Basing on the results given in [5] we get

Proposition 18: There exists a weakly observable and accessible realization (2) of (1) with $\tilde{n} < n$ if and only if (1) is non-accessible and $\mathcal{H}_{s+1}$ is integrable.

A. Relation between rings and modules for an i/o equation and its realization

Let (5) be a realization of i/o equation (4). Then $\mathcal{A}$ is the ring of functions depending on a finite number of variables from the set $Y \cup U$ and $\hat{\mathcal{A}}$ is the ring of functions in finite number of variables from the set $\{x[j], u^{(i)}[j]; i, j \in \{0\}\cup\mathbb{N}\}$. In rings $\mathcal{A}$ and $\hat{\mathcal{A}}$ we have respectively a time-derivative operators $\mathrm{d}/\mathrm{d}t : \mathcal{A} \to \mathcal{A}$, $\mathrm{d}\hat{/}\mathrm{d}t : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$, the delay operators $D : \mathcal{A} \to \mathcal{A}$, $\hat{D} : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$, the polynomial rings $\mathcal{A}[\vartheta]$, $\hat{\mathcal{A}}[\vartheta]$ and the modules $\mathcal{N}$, $\hat{\mathcal{N}}$ of one-forms. Observe that the operators $\mathrm{d}\hat{/}\mathrm{d}t$ and $\hat{D}$ act on $\hat{\mathcal{A}}$ similarly as $\mathrm{d}/\mathrm{d}t$ and D act on $\mathcal{A}$.

Note that $\xi : \mathcal{A} \to \hat{\mathcal{A}}$ defined by (28) is a homomorphism of rings and obviously, $\xi \circ D = \hat{D} \circ \xi$. Moreover, one can easily show the following proposition holds.

Proposition 19: Let $G \in \mathcal{A}$. Then $\xi\left(\frac{\mathrm{d}}{\mathrm{d}t}(G)\right) = \frac{\hat{\mathrm{d}}}{\mathrm{d}t}(\xi(G))$.

There exists a natural extension of the homomorphism ξ to the polynomial rings. Namely, $\xi : \mathcal{A}[\vartheta] \to \hat{\mathcal{A}}[\vartheta]$ is a homomorphism of rings defined by

$$\xi\left(\sum_{i=0}^{\eta}\alpha_i\vartheta^i\right) := \sum_{i=0}^{\eta}\xi\left(\alpha_i\right)\vartheta^i\,, \tag{30}$$

where $\alpha_i \in \mathcal{A}$, $\eta \geq 0$ and $i = 0,\dots,\eta$. Moreover, one can extend the map ξ to the modules and $\xi : \mathcal{N} \to \hat{\mathcal{N}}$ is a map of modules defined as follows

$$\begin{aligned}&\xi\left(\sum_{i=0}^{n-1}a_i\mathrm{d}y^{(i)} + \sum_{j=0}^{\kappa}b_j\mathrm{d}u^{(j)}\right)\\&\quad := \sum_{i=0}^{n-1}\xi\left(a_i\right)\mathrm{d}\left(\xi\left(y^{(i)}\right)\right) + \sum_{j=0}^{\kappa}\xi\left(b_j\right)\mathrm{d}u^{(j)}\,,\end{aligned} \tag{31}$$

where $a_i, b_j \in \mathcal{A}[\vartheta]$, $i = 0,\dots,n-1$, $\kappa \geq 0$ and $j = 0,\dots,\kappa$. Additionally, it is easy to show using induction principle that the following proposition holds.

Proposition 20: Let $\nu \in \mathcal{N}$. Then $\xi(\mu^k(\nu))) = \hat{\mu}^k(\xi(\nu))$ for $k \geq 1$.

Now, we give relation between submodules $\mathcal{H}_\infty$ and $\hat{\mathcal{H}}_\infty$ that are associated with accessibility property, see Corollary 14.

Proposition 21: Let $\mathcal{H}_\infty \subset \mathcal{N}$ and $\hat{\mathcal{H}}_\infty \subset \hat{\mathcal{N}}$ be submodules given by (21) and (22), respectively. Then

$$\xi(\mathcal{H}_\infty) \subseteq \hat{\mathcal{H}}_\infty\,. \tag{32}$$

Proof: Let $\nu \in \mathcal{H}_\infty$. By Lemma 4, $\omega \in \mathcal{H}_\infty$ if and only if there exists $k \in \mathbb{N}$ such that $\omega,\dots,\mu^k(\nu)$ are linearly dependent over $\mathcal{A}[\vartheta]$, i.e. $\sum_{i=0}^{k}\alpha_i(\vartheta)\mu^i(\nu) = 0$. By (28) we get $\xi(0) = 0$ and consequently, $\xi\left(\sum_{i=0}^{k}\alpha_i(\vartheta)\mu^i(\nu)\right) = 0$. It is equivalent to $\sum_{i=0}^{k}\xi(\alpha_i(\vartheta))\xi(\mu^i(\nu)) = 0$. Using the fact that $\xi(\mu^i(\nu)) = \hat{\mu}^i(\xi(\nu))$ we get $\sum_{i=0}^{k}\xi(\alpha_i(\vartheta))\hat{\mu}^i(\xi(\nu)) = 0$. Hence by Lemma 4, $\xi(\nu) \in \hat{\mathcal{H}}_\infty$. ■

If M is a subset of the module $\hat{\mathcal{N}}$, then $\hat{\mathcal{A}}[\vartheta]M$ means the submodule of $\hat{\mathcal{N}}$ generated by M.

Corollary 22: By relation (32) given in Proposition 21 and the fact that $\hat{\mathcal{H}}_\infty$ is closed one gets

$$cl_{\hat{\mathcal{A}}[\vartheta]}\hat{\mathcal{A}}[\vartheta]\xi(\mathcal{H}_\infty) \subset \hat{\mathcal{H}}_\infty\,. \tag{33}$$

Now, let us study the properties of the map ξ with regard to observability.

Proposition 23: If realization (5) of (4) is weakly observable, then $\tilde{n} \leq n$.

Proof: Weak observability means that for $i = 1,\dots,\tilde{n}$ there are $\alpha_i[\vartheta], \beta_{ij}[\vartheta], \gamma_{ik}[\vartheta] \in \hat{\mathcal{A}}[\vartheta]$, $j = 0,\dots,\tilde{n}-1$, $k = 0,\dots,\tilde{n}-2$ such that $\alpha_i[\vartheta]dx_i = \sum_{j=0}^{\tilde{n}-1}\beta_{ij}[\vartheta]dh^{(j)} + \sum_{k=0}^{\tilde{n}-2}\gamma_{ik}[\vartheta]du^{(k)}$. Since $dh^{(j)}$, $j = 0,\dots,\tilde{n}-1$, contain dx_i, $i = 1,\dots,\tilde{n}$, and $\alpha_i[\vartheta]dx_i$ are linearly independent, $dh^{(j)}$, $j = 0,\dots,\tilde{n}-1$, must also be linearly independent. From the definition of realization $dh,\dots,dh^{(n)}$ are linearly independent over $\hat{\mathcal{A}}[\vartheta]$ (modulo $du^{(k)}$). So $\tilde{n} \leq n$. ■

Observe that Proposition 23 gives only a necessary condition for the weak observability of a realization. The sufficient and necessary condition for this property can be expressed by using modules $\mathcal{N}$ and $\hat{\mathcal{N}}$ as follows:

Theorem 24: Realization (5) of (4) is weakly observable if and only if

$$cl_{\hat{\mathcal{A}}[\vartheta]}\hat{\mathcal{A}}[\vartheta]\xi(\mathcal{N}) = \hat{\mathcal{N}}\,. \tag{34}$$

Proof: "⇒" Assume that the realization is weakly observable. Then by Proposition 23, $\tilde{n} \leq n$ and for $i = 1,\dots,\tilde{n}$ there is $\alpha_i[\vartheta] \in \hat{\mathcal{A}}[\vartheta]$ such that $\alpha_i[\vartheta]dx_i \in \mathrm{span}_{\hat{\mathcal{A}}[\vartheta]}\{\xi(dy^{(j)}), \xi(du^{(k)}), j = 0,\dots,\tilde{n}-1, k = 0,\dots,\tilde{n}-2\} \subseteq \mathrm{span}_{\hat{\mathcal{A}}[\vartheta]}\xi(\mathcal{N}) = \hat{\mathcal{A}}[\vartheta]\xi(\mathcal{N})$. Moreover $du^{(k)} \in \hat{\mathcal{A}}[\vartheta]\xi(\mathcal{N})$ for $k \geq 0$. Therefore $\hat{\mathcal{N}} \subseteq cl_{\hat{\mathcal{A}}[\vartheta]}\hat{\mathcal{A}}[\vartheta]\xi(\mathcal{N})$. Because $\hat{\mathcal{N}}$ is closed and $\xi(\mathcal{N}) \subset \hat{\mathcal{N}}$, then $cl_{\hat{\mathcal{A}}[\vartheta]}\hat{\mathcal{A}}[\vartheta]\xi(\mathcal{N}) \subseteq \hat{\mathcal{N}}$. This gives (34).

"$\Leftarrow$" Assume that (34) holds. This implies that for $i = 1,\dots,\tilde{n}$ there is $\alpha_i[\vartheta] \in \hat{\mathcal{A}}[\vartheta]$ such that

$$\begin{aligned}\alpha_i[\vartheta]dx_i &= \sum_{j=0}^{n-1}\beta_{ij}[\vartheta]\xi(dy^{(j)}) + \sum_{k=0}^{\kappa}\gamma_{ik}[\vartheta]\xi(du^{(k)}) \\ &= \sum_{j=0}^{n-1}\beta_{ij}[\vartheta]dh^{(j)} + \sum_{k=0}^{\kappa}\gamma_{ik}[\vartheta]du^{(k)}\end{aligned} \tag{35}$$

for some $\kappa \geq 0$, $\beta_{ij}[\vartheta], \gamma_{ik}[\vartheta] \in \hat{\mathcal{A}}[\vartheta]$. Assume that $n \leq \tilde{n}$. Since in $dh^{(j)}$ only $du,\dots,du^{(j-1)}$ can appear, (35) implies that $\kappa \leq n-2 \leq \tilde{n}-2$. This means that the realization is weakly observable. Now assume that $\tilde{n} < n$. First observe that for $l \geq \tilde{n}$ $dh,\dots,dh^{(\tilde{n}-1)}, dh^{(l)}$ are linearly dependent modulo $du^{(k)}$, $k \geq 0$, so for some $\varepsilon_{lj}[\vartheta] \in \hat{\mathcal{A}}[\vartheta]$, $j = 0,\dots,\tilde{n}-1$, $l \geq \tilde{n}$, we get $\varepsilon_{l0}[\vartheta]dh + \dots \varepsilon_{l,\tilde{n}-1}[\vartheta]dh^{(\tilde{n}-1)} + \varepsilon_{ll}[\vartheta]dh^{(l)} = 0$ (mod $du^{(k)}$, $k \geq 0$). Using left fractions of the ring $\hat{\mathcal{A}}[\vartheta]$ we get $dh^{(l)} = -\varepsilon_{ll}[\vartheta]^{(-1)}\varepsilon_{l0}[\vartheta]dh + \dots + \varepsilon_{ll}[\vartheta]^{(-1)}\varepsilon_{l,\tilde{n}-1}[\vartheta]dh^{(\tilde{n}-1)}$ for $l \geq \tilde{n}$. After substituting it to (35) we can express $\alpha_i[\vartheta]dx_i$ as a linear combinations of $dh,\dots,dh^{(\tilde{n}-1)}$ and $du^{(k)}$, $k \geq 0$. Now observe that $\beta_{ij}[\vartheta]\varepsilon_{jj}[\vartheta]^{-1} = \tilde{\varepsilon}_{jj}[\vartheta]^{-1}\tilde{\beta}_{ij}[\vartheta]$ for some $\tilde{\varepsilon}_{jj}[\vartheta], \tilde{\beta}_{ij}[\vartheta] \in \hat{\mathcal{A}}[\vartheta]$ (from left Ore property). Multiplying both sides of (35) by $\tilde{\varepsilon}_{jj}[\vartheta]$ from the left allows to eliminate this denominator from the right-hand side of (35). Continuing this procedure allows for elimination of other denominators. This means weak observability of the realization. ■

From Proposition 23 we get the following implication:

Corollary 25: If a realization is strongly or regularly observable, then $\tilde{n} \leq n$.

Similarly as for the weak observability modules $\mathcal{N}$ and $\tilde{\mathcal{N}}$ can be used to check whether a realization is regularly observable and the following theorem holds:

Theorem 26: A realization (5) of (4) is regularly observable if and only if

$$\xi(\mathcal{N}) = \tilde{\mathcal{N}} \tag{36}$$

after possibly reducing the state space of the realization.

Proof: "$\Leftarrow$" Assume that $\xi(\mathcal{N}) = \hat{\mathcal{N}}$. Then for every $i = 1,\dots,\tilde{n}$ $dx_i = \sum_{j=0}^{n-1}\xi(\alpha_{ij}[\vartheta])\xi(dy^{(j)}) + \sum_{k=0}^{\kappa}\xi(\beta_{ik}[\vartheta])\xi(du^{(k)}) = \sum_{j=0}^{n-1}\hat{\alpha}_{ij}[\vartheta]dh^{(j)} + \sum_{k=0}^{\kappa}\hat{\beta}_{ik}[\vartheta]du^{(k)}$ for some $\alpha_{ij}[\vartheta], \beta_{ik}[\vartheta] \in \mathcal{A}[\vartheta]$ and $\hat{\alpha}_{ij}[\vartheta] = \xi(\alpha_{ij}[\vartheta]) \in \hat{\mathcal{A}}[\vartheta]$, $\hat{\beta}_{ik}[\vartheta] = \xi(\beta_{ik}[\vartheta]) \in \hat{\mathcal{A}}[\vartheta]$. As in the proof of Proposition 23 and Theorem 24 we can show that $dh,\dots,dh^{(\tilde{n}-1)}$ must be linearly independent and $\kappa \leq n-2$. Then $n \geq \tilde{n}$ and the realization is regularly observable.

"$\Rightarrow$" Assume that the realization is regularly observable. Then, from Proposition 23, $n \geq \tilde{n}$, and for $i = 1,\dots,\tilde{n}$ there are $\hat{\alpha}_{ij}[\vartheta], \hat{\beta}_{ik}[\vartheta] \in \hat{\mathcal{A}}[\vartheta]$ such that

$$dx_i = \sum_{j=0}^{m-1}\hat{\alpha}_{ij}[\vartheta]dh^{(j)} + \sum_{k=0}^{m-2}\hat{\beta}_{ik}[\vartheta]du^{(k)} \tag{37}$$

for $m \geq \tilde{n}$. For $j \geq n$, $dh^{(j)}$ is a linear combination of $dh,\dots,dh^{(n-1)}, du\dots,du^{(n-2)}$, so we can assume that $m \leq n$. The right-hand side of (37) belongs to a codistribution spanned by $dh^{(j)}[r]$, $du^{(k)}[r]$, where $j = 0,\dots,n-1$, $k = 0,\dots,n-2$, $r = 0,\dots,s$. After restricting to some smaller set we may assume that this codistribution has a constant dimension. Using Lemma 6.2 in [8] we get $x_i = \phi_i(dh^{(j)}[r], u^{(k)}[r], j = 0,\dots,n-1, k = 0,\dots,n-2, r = 0,\dots,s) = \xi(\phi_i(dy^{(j)}[r], u^{(k)}[r], j = 0,\dots,n-1, k = 0,\dots,n-2, r = 0,\dots,s))$ for some analytic functions ϕ_i, $i = 1,\dots,\tilde{n}$ and some $s \geq 0$. This means that $\xi(\mathcal{A}) = \hat{\mathcal{A}}$, so also $\xi(\mathcal{N}) = \hat{\mathcal{N}}$. ■

From the proof of Theorem 24 it follows that when the realization (5) is regulary observable, then $\xi: \mathcal{A} \to \hat{\mathcal{A}}$ is an epimorphism. Moreover, the following Proposition holds:

Proposition 27: If system (5) is a regularly observable realization of (4), then

$$\xi(\mathcal{H}_\infty) = \hat{\mathcal{H}}_\infty\,. \tag{38}$$

Proof: By Proposition 21 one gets (32). Let us prove $\hat{\mathcal{H}}_\infty \subseteq \xi(\mathcal{H}_\infty)$. Let $\hat{v} \in \hat{\mathcal{H}}_\infty$. Then using the fact that $\xi(\mathcal{N}) = \hat{\mathcal{N}}$ and $\xi(\mathcal{A}) = \hat{\mathcal{A}}$, one gets $\xi(v) = \hat{v}$ for some $v \in \mathcal{N}$. Since for some $k \in \mathbb{N}$ and $\hat{\alpha}_i \in \hat{\mathcal{A}}[\vartheta]$, $i = 0,\dots,k$, $\sum_{i=0}^{k}\hat{\alpha}_i(\vartheta)\hat{\mu}^i(\hat{v}) = 0$ and $\hat{\alpha}_i = \xi(\alpha_i)$ for $\alpha_i \in \mathcal{A}[\vartheta]$, one gets $\xi(\sum_{i=0}^{k}\alpha_i(\vartheta)\mu^i(v)) = 0$. Hence $\sum_{i=0}^{k}\alpha_i(\vartheta)\mu^i(v) \in \mathcal{H}_\infty$. By Lemma 4 there exist $\beta_j \in \mathcal{A}[\vartheta]$, $j = 0,\dots,\ell$ such that $\sum_{j=0}^{\ell}\beta_j(\vartheta)\mu^j\left(\sum_{i=0}^{k}\alpha_i(\vartheta)\mu^i(v)\right) = 0$. Then

$$\sum_{j=0}^{\ell}\sum_{i=0}^{k}\sum_{s=0}^{j}\beta_j(\vartheta)\binom{j}{s}\frac{\mathrm{d}^s}{\mathrm{d}t^s}(\alpha_i)(\vartheta)\mu^{i+j-s}(v) = 0$$

and consequently, by Lemma 4 we get $v \in \mathcal{H}_\infty$. ■

Proposition 28: If system (5) is a weakly observable realization of (4), then

$$cl_{\hat{\mathcal{A}}[\vartheta]}\hat{\mathcal{A}}[\vartheta]\xi(\mathcal{H}_\infty) = \hat{\mathcal{H}}_\infty\,. \tag{39}$$

Proof: By Corollary 22 one gets (33). Therefore one has to show that $\hat{\mathcal{H}}_\infty \subseteq cl_{\hat{\mathcal{A}}[\vartheta]}\hat{\mathcal{A}}[\vartheta]\xi(\mathcal{H}_\infty)$. Assume that $\hat{\omega} \in \hat{\mathcal{H}}_\infty$. Then there are $\hat{\alpha}[\vartheta], \hat{\beta}[\vartheta] \in \hat{\mathcal{A}}[\vartheta]$ and $\omega \in \mathcal{N}$ such that $\hat{\alpha}[\vartheta]\hat{\omega} = \hat{\beta}[\vartheta]\xi(\omega)$. As $\hat{\omega} \in \hat{\mathcal{H}}_\infty$, then $\tilde{\omega} := \hat{\alpha}[\vartheta]\hat{\omega} \in \hat{\mathcal{H}}_\infty$ as well. This means that $\tilde{\omega}^{(i)} \in \hat{\mathcal{H}}_1 = \mathrm{span}_{\hat{\mathcal{A}}[\vartheta]}\{dx_j, j = 1,\dots,\tilde{n}\}$ for $i \geq 0$. Thus $(\hat{\beta}[\vartheta]\xi(\omega))^{(i)} \in \hat{\mathcal{H}}_1$ for $i \geq 0$. From $\hat{\beta}[\vartheta]\xi(\omega) \in \hat{\mathcal{H}}_1$ it follows that $\xi(\omega) \in \hat{\mathcal{H}}_1$ ($\hat{\mathcal{H}}_1$ is closed). Then, similarly $\hat{\mu}(\xi(\omega)) \in \hat{\mathcal{H}}_1$ and consequently $(\xi(\omega))^{(i)} \in \hat{\mathcal{H}}_1$ for $i \geq 0$. But $(\xi(\omega))^{(i)} = \xi(\omega^{(i)})$, so $\xi(\omega^{(i)}) \in \hat{\mathcal{H}}_1$. Weak observability of realization (5) implies that $\omega^{(i)} \in \mathcal{H}_1$ for $i \geq 0$, which means that $\omega \in \mathcal{H}_\infty$. ■

Proposition 29: If $\xi: \mathcal{N} \to \hat{\mathcal{N}}$ is injective, then $\tilde{n} \geq n$.

Proof: Assume that $\xi: \mathcal{N} \to \hat{\mathcal{N}}$ is injective. Then $dh^{(k)} = \xi(dy^{(k)})$, $k = 0,\dots,n-1$ are linearly independent. Note that $dh = \sum_{i=1}^{\tilde{n}}\hat{\alpha}_{ki}[\vartheta]dx_i$ and $dh^{(k)} = \sum_{i=1}^{\tilde{n}}\hat{\alpha}_{ki}[\vartheta]dx_i + \sum_{j=0}^{k-1}\hat{\beta}_{kj}[\vartheta]du^{(j)}$, $k = 1,\dots,n-1$ for

$\hat{\alpha}_{ki}[\vartheta], \hat{\beta}_{kj}[\vartheta] \in \hat{\mathcal{A}}[\vartheta]$ and one forms: dx_i, $i = 1, \ldots, \tilde{n}$, $du^{(j)}$, $j = 0, \ldots, n-2$ are linearly independent. Hence we get $\tilde{n} \geq n$. ■

Proposition 30: If realization (5) of (4) is weakly observable and $\xi : \mathcal{N} \to \hat{\mathcal{N}}$ is injective, then $\tilde{n} = n$ and $\operatorname{rank} \mathcal{H}_\infty = \operatorname{rank} \hat{\mathcal{H}}_\infty$.

Proof: Weak observability of realization and injectivity of ξ imply that $\tilde{n} = n$. Moreover, $\xi(\mathcal{H}_\infty)$ is a free module (over $\mathcal{A}[\vartheta]$) whose rank is equal to $\operatorname{rank} \mathcal{H}_\infty$. Since $\operatorname{span}_{\hat{\mathcal{A}}[\vartheta]} \xi(\mathcal{H}_\infty)$ is also a free module (over $\hat{\mathcal{A}}[\vartheta]$) of the same rank, then $\operatorname{rank} \hat{\mathcal{H}}_\infty = \operatorname{rank} cl_{\hat{\mathcal{A}}[\vartheta]} \operatorname{span}_{\hat{\mathcal{A}}[\vartheta]} \xi(\mathcal{H}_\infty) = \operatorname{rank} \mathcal{H}_\infty$. ■

Corollary 31: If $\xi : \mathcal{N} \to \hat{\mathcal{N}}$ is bijective, then $\tilde{n} = n$.

From Propositions 21 and 29 one gets the following property:

Proposition 32: If realization (5) of (4) is accessible and ξ is injective, then i/o system (4) is accessible.

Theorem 33: If realization (5) is strongly observable, then

$$\operatorname{rank} \mathcal{H}_\infty = \operatorname{rank} \hat{\mathcal{H}}_\infty + n - \tilde{n}\,. \tag{40}$$

Proof: Strong observability implies that $\xi(\mathcal{H}_\infty) = \hat{\mathcal{H}}_\infty$. Then $\hat{\mathcal{H}}_\infty \cong \mathcal{H}_\infty / \ker \xi$, where $\ker \xi = \{\omega \in \mathcal{H}_\infty : \xi(\omega) = 0\}$. Since $\hat{\mathcal{H}}_\infty$ and $\mathcal{H}_\infty$ are free, so is $\ker \xi$. Let us find its basis. From strong observability we get that $dh^{(i)} = \sum_{j=0}^{\tilde{n}-1} \hat{\alpha}_{ij}[\vartheta] dh^{(j)} + \sum_{k=0}^{s-1} \hat{\beta}_{ik}[\vartheta] du^{(k)}$ for $i = \tilde{n}, \ldots, n-1$ and $\hat{\alpha}_{ij}[\vartheta], \hat{\beta}_{ik}[\vartheta] \in \hat{\mathcal{A}}[\vartheta]$. Therefore $\hat{\alpha}_{ij}[\vartheta] = \xi(\alpha_{ij}[\vartheta])$ and $\hat{\beta}_{ik}[\vartheta] = \xi(\beta_{ik}[\vartheta])$ for $\alpha_{ij}[\vartheta], \beta_{ik}[\vartheta] \in \mathcal{A}[\vartheta]$, and

$$\omega_i := dy^{(i)} - \sum_{j=0}^{\tilde{n}-1} \alpha_{ij}[\vartheta] dy^{(j)} - \sum_{k=0}^{s-1} \beta_{ik}[\vartheta] du^{(k)} \in \ker \xi$$

for $i = \tilde{n}, \ldots, n-1$. Observe that $\omega_{\tilde{n}}, \ldots, \omega_{n-1}$ are linearly independent. It can be shown that they belong to $\mathcal{H}_\infty$ and they span $\ker \xi$. This implies that $\operatorname{rank} \ker \xi = n - \tilde{n}$ and $\operatorname{rank} \mathcal{H}_\infty = \operatorname{rank} \hat{\mathcal{H}}_\infty + n - \tilde{n}$. ■

Remark 34: Using similar techniques one can show that (40) holds under the assumption that realization (5) is weakly observable. This fact was earlier proved in [5] using different methods.

Example 35: [5] Consider the retarded second order i/o equation

$$\ddot{y}(t) = 2u(t)^2 + \frac{(\dot{y}(t) - u(t-1))\dot{u}(t)}{u(t)} + \dot{u}(t-1)\,. \tag{41}$$

From [5] the system

$$\begin{aligned} \dot{x}_1(t) &= 2u(t)x_2(t) + u(t-1) \\ \dot{x}_2(t) &= u(t) \\ y(t) &= x_1(t) \end{aligned} \tag{42}$$

is a realization of (41). It is strongly observable but not accessible, since $dx_1 - (2x_2 + \vartheta)dx_2$ is an autonomous one-form of (42). Then $\xi : \mathcal{A} \to \hat{\mathcal{A}}$ is given by

$$\begin{aligned} &\xi(y[j]) := x_1[j], \\ &\xi(y^{(1)}[j]) := u[j+1] + 2u[j]x_2[j], j \in \{0\} \cup \mathbb{N}, \\ &\xi(u^{(i)}[j]) := u^{(i)}[j], i, j \in \{0\} \cup \mathbb{N}\,. \end{aligned}$$

Note that ξ is an isomorphism of rings. Moreover $\hat{\mathcal{H}}_1 = \operatorname{span}_{\hat{\mathcal{A}}[\vartheta]}\{dx_1, dx_2\} = \operatorname{span}_{\hat{\mathcal{A}}[\vartheta]}\{\xi(dy), \xi(d\psi)\}$, where $\psi(y^{(1)}, u, u[1]) := \frac{y^{(1)} - u[1]}{2u}$ and $\hat{\mathcal{H}}_\infty = \operatorname{span}_{\hat{\mathcal{A}}[\vartheta]}\{dx_1 - (2x_2 + \vartheta)dx_2\} = \operatorname{span}_{\hat{\mathcal{A}}[\vartheta]}\{\xi(dy - (2\psi + \vartheta)d\psi)\}$. Let $\upsilon := y - \psi^2 - \psi[1]$. Then $\mathcal{H}_\infty := \operatorname{span}_{\mathcal{A}[\vartheta]}\{d\upsilon\}$. Since $d\upsilon = dy - (2\psi + \vartheta)d\psi$, one gets $\xi(\mathcal{H}_\infty) = \hat{\mathcal{H}}_\infty$, see Proposition 27. Then $d\upsilon$ is an autonomous one-form of (41) and ξ is an isomorphism of modules $\mathcal{H}_\infty$ and $\hat{\mathcal{H}}_\infty$. Let us consider now a weakly observable and accessible realization of (41) given by

$$\begin{aligned} \dot{x}(t) &= u(t) \\ y(t) &= x(t)^2 + x(t-1) \end{aligned} \tag{43}$$

(see [5]). Then

$$\begin{aligned} \xi(y[j]) &:= x[j]^2 + x[j+1], \\ \xi(y^{(1)}[j]) &= 2x[j]u[j] + u[j+1], \\ \xi(u^{(i)}[j]) &:= u^{(i)}[j], i, j \in \{0\} \cup \mathbb{N}\,. \end{aligned}$$

Then one gets $\hat{\mathcal{H}}_1 = \operatorname{span}_{\hat{\mathcal{A}}[\vartheta]}\{dx\} = \operatorname{span}_{\hat{\mathcal{A}}[\vartheta]}\{\xi(d\psi)\}$ and $\hat{\mathcal{H}}_\infty = \{0\}$. Note that $\xi(d\upsilon) = \xi(dy - d\psi^2 - d\psi[1]) = d\xi(y) - dx^2 - dx[1] = 0$, so $\xi(\mathcal{H}_\infty) = \{0\}$ and Proposition 28 holds. Then ξ is not injective and $\ker \xi|_{\mathcal{H}_\infty} = \mathcal{H}_\infty$.

V. CONCLUSIONS

The paper addresses the problem of relations between the rings and modules associated with single-input single-output time-delay nonlinear i/o equations and their realizations. The algebraic approach based on polynomial tools and modules of differential one-forms is used to study those relations. We plan to use these methods to transform a given realization to a better one, e.g. observable or accessible.

REFERENCES

[1] Z. Bartosiewicz, A. Kaldmäe, Ü. Kotta, E. Pawluszewicz, A. Simha, and M. Wyrwas. Accessibility and system reduction of nonlinear time-delay control systems. *IEEE Transactions on Automatic Control*, 66(8):3781-3788, 2021. DOI:10.1109/tac.2020.3028566

[2] M. Bronstein, M. Petkovšek, An introduction to pseudo-linear algebra, Theoret. Comput. Sci. 157(1):3–33, 1996.

[3] C. Califano and C. H. Moog. Observability of nonlinear time-delay systems and its application to their state realization. IEEE Control Systems Letters, 4(4):803-808, 2020.

[4] M. Halas and M. Anguelova. When retarded nonlinear time-delay systems admit an input-output representation of neutral type. Automatica, 49:561-567, 2013.

[5] A. Kaldmae, Z. Bartosiewicz, U. Kotta, and M. Wyrwas. Minimal Realizations of Nonlinear Time-delay Systems. *IEEE Transactions on Automatic Control*, published online, 26 January 2023, DOI: 10.1109/TAC.2023.3240124.

[6] A. Kaldmäe, C. Califano, and C. H. Moog. Integrability for nonlinear time-delay systems. *IEEE Transactions on Automatic Control*, 61(7):1912-1917, 2016.

[7] A. Kaldmäe and Ü. Kotta. Realization of time-delay systems. *Automatica*, 90:317-320, 2018.

[8] A. J. van der Schaft. On realization of nonlinear systems described by higher-order differential equations, Mathematical Systems Theory, 19(3):239-275, 1987.

[9] X. Xia, L. A. Márquez-Martínez, P. Zagalak, and C. H. Moog. Analysis of nonlinear time-delay systems using modules over non-commutative rings. *Automatica*, 38(9):1549-1555, 2002.