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Distributed Dynamic Event-Triggered Communication Mechanisms for Dynamic Average Consensus

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Abstract— **This paper studies the dynamic average consensus problem of multi-agent systems under event-triggered communication. In this problem, each agent has access to a time-varying reference signal and aims to track the average of all reference signals. Distributed algorithms with event-triggered communication have been developed to achieve dynamic average consensus. Nevertheless, these existing event-triggered communication mechanisms cannot guarantee the existence of a designable positive minimum inter-event time (MIET), which is important in their practical implementation. Motivated by this observation, we propose a distributed dynamic event-triggered communication mechanism (ETCM) for each agent. It is shown that the proposed ETCM guarantees the existence of a positive MIET that is locally adjustable by tuning design parameters. It is also shown that the dynamic average consensus is achieved with any pre-specified level of accuracy. As an illustrative example, the theoretical results are applied to a networked battery energy storage system for state-of-charge balancing and desired total power tracking.**

Index Terms— **Distributed algorithms, dynamic average consensus, event-triggered communication, multi-agent systems.**

I. INTRODUCTION

The dynamic average consensus problem of a multi-agent system requires agents to reach an agreement on the average of some time-varying reference signals, given that each agent has access to only one reference signal [1]. The distributed nature of the reference signals motivates the development of distributed solutions to the dynamic average consensus problem that only rely on local interaction and decision among agents (see, for example, [2–11]). Distributed solutions to the dynamic average consensus problem find their applications in various problems including distributed formation control of networked mobile robots [1], distributed state estimation of wireless sensors [12], distributed resource allocation [13, 14], distributed optimization [15] and distributed learning [16].

Distributed solutions to the dynamic average consensus problem evolve either in continuous time [2–8] or in discrete time [9–11]. Continuous-time solutions, which are analyzed and designed using control-perspective methods such as root locus, Nyquist criterion, and Lyapunov methods, typically exhibit good convergence properties and offer flexible design choices. However, the need for continuous communication among agents makes such algorithms impractical in many real-world scenarios. Discrete-time algorithms, whether obtained by discretizing continuous-time algorithms or designed using discrete-time approaches, require agents to communicate only at discrete-time steps, with a fixed step size. The design of appropriate step size is essential to ensure convergence but choosing a fixed step size may lead to inefficient use of the network resources.

To further improve communication efficiency while preserving the benefits of continuous-time algorithms, [17] proposes distributed dynamic average consensus algorithms with continuous-time computation and event-triggered communication. These algorithms rely on the ability of advanced processors to approximate continuous-time computation and an event-triggered communication mechanism (ETCM) to initiate communication at certain time instants. ETCMs can be either static or dynamic (see, for example, [18–21]), with the latter incorporating an extra dynamic trigger variable. For the dynamic average consensus problem, [17] designs a static ETCM with a positive constant triggering threshold for a strongly connected and weight-balanced network and a static state-dependent ETCM for an undirected and connected network. Practical tracking of the averaged signal is achieved under these two ETCMs. [22] proposes a dynamic ETCM for an undirected and connected network that achieves perfect tracking of the averaged signal.

For practical implementation, ETCMs should guarantee the existence of a locally designable positive minimum inter-event time (MIET). In discrete-time algorithms with event-triggered communication, the sampling period naturally acts as a positive MIET. Nevertheless, the sampling is required to be globally synchronized for all agents [23]. Although the ETCMs for continuous-time algorithms such as those in [17] and [22] guarantee a positive MIET, the MIET in [22] is not designable, and the MIET in [17] is determined by global information on the communication network and initial conditions of global states and thus not locally designable. Motivated by this observation, we propose a distributed dynamic ETCM for continuous-time dynamic average consensus. We will establish the existence of a positive MIET that is locally adjustable by the tuning design parameters. The tracking of the averaged signal will be shown to be achieved with any pre-specified level of accuracy.

The remainder of this paper is outlined as follows. In

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Section II, some preliminaries, including the notation, system dynamics and communication network, are provided, and the dynamic average consensus problem of a multi-agent system is formulated. Section III reviews an existing dynamic average consensus algorithm. Section IV presents the proposed distributed dynamic ETCMs with positive MIET guarantees. In Section V, a simulation example is given to validate the effectiveness of the proposed design. Finally, Section VI concludes this paper.

II. Preliminaries and Problem Statement

A. Notation

Let $\mathbb{R}$ and $\mathbb{N}$ be the set of real numbers and the set of non-negative integers, respectively. For a scalar a, $|a|$ represents its absolute value. For a vector v, $\|v\|$ represents its Euclidean norm. For a matrix A, A^{T} represents its transpose. Let $\mathbf{1}_N$ be the column vector with N elements of 1, $\mathbf{0}_N$ be the column vector with N elements of 0, and I_N be the identity matrix of dimensions $N \times N$. Given scalars $a_1, a_2, \cdots, a_N$, $\mathrm{diag}\{a_1, a_2, \cdots, a_N\}$ represents the diagonal matrix with the diagonal elements being $a_1, a_2, \cdots, a_N$.

B. System Dynamics

Consider a multi-agent system with N agents, whose dynamics is described by

$$\dot{x}_i(t) = u_i(t), t \geq 0, i = 1, 2, \cdots, N, \tag{1}$$

where $x_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ are the state and the driving command of agent i, respectively. The agent dynamics under consideration is simple. This is because the state $x_i(t)$ corresponds to some local variable such as the decision variable of a distributed estimator, rather than some physical variable governed by complex agent dynamics. In our problem setting, each agent i has access to a time-varying reference signal $r_i(t) \in \mathbb{R}$, and is able to communicate with its neighbors through a communication network.

C. Communication Network

The communication network among all agents is represented by an undirected graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}\}$, where $\mathcal{N} = \{1, 2, \cdots, N\}$ is the node set and $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is the edge set. An edge $(i, j) \in \mathcal{E}$ indicates that node i is able to send information to node j. Let the set of neighbors of node i be $\mathcal{N}_i = \{j \in \mathcal{N} : (j, i) \in \mathcal{E}\}$. A graph $\mathcal{G}$ is said to be undirected if $(i, j) \in \mathcal{E}$ implies that $(j, i) \in \mathcal{E}$. An undirected graph $\mathcal{G}$ is said to be connected if every pair of nodes in graph $\mathcal{G}$ are connected by a sequence of edges.

Let the adjacency matrix of graph $\mathcal{G}$ be $A = [a_{ij}] \in \mathbb{R}^{N\times N}$, where $a_{ii} = 0$, $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $i \neq j$, and $a_{ij} = 0$ if $(j, i) \notin \mathcal{E}$ and $i \neq j$. For an undirected graph $\mathcal{G}$, $a_{ij} = a_{ji}$. Let the Laplacian matrix of graph $\mathcal{G}$ be $L = [l_{ij}] \in \mathbb{R}^{N\times N}$, where $l_{ij} = -a_{ij}$ for $i \neq j$ and $l_{ii} = \sum_{k=1,k\neq i}^{N} a_{ik}$. Let the eigenvalues of the Laplacian matrix L be λ_i, $i = 1, 2, \cdots, N$. For a connected undirected graph $\mathcal{G}$, $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$, $\mathbf{1}_N^{\mathrm{T}} L = L\mathbf{1}_N = \mathbf{0}$, and $x^{\mathrm{T}} L x \geq \lambda_2 \|x\|^2$ holds for any column vector x satisfying $\mathbf{1}_N^{\mathrm{T}} x = 0$. These properties of the Laplacian matrix L can be found in [24].

D. Problem Statement

The objective of this paper is to design a distributed ETCM for each agent such that the dynamic average consensus can be achieved among all agents.

Before proceeding, we make the following mild assumptions on the reference signals and the communication network.

Assumption 1: For each agent i, the reference signal $r_i(t)$ satisfies $|\dot{r}_i(t)| \leq \bar{r}_i$, where $\bar{r}_i$ is a positive constant.

Assumption 2: The graph $\mathcal{G}$ that represents the communication network is undirected and connected.

The problem to be studied is stated as follows.

Problem 1: Consider a multi-agent system (1) under Assumptions 1 and 2. Design a distributed ETCM for each agent, such that its state tracks the average of all reference signals with a level of accuracy specified by $\varepsilon_{\mathrm{r}} > 0$, that is,

$$\limsup_{t\to\infty} |x_i(t) - r_{\mathrm{a}}(t)| \leq \varepsilon_{\mathrm{r}}, i = 1, 2, \cdots, N, \tag{2}$$

where $r_{\mathrm{a}}(t) \triangleq \frac{1}{N}\sum_{i=1}^{N} r_i(t)$.

Remark 1: As observed in [8], for *linear* dynamic average consensus algorithms such as those in [3] and [5], exact convergence cannot be achieved. In [3] and [5], the steady-state error ε_{r} is dependent on the rate of variation of the reference signals $r_i(t)$, $i = 1, 2, \cdots, N$. Because it takes time for the given dynamics of the agents to react to the variation of the reference signals, as in [3] and [5], we make Assumption 1 on the bound of the variation rate of $r_i(t)$.

III. Dynamic Average Consensus Algorithm

The distributed dynamic average consensus algorithm proposed in [5] is given by

$$\dot{x}_i(t) = \dot{r}_i(t) - \alpha\left(x_i(t) - r_i(t)\right) - \beta\sum_{j=1}^{N} a_{ij}\left(x_i(t) - x_j(t)\right) - v_i(t), \tag{3a}$$

$$\dot{v}_i(t) = \alpha\beta\sum_{j=1}^{N} a_{ij}\left(x_i(t) - x_j(t)\right),\ v_i(0) = 0, \tag{3b}$$

where $\alpha, \beta > 0$ are design parameters and $v_i(t)$ is an internal state. By Corollary 4.1 in [5], the state $x_i(t)$ in the algorithm (3a)-(3b) exponentially converges to an adjustable neighborhood of the average of the reference signals.

Let $\{t_k^i : k \in \mathbb{N}\}$ be the sequence of event times at which agent i communicates with its neighbors. Then, based on event-triggered communication, the dynamic average consensus algorithm (3a)-(3b) is implemented as [17]

$$\dot{x}_i(t) = \dot{r}_i(t) - \alpha(x_i(t) - r_i(t)) - \beta\sum_{j=1}^{N} a_{ij}\left(\hat{x}_i(t) - \hat{x}_j(t)\right) - v_i(t), \tag{4a}$$

$$\dot{v}_i(t) = \alpha\beta\sum_{j=1}^{N} a_{ij}(\hat{x}_i(t) - \hat{x}_j(t)),\ v_i(0) = 0, \tag{4b}$$

where $\hat{x}_i(t) = x_i(t_k^i)$, $t \in \left[t_k^i, t_{k+1}^i\right)$. The event times t_k^i, $k \in \mathbb{N}$, will be determined in the next section.

IV. Dynamic Event-Triggered Communication Mechanism

In this section, we provide our distributed dynamic ETCMs with positive MIET guarantees.

A. Event-Triggered Communication Mechanism

For each agent i, we will design a distributed dynamic ETCM in order to determine the time instants when agent i broadcasts $x_i(t)$ to its neighbors j, $j \in \mathcal{N}_i$.

First, we define the following measurement error:

$$e_i(t) = \hat{x}_i(t) - x_i(t) = x_i(t_k^i) - x_i(t), t \in \left[t_k^i, t_{k+1}^i\right). \quad (5)$$

Note that $e_i(t)$ is reset to 0 at each event time t_k^i. With the measurement error $e_i(t)$, we define a dynamic trigger variable governed by

$$\begin{cases} \dot{\xi}_i(t) = \min\{\varpi_i(t), 0\} - \sigma_i, \text{ for } t \in (t_k^i, t_{k+1}^i), \\ \xi_i(t) = \bar{\xi}_i, \text{ for } t = t_k^i, \end{cases} \quad (6)$$

where $\bar{\xi}_i, \sigma_i > 0$ are design parameters. Here, $\varpi_i(t)$ is defined as follows,

$$\varpi_i(t) = \frac{\beta}{8\left|e_i(t)\right|^2} \sum_{j=1}^{N} a_{ij}\left(\hat{x}_i(t) - \hat{x}_j(t)\right)^2 - \mu_i \xi_i^2(t) - \kappa_i \xi_i(t) - \beta d_i, \text{ if } |e_i(t)| \neq 0, \quad (7)$$

with $d_i \triangleq \sum_{i=1}^{N} a_{ij}$ and $\mu_i \triangleq 8\beta d_i + 2\epsilon_i^2 + 2$, and $\varpi_i(t) = 0$ if $|e_i(t)| = 0$. In (7), $\epsilon_i > 0$ and $\kappa_i > 0$ are design parameters and κ_i is chosen such that $0 < \kappa_i \leq 2\sqrt{\beta\mu_i d_i}$. With the trigger variable $\xi_i(t)$, we design a distributed dynamic ETCM for each agent i as

$$t_{k+1}^i = \inf\left\{t > t_k^i \mid \xi_i(t) \leq 0\right\}, \quad (8)$$

with $t_0^i = 0$. This ETCM is said to be dynamic because of the presence of an additional dynamic variable $\xi_i(t)$. The dynamics of $\xi_i(t)$ only depends on information of agent i and its immediate neighbors. In this sense, the ETCM (8) involving $\xi_i(t)$ is distributed. Note that the expression of $\varpi_i(t)$ in (7) is derived to guarantee the convergence of the algorithm (4a)-(4b) and the existence of a positive MIET.

B. Lower Bounds of the Inter-Event Times

The following technical lemma provides a positive lower bound of the inter-event times for each agent under ETCM (8).

Lemma 1: Consider the distributed dynamic ETCM (8) with the trigger variable $\xi_i(t)$ updated by (6). Then, the trigger variable $\xi_i(t)$ satisfies $0 \leq \xi_i(t) \leq \bar{\xi}_i$, for all $t \geq 0$. Moreover, the inter-event times $t_{k+1}^i - t_k^i$, for each agent i, have a positive lower bound

$$\tau_i = \frac{1}{\sqrt{\mu_i c_i}}\left[\arctan\left(\sqrt{\frac{\mu_i}{c_i}}\left(\bar{\xi}_i + \frac{\kappa_i}{2\mu_i}\right)\right) - \arctan\left(\sqrt{\frac{\mu_i}{c_i}}\frac{\kappa_i}{2\mu_i}\right)\right], \quad (9)$$

where $c_i \triangleq \sigma_i + \beta d_i - \frac{\kappa_i^2}{4\mu_i} > 0$.

Proof: According to the trigger dynamics (6), $\dot{\xi}_i(t) < 0$ for $t \in (t_k^i, t_{k+1}^i)$ and $\xi_i(t_k^{i+}) = \bar{\xi}_i$. Hence, $\xi_i(t)$ drops from $\bar{\xi}_i$ during each time interval $\left[t_k^i, t_{k+1}^i\right)$. ETCM (8) enforces that $\xi_i(t) \geq 0$. As a result, $\xi_i(t)$ satisfies $0 \leq \xi_i(t) \leq \bar{\xi}_i$, for all $t \geq 0$. To calculate a lower bound of inter-event times, we evaluate the trigger variable $\xi_i(t)$ during each time interval $\left[t_k^i, t_{k+1}^i\right)$. Specifically, we consider the following two cases over the time interval $\left[t_k^i, t_{k+1}^i\right)$. In the case of $|e_i(t)| \neq 0$, $\varpi_i(t)$ is lower-bounded by

$$\varpi_i(t) \geq -\mu_i\left(\xi_i(t) + \frac{\kappa_i}{2\mu_i}\right)^2 + \frac{\kappa_i^2}{4\mu_i} - \beta d_i. \quad (10)$$

Recalling $0 < \kappa_i \leq 2\sqrt{\beta\mu_i d_i}$, we have $\frac{\kappa_i^2}{4\mu_i} - \beta d_i \leq 0$. Furthermore, in view of (10), $\dot{\xi}_i(t)$ satisfies

$$\dot{\xi}_i(t) \geq -\mu_i\left(\xi_i(t) + \frac{\kappa_i}{2\mu_i}\right)^2 - c_i, \quad (11)$$

where c_i is given in (9). Because $\frac{\kappa_i^2}{4\mu_i} - \beta d_i \leq 0$ and $\sigma_i > 0$, we have $c_i \geq \sigma_i > 0$. Thus, in the case of $|e_i(t)| = 0$, $\dot{\xi}_i(t)$ satisfies

$$\dot{\xi}_i(t) = -\sigma_i \geq -\mu_i\left(\xi_i(t) + \frac{\kappa_i}{2\mu_i}\right)^2 - c_i. \quad (12)$$

Combining (11) and (12) yields that in both cases,

$$\dot{\xi}_i(t) \geq -\mu_i\left(\xi_i(t) + \frac{\kappa_i}{2\mu_i}\right)^2 - c_i, \quad (13)$$

for $t \in \left[t_k^i, t_{k+1}^i\right)$. Thus, by the comparison lemma, we obtain that $\xi_i(t) \geq \phi_i(t)$, for $t \in \left[t_k^i, t_{k+1}^i\right)$, where $\phi_i(t)$ is the solution of the differential equation

$$\dot{\phi}_i(t) = -\mu_i\left(\phi_i(t) + \frac{\kappa_i}{2\mu_i}\right)^2 - c_i, \quad (14)$$

with $\phi_i(t_k^{i+}) = \bar{\xi}_i$. Denote $\chi_i(t) = \sqrt{\frac{\mu_i}{c_i}}\left(\phi_i(t) + \frac{\kappa_i}{2\mu_i}\right)$. Then, it follows from (14) that

$$-\frac{1}{\sqrt{\mu_i c_i}}\frac{\dot{\chi}_i(t)}{\chi_i^2(t) + 1} = 1. \quad (15)$$

Integrating both sides of (15) from t_k^i to t yields

$$t - t_k^i = \frac{1}{\sqrt{\mu_i c_i}}\left(\arctan(\chi_i(t_k^i)) - \arctan(\chi_i(t))\right), \quad (16)$$

for $t \in \left[t_k^i, t_{k+1}^i\right)$. Because $\xi_i(t) \geq \phi_i(t)$, the inter-event time $t_{k+1}^i - t_k^i$ is lower-bounded by the amount of time that it takes for $\phi_i(t)$ to drop from $\bar{\xi}_i$ to 0. Then, we conclude from (16) that $t_{k+1}^i - t_k^i \geq \tau_i > 0$. ■

In Lemma 1, the positive lower bound τ_i is referred to as a positive MIET. The MIET τ_i only relies on the design parameters $\beta, \bar{\xi}_i, \sigma_i, \epsilon_i$ and κ_i, and hence can be pre-specified locally by the designer.

C. Exponential Convergence

We are now in a position to analyze the exponential convergence of the dynamic average consensus algorithm (4a)-(4b) under ETCM (8).

Let $\{\bar{t}_m : m \in \mathbb{N}\}$ be the strictly increasing sequence of event times at which at least one agent is triggered. By Lemma 1, the Zeno behavior is excluded. This indicates that $\lim_{k\to\infty} t_k^i = \infty$, for $i = 1, 2, \cdots, N$. Hence, $\lim_{m\to\infty} \bar{t}_m = \infty$. Let $x(t) = [x_1(t), x_2(t), \cdots, x_N(t)]^{\mathrm{T}}$, $r(t) = [r_1(t), r_2(t), \cdots, r_N(t)]^{\mathrm{T}}$, $v(t) = [v_1(t), v_2(t), \cdots, v_N(t)]^{\mathrm{T}}$, and $e(t) = [e_1(t), e_2(t), \cdots, e_N(t)]^{\mathrm{T}}$. As in [17], using the orthogonal transfer matrix $\Xi = [\eta, R] \in \mathbb{R}^{N\times N}$, with $\eta = \frac{1}{\sqrt{N}}\mathbf{1}_N$, we define the following variable transformations:

$$z_1(t) = \eta^{\mathrm{T}} y(t), z_{2:N}(t) = R^{\mathrm{T}} y(t),$$
$$q_1(t) = \eta^{\mathrm{T}} w(t), q_{2:N}(t) = \alpha R^{\mathrm{T}} y(t) + R^{\mathrm{T}} w(t),$$

where $y(t) \triangleq x(t) - \mathbf{1}_N r_{\mathrm{a}}(t)$ and $w(t) \triangleq v(t) - \alpha \Pi r(t)$, with $\Pi = I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}$. As shown in [17], for all $t \geq 0$, $q_1(t) = 0$ and $q_{2:N}(t) = q_{2:N}(0)\mathrm{e}^{-\alpha t}$. Then, the system composed of $z_1(t)$, $z_{2:N}(t)$ and $e(t)$, for $t \in [\bar{t}_m, \bar{t}_{m+1})$, is

$$\dot{z}_1(t) = -\alpha z_1(t), \tag{17a}$$

$$\dot{z}_{2:N}(t) = -\beta R^{\mathrm{T}} L R z_{2:N}(t) - \beta R^{\mathrm{T}} L e(t) + R^{\mathrm{T}}\dot{r}(t) - q_{2:N}(t), \tag{17b}$$

$$\dot{e}(t) = \beta L e(t) + \alpha\eta z_1(t) + \beta L R z_{2:N}(t) + R q_{2:N}(t) - \dot{r}(t). \tag{17c}$$

By analyzing the stability of the system in (17a)-(17c), we are ready to present the convergence result of the dynamic average consensus algorithm (4a)-(4b) under ETCM (8).

Theorem 1: Consider the distributed event-triggered dynamic average consensus algorithm (4a)-(4b) under ETCM (8). There exists constant $\gamma > 0$ such that, for any $\alpha > 0$ and $\beta > 0$, the state $x_i(t)$ exponentially converges to a neighborhood of $r_{\mathrm{a}}(t)$, that is,

$$\limsup_{t\to\infty} |x_i(t) - r_{\mathrm{a}}(t)| \leq \frac{\gamma}{\theta\beta\lambda_2} + \gamma\left[\left(\frac{1}{\theta\beta\lambda_2}\right)^2 + \frac{1}{\epsilon_{\min}^2\theta\beta\lambda_2}\right]^{\frac{1}{2}}, \tag{18}$$

where $\epsilon_{\min} = \min_i \epsilon_i > 0$ and $0 < \theta < 1$, and the rate of exponential convergence is $\varrho = \min\{\alpha, (1-\theta)\beta\lambda_2, \kappa_{\min}\}$, with $\kappa_{\min} = \min_i \kappa_i$.

Proof: Consider the Lyapunov function candidate

$$W(z_1(t), z_{2:N}(t), e(t)) = \frac{1}{2}\alpha|z_1(t)|^2 + \frac{1}{2}z_{2:N}^{\mathrm{T}}(t)z_{2:N}(t) + \sum_{i=1}^{N}\xi_i(t)|e_i(t)|^2. \tag{19}$$

At time $t = \bar{t}_m$, at least one agent, say agent i, is triggered. In this case, $\xi_i(\bar{t}_m^-) = 0$, $\xi_i(\bar{t}_m^+) = \bar{\xi}_i$, and $e_i(\bar{t}_m^+) = 0$. Thus, $\xi_i(t)|e_i(t)|^2$ is continuous at $t = \bar{t}_m$ for any $m \in \mathbb{N}$. Furthermore, $\xi_i(t)|e_i(t)|^2$ is continuous over the time interval $[0, \infty)$. Note that $z_1(t)$ and $z_{2:N}(t)$ are always continuous. Thus, $W(t)$ is continuous over the time interval $[0, \infty)$. Under Assumption 1, there exists constant $\gamma > 0$ such that $\|R^{\mathrm{T}}\dot{r}(t)\| \leq \gamma$ and $\|\dot{r}(t)\| \leq \gamma$. The time derivative of $W(t)$ along the trajectories of (17a)-(17c) over each time interval $[\bar{t}_m, \bar{t}_{m+1})$ is evaluated as

$$\begin{aligned}\dot{W}(t) \leq{}& -\alpha^2|z_1(t)|^2 - \frac{1}{2}\beta z_{2:N}^{\mathrm{T}}(t)R^{\mathrm{T}}LRz_{2:N}(t) \\ &- \frac{1}{2}\beta\hat{x}^{\mathrm{T}}(t)L\hat{x}(t) + \frac{1}{2}\beta e^{\mathrm{T}}(t)Le(t) \\ &+ \left(\gamma + \|q_{2:N}(0)\|\mathrm{e}^{-\alpha t}\right)\|z_{2:N}(t)\| \\ &+ 2\beta e^{\mathrm{T}}(t)\Xi(t)L\hat{x}(t) \\ &+ 2e^{\mathrm{T}}(t)\Xi(t)\left(\alpha\eta z_1(t) + Rq_{2:N}(t)\right) \\ &- 2e^{\mathrm{T}}(t)\Xi(t)\dot{r}(t) + \sum_{i=1}^{N}\dot{\xi}_i(t)|e_i(t)|^2,\end{aligned} \tag{20}$$

where $\Xi(t) \triangleq \mathrm{diag}\{\xi_1(t), \xi_2(t), \cdots, \xi_N(t)\}$ and we have used $RR^{\mathrm{T}} = \Pi$, $L\Pi = \Pi L = L$, and $\mathbf{1}_N^{\mathrm{T}}L = L\mathbf{1}_N = \mathbf{0}$. Using Young's inequality $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$, with $\varepsilon > 0$, leads to

$$\begin{aligned}&\frac{1}{2}\beta e^{\mathrm{T}}(t)Le(t) \\ &\leq \frac{1}{2}\beta\sum_{i=1}^{N}d_i|e_i(t)|^2 + \frac{1}{4}\beta\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left(|e_i(t)|^2 + |e_j(t)|^2\right) \\ &= \beta\sum_{i=1}^{N}d_i|e_i(t)|^2,\end{aligned} \tag{21}$$

$$\begin{aligned}&2\beta e^{\mathrm{T}}(t)\Xi(t)L\hat{x}(t) \\ &\leq 8\beta\sum_{i=1}^{N}d_i\xi_i^2(t)|e_i(t)|^2 + \frac{1}{8}\beta\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left(\hat{x}_i(t) - \hat{x}_j(t)\right)^2,\end{aligned} \tag{22}$$

$$\begin{aligned}&2e^{\mathrm{T}}(t)\Xi(t)\left(\alpha\eta z_1(t) + Rq_{2:N}(t)\right) - 2e^{\mathrm{T}}(t)\Xi(t)\dot{r}(t) \\ &\leq \frac{1}{2}\alpha^2|z_1(t)|^2 + \sum_{i=1}^{N}\left(2\epsilon_i^2 + 2\right)\xi_i^2(t)|e_i(t)|^2 \\ &\quad + \frac{1}{2\epsilon_{\min}^2}\left(\gamma + \|q_{2:N}(0)\|\mathrm{e}^{-\alpha t}\right)^2,\end{aligned} \tag{23}$$

where we have used $\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}|e_j(t)|^2 = \sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}|e_i(t)|^2$, as $a_{ij} = a_{ji}$. Substituting (21), (22) and (23) in (20) and recalling the property of the Laplacian matrix L in Section II.C yield

$$\begin{aligned}\dot{W}(t) \leq{}& -\frac{1}{2}\alpha^2|z_1(t)|^2 - \frac{1}{2}(1-\theta)\beta\lambda_2\|z_{2:N}(t)\|^2 \\ &- \frac{1}{8}\beta\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left(\hat{x}_i(t) - \hat{x}_j(t)\right)^2 + \beta\sum_{i=1}^{N}d_i|e_i(t)|^2 \\ &+ \sum_{i=1}^{N}\left(8\beta d_i + 2\epsilon_i^2 + 2\right)\xi_i^2(t)|e_i(t)|^2 \\ &+ \sum_{i=1}^{N}\dot{\xi}_i(t)|e_i(t)|^2 + \Delta,\end{aligned} \tag{24}$$

where $\Delta \triangleq -\frac{1}{2}\theta\beta\lambda_2\|z_{2:N}(t)\|^2 + (\gamma + \|q_{2:N}(0)\|\mathrm{e}^{-\alpha t})\|z_{2:N}(t)\| + \frac{1}{2\epsilon_{\min}^2}(\gamma + \|q_{2:N}(0)\|\mathrm{e}^{-\alpha t})^2$. Note that $\Delta \leq 0$ if $\|z_{2:N}(t)\|$ satisfies

$$\begin{aligned}\|z_{2:N}(t)\| \geq{}& \frac{\gamma + \|q_{2:N}(0)\|\mathrm{e}^{-\alpha t}}{\theta\beta\lambda_2} + \left(\gamma + \|q_{2:N}(0)\|\mathrm{e}^{-\alpha t}\right) \\ &\times\left[\left(\frac{1}{\theta\beta\lambda_2}\right)^2 + \frac{1}{\epsilon_{\min}^2\theta\beta\lambda_2}\right]^{\frac{1}{2}} \triangleq \zeta(t).\end{aligned} \tag{25}$$

Thus, we can obtain from (6) and (24) that

$$\begin{aligned}&\dot{W}(t) \\ &\leq -\frac{1}{2}\alpha^2|z_1(t)|^2 - \frac{1}{2}(1-\theta)\beta\lambda_2\|z_{2:N}(t)\|^2 \\ &\quad - \frac{1}{8}\beta\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left(\hat{x}_i(t) - \hat{x}_j(t)\right)^2 + \beta\sum_{i=1}^{N}d_i|e_i(t)|^2 \\ &\quad + \sum_{i=1}^{N}\left(8\beta d_i + 2\epsilon_i^2 + 2\right)\xi_i^2(t)|e_i(t)|^2 + \sum_{i=1}^{N}\dot{\xi}_i(t)|e_i(t)|^2 \\ &\leq -\frac{1}{2}\alpha^2|z_1(t)|^2 - \frac{1}{2}(1-\theta)\beta\lambda_2\|z_{2:N}(t)\|^2 \\ &\quad - \sum_{i=1}^{N}\kappa_i\xi_i(t)|e_i(t)|^2 \\ &\leq -\varrho W(t),\ t \in [\bar{t}_m, \bar{t}_{m+1}),\end{aligned} \tag{26}$$

as long as $\|z_{2:N}(t)\| \geq \zeta(t)$. This implies that when

$\|z_{2:N}(t)\| \geq \zeta(t)$, $W(t) \leq W(0)\mathrm{e}^{-\varrho t}$ and, hence, $\|z_{2:N}(t)\| \leq 2W(t) \leq 2W(0)\mathrm{e}^{-\varrho t}$. Consequently, $\|z_{2:N}(t)\|$ is bounded by

$$\limsup_{t\to\infty} \|z_{2:N}(t)\| \leq \frac{\gamma}{\theta\beta\lambda_2} + \gamma\left[\left(\frac{1}{\theta\beta\lambda_2}\right)^2 + \frac{1}{\epsilon_{\min}^2\theta\beta\lambda_2}\right]^{\frac{1}{2}}. \quad (27)$$

Denote $z(t) = [z_1(t)\ z_{2:N}^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{N\times 1}$. Recalling $z_1(t) = z_1(0)\mathrm{e}^{-\alpha t}$, we obtain that $\|z(t)\|$ is bounded. Note that

$$\|x_i(t) - r_{\mathrm{a}}(t)\| \leq \|x(t) - \mathbf{1}_N r_{\mathrm{a}}(t)\| = \|y(t)\| = \|z(t)\|. \quad (28)$$

Combining (27) and (28), we arrive at (18). ■

Remark 2: By increasing the values of the design parameters β and ϵ_i, the steady-state error in (18) can be made arbitrarily small.

Remark 3: The two types of static ETCMs proposed in [17] for the dynamic average consensus algorithm in (4a)-(4b) are

$$t_{k+1}^i = \inf\left\{t > t_k^i \mid |e_i(t)|^2 \geq \frac{1}{4d_i}\sum_{j=1}^N a_{ij}(\hat{x}_i(t) - \hat{x}_j(t))^2 + \rho_i\right\}, \quad (29)$$

$$t_{k+1}^i = \inf\left\{t > t_k^i \mid |e_i(t)|^2 \geq \rho_i\right\}, \quad (30)$$

where $\rho_i > 0$. Compared to ETCM (30), ETCM (29) can lead to fewer communication events while ensuring a similar consensus performance, as shown in [17]. The MIETs guaranteed by (29) and (30) rely on the initial values of the global states and the global information on the communication network and, hence, cannot be pre-specified locally by the designer, in contrast to MIET (9).

V. Simulation Studies

In this section, we validate our distributed dynamic ETCM on a networked battery energy storage system consisting of ten battery units. The state-of-charge (SoC) of each battery unit i (see [13, 14, 25] for details) is determined by

$$s_i(t) = s_i(0) - \frac{1}{C_i}\int_0^t i_i(\varsigma)d\varsigma,\ i = 1, 2, \cdots, 10, \quad (31)$$

where $s_i(t)$ is the SoC, $i_i(t)$ is the output current, and C_i is the battery capacity. Given a constant output voltage V_i, the output power $p_i(t)$ of battery unit i is $p_i(t) = V_i i_i(t)$. Then, the SoC dynamics can be written as

$$\dot{s}_i(t) = -\frac{1}{C_i V_i} p_i(t). \quad (32)$$

The objective is to achieve SoC balancing among all battery units while delivering the desired total power p_{ref} in discharging or charging mode, that is, $\lim_{t\to\infty}(s_i(t) - s_j(t)) = 0$ and $\lim_{t\to\infty} p_{\Sigma}(t) = p_{\mathrm{ref}}$, where $p_{\Sigma}(t) = \sum_{i=1}^{10} p_i(t)$ is the total power. To achieve this objective, a power control law was designed in [25] as

$$p_i(t) = \frac{r_i(t)}{x_i(t)} p_{\mathrm{a}}, \quad (33)$$

where $p_{\mathrm{a}} = \frac{1}{10}p_{\mathrm{ref}}$ is the average desired total power, $x_i(t)$ is the estimate of $r_{\mathrm{a}}(t) = \frac{1}{10}\sum_{i=1}^N r_i(t)$, and $r_i(t)$ is the unit state of battery unit i defined as

$$r_i(t) = \begin{cases} C_i V_i s_i(t) \text{ (in discharging mode)}, \\ C_i V_i (1 - s_i(t)) \text{ (in charging mode)}. \end{cases} \quad (34)$$

We conduct the dynamic average consensus algorithm (4a)-(4b) under the proposed ETCM (8) in the following case study. The desired total power p_{ref} is 14000 W. The capacities C_i, $1, 2, \cdots, 10$ are $(190, 210, 230, 200, 220, 180, 195, 215, 225, 205)$ Ah. The output voltages V_i, $i = 1, 2, \cdots, 10$, are all 50 V. The initial SoC values are $(0.90, 0.81, 0.75, 0.85, 0.88, 0.76, 0.83, 0.73, 0.75, 0.88)$. The communication topology among ten battery units is shown in Fig. 1. The initial values of the dynamic variables in (4a), (4b) and (8) are $x_i(0) = 5000$, $v_i(0) = 0$ and $\xi_i(0) = \bar{\xi}_i$, $i = 1, 2, \cdots, 10$. The design parameters are chosen as follows: 1) $\alpha = 2$, $\beta = 12$; 2) $\bar{\xi}_i = 5 + 0.1 \times i$, $\sigma_i = 0.1 \times i$, $\epsilon_i = 5 + 0.2 \times i$, $\kappa_i = 2 + 0.1 \times i$, $i = 1, 2, \cdots, 10$. According to Lemma 1, the guaranteed MIETs τ_i, $i = 1, 2, \cdots, 10$, given by (9), are $(0.0194, 0.0134, 0.0190, 0.0132, 0.0186, 0.0184, 0.0182, 0.0128, 0.0178, 0.0126)$ h.

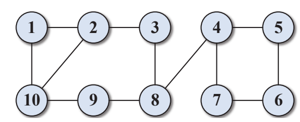

Fig. 1. The communication topology.

The simulation results in the discharging mode are shown in Figs. 2-5. As seen in Figs. 2 and 3, the SoC balancing and the desired total power tracking are achieved with a high level of accuracy, under the power control law (33). Shown in Fig. 4 are the average unit state estimates of all battery units. These estimates are obtained by the dynamic average consensus algorithm (4a)-(4b) under ETCM (8). It can be observed that the average unit state estimates quickly achieve the dynamic average consensus. To illustrate the triggering mechanism, the trigger variable of battery unit 5 is displayed in Fig. 5. As seen, the trigger variable ξ_5 drops from $\bar{\xi}_5$ to 0 and is reset to $\bar{\xi}_5$ whenever ξ_5 reaches 0.

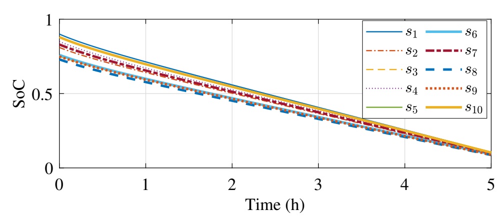

Fig. 2. The SoC of all battery units.

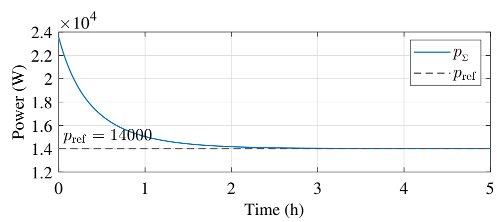

Fig. 3. The total power and the desired power.

Finally, we make a comparison between ETCMs (8) and (29). For comparison, we choose $\rho_i = 20$ in ETCM (29) such that the resulting estimation performance is similar to that of ETCM (8) in Fig. 4. Table I shows the number of events (NUM), the observed MIET and the average inter-event time

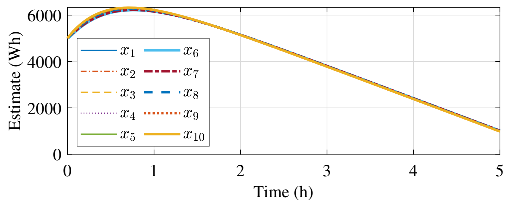

Fig. 4. The average unit state estimates of all battery units.

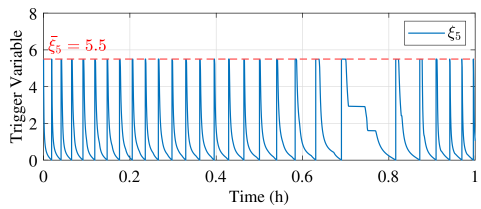

Fig. 5. The trigger variable of battery unit 5.

(AIET) during the first 5 h for battery units $1, 2, \cdots, 5$ under ETCMs (8) and (29). In Table I, it is observed that ETCM (8) can significantly reduce the communication load when compared to ETCM (29).

TABLE I
SIMULATION RESULTS UNDER ETCMS (8) AND (29).

Unit	NUM		Observed MIET (h)		AIET (h)	
	(8)	(29)	(8)	(29)	(8)	(29)
Unit 1	235	1379	0.0200	0.0010	0.0212	0.0036
Unit 2	318	1283	0.0135	0.0010	0.0157	0.0039
Unit 3	225	1083	0.0198	0.0009	0.0222	0.0046
Unit 4	295	986	0.0136	0.0010	0.0169	0.0051
Unit 5	217	899	0.0199	0.0007	0.0230	0.0056

VI. Conclusions

We investigated the dynamic average consensus problem of multi-agent systems under event-triggered communication. In order to address this problem, we reviewed one existing dynamic average consensus algorithm. Based on this algorithm, we designed a distributed dynamic ETCM for each agent such that it can determine when to communicate with its neighbors. An appealing feature of the proposed ETCM for each agent is the guarantee of the existence of a positive MIET that is locally adjustable by tuning the design parameters. It was shown that the dynamic average consensus algorithm under the proposed ETCM exponentially converges to an arbitrarily small neighborhood of the average of all reference signals. The obtained theoretical results were applied to achieve both SoC balancing and desired total power tracking for a networked battery energy storage system.

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