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# Asymptotic Stability Preservation of Input Delayed Nonlinear Systems under Sampled-Data Feedback

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***Abstract*— We study the problem of global asymptotic stability preservation (GASP) for $C^0$ non-smoothly stabilizable systems with input delay under sampled-data feedback. With the aid of Halanay inequality and the notion of homogeneity, the following sampled-data control results are established under a fast sampling: 1) GAS is preservable if the nonlinear system is homogeneous of degree zero and globally asymptotically stabilizable by homogeneous feedback; 2) As a consequence, GAS via sampled-data feedback is achieved for a class of non-smoothly stabilizable systems with input delay in a lower-triangular or upper-triangular form.**

## I. Introduction and Problem Statement

In [29], the stability preservation problem was studied for the nonlinear system

$$\dot{x}(t) = f(x(t), u(t)), \quad f(0,0) = 0, \tag{1.1}$$

under sampled-delayed input, where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and input, respectively.

The main outcomes of [29] are two-fold: i) GASP under sampled-delayed input is possible if the nonlinear system (1.1) is globally Lipschitz continuous (GLC) and globally exponentially stabilizable (GES) by *smooth state* feedback; ii) Semi-GASP is possible if the $C^1$ nonlinear system (1.1) is globally asymptotically locally exponentially stabilizable (GALES) by *smooth feedback*, under a fast sampling and limited input delay.

Following the development of [29], we address in this paper the problem of GASP under sampled-delayed input for the $C^0$ *non-smoothly stabilizable* system (1.1). For technical convenience, the GASP problem is recalled below.

Assume that $u = \alpha(x)$ with $\alpha(0) = 0$ is a $C^0$ globally asymptotically stabilizing (GAS) controller for the $C^0$ system (1.1). That is, the continuous-time closed-loop system $\dot{x} = f(x, \alpha(x))$ is GAS at $x = 0$. When taking into account a delay in the input, the closed-loop system becomes $\dot{x}(t) = f(x(t), \alpha(x(t-d)))$. In practice, the control action is often implemented by digital computer or by "sample and hold" signals $x(t_k)$ at the sampling time $t_k = kT$ for $k = 0, 1, 2, \cdots$, where $T > 0$ is a sampling period. In this case, the feedback controller is generated by $u(t) = u(t_k) = \alpha(x(t_k)),\ t \in [t_k,\ t_{k+1})$. This, combined with the factor of input delay, yields

$$u(t-d) = \alpha(x(t_k)), \qquad t - d \in [t_k,\ t_{k+1}). \tag{1.2}$$

Consequently, the resulting hybrid closed-loop system with delay is described by

$$\dot{x}(t) = f(x(t), \alpha(x(t_k))), \qquad t \in [t_k + d,\ t_{k+1} + d). \tag{1.3}$$

With the help of (1.3) and discussions above, the GASP problem under sampled-delayed input can be formally formulated as follows.

*Definition 1.1:* The $C^0$ non-smoothly stabilizable system (1.1) is called *global asymptotic stability preservable* (GASP) under sampled-delayed input if there is a real constant $\delta^* > 0$ such that the hybrid closed-loop system (1.3) with delay is GAS at $x = 0$ for $(T + d) \in (0, \delta^*]$ and any initial condition $x_0 = \mu \in C([-(d+T), 0], \mathbb{R}^n)$.

To the literature we are aware of, various interesting and important results have been obtained for sampled-data control of time-delay nonlinear systems over the years, as documented, e.g., in [1], [2], [7], [8], [16], [17], [19], [14], [15], and [3], [12], [18], [4], [10], [13], [26], [27] as well as the references therein. One of effective methods for the design of sampled-data feedback controllers is the so-called emulation technique. Roughly speaking, one designs sampled-data feedback controllers by discretizing the corresponding continuous-time controllers with appropriate sampling periods, to achieve local, semi-global and global stability for time-delay nonlinear systems [7], [17], [16], [19], [10]. Both memorized and memoryless sampled-data control schemes have been developed so far.

For example, predictor based sampled-data control strategies were developed in [7], [17] to deal with linear systems or forward complete nonlinear systems with input delay, while the problem of global asymptotic stabilization (GAS) of nonlinear systems with affine input was addressed in [16], using sampled-data memory state feedback. It was showed that, with the aid of some restrictive assumptions, the property of GAS is preservable if the sampling period and input are limited. For time-delay nonlinear systems with the global Lipschitz continuity (GLC), the GAS property was also proved to be preservable via sampled-data feedback with a fast sampling [19]. This was obtained under the very demanding condition that the time-delay systems are globally exponentially stabilizable by memory GLC state feedback in continuous-time. Recently, semi-global exponential stability (SGES) of locally Lipschitz nonlinear systems with *state delay* has been established by memory sampled-data feedback [1], [2], under a time-delay type (infinite-dimensional or FDE version) of the global asymptotic local exponential stabilizability (GALES) characterized in [10] for finite-dimensional systems described

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by nonlinear ODE.

In contrast to the previous works [10], [28], [2], [29] which concentrated primarily on the *semi-global* asymptotic stabilization (SGAS) via sampled-data feedback or *semi-global* input delay tolerance of nonlinear systems, the focus of this paper is on the problem of GAS preservation for non-smoothly stabilizable nonlinear systems subject to sampled-delay input. A Halanay inequality approach is presented for the nonsmooth analysis and synthesis of the GAS property of the time-delay hybrid closed-loop system. Specifically, following the idea of [29], together with the nonsmooth analysis tools, homogeneity, and Halanay inequality, we prove that the time-delay hybrid closed-loop nonlinear system induced by sampled-delayed actuator maintains the GAS property under a fast sampling and limited input delay. The main conclusion is that global asymptotic stability is preserved if the nonlinear system has a dominated homogeneity with zero degree. As a byproduct, global asymptotic stabilization of non-smoothly stabilizable systems with input delay in a lower-triangular or upper-triangular form is shown to be possible by sampled-data feedback with a fast sampling.

## II. Preliminary and Tools

This section review briefly some tools to be used in this research, including the notions of homogeneity with respect to a family of dilations, homogeneous function and homogeneous vector field, and related properties. The reader is referred to the papers [6], [9], [25], [22], [23], [32], [31], the survey papers [5], [24], the book [34] and the references therein for further details.

Consider the autonomous system $\dot{x} = f(x),\ f(0) = 0$, with $f : \mathbb{R}^n \to \mathbb{R}^n$ being a continuous vector field. The following concepts are fundamental in studying homogeneous systems [34], [6], [9], [23], [24], [32], [31].

(a) For $r_i > 0$, $i = 1, \cdots, n$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, the dilation $\Delta_\varepsilon^r$ is defined by $\Delta_\varepsilon^r(x) = (\varepsilon^{r_1}x_1, \cdots, \varepsilon^{r_n}x_n)$, $\forall \varepsilon > 0$, where $r_i$ is the weight of $x_i$ and $r = (r_1, \cdots, r_n)$ is a dilation weight.

(b) A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous of degree $\tau$ if there is a real constant $\tau$, such that $\forall x \in \mathbb{R}^n\backslash\{0\}$, $f_i(\Delta_\varepsilon^r(x)) = \varepsilon^{\tau+r_i}f_i(x)$, for $i = 1, \cdots, n$.

(c) $\dot{x} = f(x) = [f_1(x) \cdots f_n(x)]^T$ is a homogeneous system of degree $\tau$ if the vector field $f$ is homogeneous of degree $\tau$ with respect to the dilation $\Delta_\varepsilon^r(x)$.

(d) For a real number $p \geq \max\{r_i,\ i = 1, \cdots, n\}$, a homogeneous $p-$norm is defined by $\|x\|_{\Delta,p} = (\sum_{i=1}^n |x_i|^{p/r_i})^{1/p}$, $\forall x \in \mathbb{R}^n$. For simplicity, $\|x\|_\Delta$ stands for $\|x\|_{\Delta,p}$ in this paper.

(e) A function $V : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree $k$ if there is a real constant $k$, such that $\forall x \in \mathbb{R}^n\backslash\{0\}$, $V(\Delta_\varepsilon^r(x)) = \varepsilon^k V(x)$.

A homogeneous Lyapunov function has the following important properties.

*Lemma 2.1:* ([34], [6], [23], [24], [32], [31]) Assume that $V : \mathbb{R}^n \to \mathbb{R}$ is a $C^1$ homogeneous function of degree $k$ with respect to the dilation $\Delta_\varepsilon^r(x)$ and $V(0) = 0$. Then,

i) $\partial V/\partial x_i$ is homogeneous of degree $k - r_i$;
ii) $\frac{\partial V}{\partial x}f(x)$ is homogeneous of degree $k + \tau$;
iii) There is a constant $c_2 > 0$ such that $V(x) \leq c_2\|x\|_\Delta^k$;
iv) If $V(x)$ is positive definite, there is a constant $c_1 > 0$ such that $c_1\|x\|_\Delta^k \leq V(x)$;
v) Let $V_1(x)$ and $V_2(x)$ be homogeneous functions of degree $k_1$ and $k_2$ with respect to the dilation $\Delta_\varepsilon^r(x)$. Then, $V_1(x)V_2(x)$ is homogeneous of degree $k_1 + k_2$ with respect to the same dilation.

The following lemmas are very useful in addressing sampled-data control of nonlinear systems with input delay.

*Lemma 2.2:* [31] Let $g : \mathbb{R} \to \mathbb{R}^n$ be a continuous mapping on the interval $[a, b]$. Then for any $p > \max_{1\leq i\leq n}\{r_i\}$, there exists a positive constant $p_0 < p$ such that

$$\Big\| \int_a^b g(s)\mathrm{d}s\Big\|_{\Delta^r,p}^p \leq (b-a)^{\frac{p-p_0}{p_0}} \int_a^b \big\|g(s)\big\|_{\Delta^r,p}^p \mathrm{d}s. \quad (2.4)$$

*Lemma 2.3:* [31] If $r_i > 0$, $i = 1, \cdots, n$, the homogeneous $p$-norm satisfies $\|x + z\|_{\Delta^r,p} \leq c_3(\|x\|_{\Delta^r,p} + \|z\|_{\Delta^r,p})$, $\forall x, z \in \mathbb{R}^n$, where $c_3 \geq 1$ is a constant.

*Lemma 2.4:* [31] Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^0$ homogeneous vector field of degree $\tau = 0$ with respect to the dilation $\Delta_\varepsilon^r(x) = (\varepsilon^{r_1}x_1, \cdots, \varepsilon^{r_n}x_n)$. Then, for any $\rho > 0$, there is a constant $L > 0$, such that $\forall x, \tilde{x} \in \mathbb{R}^n$,

$$\|f(x) - f(\tilde{x})\|_{\Delta^r,p} \leq L\|x - \tilde{x}\|_{\Delta^r,p} + \rho\|x\|_{\Delta^r,p}. \quad (2.5)$$

*Lemma 2.5:* (Halanay inequality [19]) Let $a, b, r$ be positive real constants with $a > b$, and $t_0$ be a real number. Assume that $z : [t_0 - r, +\infty) \to \mathbb{R}_+$ is a continuous function satisfying the inequality

$$D^+z(t) \leq -az(t) + b \sup_{\theta\in[-r,0]} z(t + \theta), \quad \forall t \geq t_0.$$

Then, $z(t) \leq \sup_{\theta\in[t_0-r,t_0]} z(\theta)e^{-\lambda t}$, where $D^+$ denotes the upper right-hand Dini derivative, $\lambda > 0$ is a real solution of the equation $\lambda + be^{\lambda r} = a$.

## III. Main Result

The main result of this paper is presented in this section, which addresses the GASP problem for the input delayed nonlinear system (1.1) under sampled-data feedback via zero-order holder. The class of nonlinear systems under consideration is characterized by the following conditions.

*Assumption 3.1:* The vector field $f(x, u)$ is homogeneous of degree $\tau = 0$ with respect to the dilation $\Delta_\varepsilon^{\bar{r}}(x, u) = (\varepsilon^{r_1}x_1, \cdots, \varepsilon^{r_n}x_n, \varepsilon^{r_{n+1}}u_1, \cdots, \varepsilon^{r_{n+m}}u_m)$, i.e.,

$$\begin{aligned} &f_i(\varepsilon^{r_1}x_1, \cdots, \varepsilon^{r_n}x_n, \varepsilon^{r_{n+1}}u_1, \cdots, \varepsilon^{r_{n+m}}u_m) \\ &= \varepsilon^{r_i}f(x_1, \cdots, x_n, u_1, \cdots, u_m),\ i = 1, 2, \cdots, n \quad (3.6) \end{aligned}$$

$\forall x \in \mathbb{R}^n, \mathrm{u} \in \mathbb{R}^m$ and $\varepsilon > 0$.

*Assumption 3.2:* For the nonlinear system (1.1), there is a $C^0$ GAS controller $u = \alpha(x) = [\alpha_1(x)\ \cdots\ \alpha_m(x)]^T \in \mathbb{R}^m$ with $\alpha(0) = 0$, and $\alpha_j$, $j = 1, \cdots, m$, which is homogeneous of degree $r_{n+j}$ with respect to the dilation $\Delta_{\varepsilon}^{r}(x) = (\varepsilon^{r_1} x_1, \cdots, \varepsilon^{r_n} x_n)$, i.e.,

$$\alpha_j(\varepsilon^{r_1} x_1, \cdots, \varepsilon^{r_n} x_n) = \varepsilon^{r_{n+j}} \alpha_j(x_1, \cdots, x_n), \tag{3.7}$$

$\forall x \in \mathbb{R}^n$ and $\varepsilon > 0$, such that the $C^0$ closed-loop system $\dot{x} = f(x, \alpha(x))$ is GAS at $x = 0$.

Clearly, Assumptions 3.1-3.2 imply that the vector fields $f(x, \alpha(x))$ and $f(x, \alpha(z))$ are homogeneous of degree zero.

With the help of the notion of homogeneity and its properties related to homogeneous vector field, homogeneous function and homogeneous norm, the following result on sampled-data control of the input delayed nonlinear system (1.1) can be proved.

*Theorem 3.3:* Under the Assumptions 3.1 and 3.2, global asymptotic stabilizabiliy of the nonlinear system (1.1) is preservable under input delay and sampled-data state feedback. More precisely, there exists a constant $\delta^* > 0$ such that the sampled-data feedback controller

$$u(t) = u(t_k) = \alpha(x(t_k)), \ \forall t \in [t_k, t_{k+1}), \tag{3.8}$$

with $t_k = kT$, $k = 0, 1, \ldots$, globally asymptotically stabilizs the nonlinear system (1.1) with input delay, if $d + T < \delta^*$.

*Proof:* The proof is divided into two steps. In the first step, we establish the following result.

*Claim 1:* For any initial state $x_0(s) = \phi(s) \in C([-(d + T), 0], \mathbb{R}^n)$, there exists a constant $c_0 > 0$ such that $\|x(t)\|_{\Delta^r,\kappa}^{\kappa} \le c_0 \|\phi\|_{\Delta^r,\kappa}^{\kappa}$, $\forall t \in [0, d)$.

The claim is proved by considering two cases.

*Case A) — When $d \le T$:* In this case, it is clear that $t \in [-T + d, d)$ for $t \in [0, d)$. Thus, $\dot{x}(t) = f(x(t), \alpha(x(-T)))$ when $t \in [0, d)$. By Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
&\|x(t)\|_{\Delta^r,\kappa}^{\kappa} = \|x(0) + \int_0^t f(x(s), \alpha(x(-T)))\mathrm{d}s\|_{\Delta^r,\kappa}^{\kappa} \\
&\le c_1 \|x(0)\|_{\Delta^r,\kappa}^{\kappa} + c_1 \| \int_0^t f(x(s), \alpha(x(-T)))\mathrm{d}s\|_{\Delta^r,\kappa}^{\kappa} \\
&\le c_1 \|x(0)\|_{\Delta^r,\kappa}^{\kappa} + c_1 d^{\mu_1} \int_0^t \|f(x(s), \alpha(x(-T)))\|_{\Delta^r,\kappa}^{\kappa}\mathrm{d}s \\
&\le (c_1 + c_2 d^{\mu_1+1}) \|\phi\|_{\Delta^r,\kappa}^{\kappa} + c_2 d^{\mu_1} \int_0^t \|x(s)\|_{\Delta^r,\kappa}^{\kappa}\mathrm{d}s, 
\end{aligned} \tag{3.9}$$

where $c_1, c_2$ and $\mu_1$ are some positive constants.

It follows from (3.9) and Gronwall-Bellman inequality that $\forall t \in [0, d)$,

$$\|x(t)\|_{\Delta^r,\kappa}^{\kappa} \le (c_1 + c_2 d^{\mu_1+1}) e^{c_2 d^{\mu_1+1}} \|\phi\|_{\Delta^r,\kappa}^{\kappa}. \tag{3.10}$$

*Case B) — When $d > T$:* For $t \in [0, d)$, there is an integer $N \ge 1$ such that $(N-1)T < d \le NT$ and $[0, d) \subseteq [-NT + d, -(N-1)T + d) \cup [-(N-1)T + d, -(N-2)T + d) \cup \cdots \cup [-T + d, d)$. As a result, there must exist a positive integer $N' \le N$ such that $t \in [-N'T + d, -N'T + T + d)$. By Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}
&\|x(t)\|_{\Delta^r,\kappa}^{\kappa} \\
&= \Big\|x(0) + \int_0^{-(N-1)T+d} f(x(s), \alpha(x(-NT)))\mathrm{d}s \\
&\qquad + \int_{-(N-1)T+d}^{-(N-2)T+d} f(x(s), \alpha(x(-(N-1)T)))\mathrm{d}s \\
&\qquad + \cdots + \int_{-N'T+d}^{t} f(x(s), \alpha(x(-N'T)))\mathrm{d}s\Big\|_{\Delta^r,\kappa}^{\kappa} \\
&\le c_3 \|x(0)\|_{\Delta^r,\kappa}^{\kappa} \\
&\quad + c_3 \Big\| \int_0^{-(N-1)T+d} f(x(s), \alpha(x(-NT)))\mathrm{d}s\Big\|_{\Delta^r,\kappa}^{\kappa} \\
&\quad + c_3 \Big\| \int_{-(N-1)T+d}^{-(N-2)T+d} f(x(s), \alpha(x(-(N-1)T)))\mathrm{d}s\Big\|_{\Delta^r,\kappa}^{\kappa} \\
&\quad + \cdots + c_3 \Big\| \int_{-N'T+d}^{t} f(x(s), \alpha(x(-N'T)))\mathrm{d}s\Big\|_{\Delta^r,\kappa}^{\kappa} \\
&\le (c_3 + c_5 d^{\mu_2+1}) \|\phi\|_{\Delta^r,\kappa}^{\kappa} + c_4 d^{\mu_2} \int_0^t \|x(s)\|_{\Delta^r,\kappa}^{\kappa}\mathrm{d}s
\end{aligned} \tag{3.11}$$

where $c_3, c_4, c_5$ and $\mu_2$ are some positive constants.

Using (3.11) and Gronwall-Bellman inequality, we deduce that $\forall t \in [0, d)$,

$$\begin{aligned}
\|x(t)\|_{\Delta^r,\kappa}^{\kappa} &\le (c_3 + c_5 d^{\mu_2+1}) \|\phi\|_{\Delta^r,\kappa}^{\kappa} e^{c_4 d^{\mu_2} t} \\
&\le (c_3 + c_5 d^{\mu_2+1}) e^{c_4 d^{\mu_2+1}} \|\phi\|_{\Delta^r,\kappa}^{\kappa}.
\end{aligned} \tag{3.12}$$

Then, from (3.10) and (3.12), the Claim 1 is true.

Using Claim 1, the continuity of the solution trajectory $x(t)$ of (1.1)-(3.8), and a derivation akin to (3.9), we obtain

$$\begin{aligned}
\|x(t)\|_{\Delta^r,\kappa}^{\kappa} &\le c_6 \|x(d)\|_{\Delta^r,\kappa}^{\kappa} \\
&\qquad + c_6 \Big\| \int_d^t f(x(s), \alpha(x(0)))\mathrm{d}s\Big\|_{\Delta^r,\kappa}^{\kappa} \\
&\le (c_7 + c_7 T^{\mu_3+1}) \|\phi\|_{\Delta^r,\kappa}^{\kappa} + c_7 T^{\mu_3} \int_d^t \|x(s)\|_{\Delta^r,\kappa}^{\kappa}\mathrm{d}s,
\end{aligned} \tag{3.13}$$

where $c_6, c_7$ and $\mu_3$ are some positive constants.

By (3.13) and Gronwall-Bellman inequality, one has

$$\|x(t)\|_{\Delta^r,\kappa}^{\kappa} \le c_8 \|\phi\|_{\Delta^r,\kappa}^{\kappa}, \quad \forall t \in [d, T + d), \tag{3.14}$$

where $c_8 = (c_7 + c_7 T^{\mu_3+1}) e^{c_7 T^{\mu_3+1}}$.

From Claim 1, (3.14), and the continuity of the solution trajectory $x(t)$ of the hybrid closed-loop system (1.1)-(3.8), it is concluded that

$$\|x(t)\|_{\Delta^r,\kappa}^{\kappa} \le c_9 \|\phi\|_{\Delta^r,\kappa}^{\kappa}, \quad \forall t \in [0, T + d], \tag{3.15}$$

where $c_9 = \max\{c_0, c_8\}$.

In the second step, we consider the case when $t \ge T + d$ and prove that the hybrid closed-loop system (1.1) and (3.8) is GAS at $x = 0$ by the Halanay inequality.

By Assumptions 3.1-3.2, the continuous-time closed-loop system $\dot{x} = f(x, \alpha(x))$ is GAS and homogeneous of degree $\tau = 0$ with respect to the dilation $\Delta_{\varepsilon}^{r}(x) =$

$(\varepsilon^{r_1}x_1,\cdots,\varepsilon^{r_n}x_n)$. In view of the converse Lyapunov theorem of homogeneous systems [25], there is a $C^1$ positive definite and proper Lyapunov function $V:\mathbb{R}^n\to\mathbb{R}$, which is homogeneous of degree $\kappa>\max_{1\le i\le n}\{r_i\}$, such that

$$a_1\|x\|^\kappa_{\Delta^r,\kappa}\le V(x)\le a_2\|x\|^\kappa_{\Delta^r,\kappa} \tag{3.16}$$

$$\frac{\partial V}{\partial x}f(x,\alpha(x))\le -a_3\|x\|^\kappa_{\Delta^r,\kappa} \tag{3.17}$$

where $a_1$, $a_2$ and $a_3$ are positive constants, and the homogeneous $p$-norm with $p=\kappa$ is adopted for brevity.

Along the solution trajectories of the hybrid closed-loop system (1.1) and (3.8), It is deduced from (3.17), Lemma 2.1 and Young inequality that for $t\ge T+d$ (without loss of generality, let $t_k+d\le t<t_{k+1}+d$ for some $k\ge 1$),

$$\begin{aligned}\dot V(x(t))&\le -a_3\|x(t)\|^\kappa_{\Delta^r,\kappa}\\&\quad+\sum_{i=1}^n\Big|\frac{\partial V}{\partial x_i}\Big|\big|f_i(x(t),\alpha(x(t)))-f_i(x(t),\alpha(x(t_k)))\big|\\&\le -a_3\|x(t)\|^\kappa_{\Delta^r,\kappa}\\&\quad+b_1\sum_{i=1}^n\|x(t)\|^{\kappa-r_i}_{\Delta^r,\kappa}\cdot\big|f_i(x(t),\alpha(x(t)))-f_i(x(t),\alpha(x(t_k)))\big|\\&\le -\frac{3}{4}a_3\|x(t)\|^\kappa_{\Delta^r,\kappa}\\&\quad+b_2\|f(x(t),\alpha(x(t)))-f(x(t),\alpha(x(t_k)))\|^\kappa_{\Delta^r,\kappa},\end{aligned} \tag{3.18}$$

where $b_1$ and $b_2$ are positive constants.

Note that $f(x,\alpha(z))$ is homogeneous of degree $\tau=0$ with respect to the dilation $\Delta^{\bar{\bar r}}_\varepsilon(x,z)=(\varepsilon^{r_1}x_1,\cdots,\varepsilon^{r_n}x_n,\varepsilon^{r_1}z_1,\cdots,\varepsilon^{r_n}z_n)$. By Lemma 2.4, that there is a constant $b_3>0$ such that

$$\begin{aligned}&b_2\|f(x(t),\alpha(x(t)))-f(x(t),\alpha(x(t_k)))\|^\kappa_{\Delta^r,\kappa}\\&\quad\le\frac{1}{4}a_3\|x(t)\|^\kappa_{\Delta^r,\kappa}+b_3\|x(t)-x(t_k)\|^\kappa_{\Delta^r,\kappa}.\end{aligned} \tag{3.19}$$

Substituting (3.19) into (3.18) yields

$$\dot V(x(t))\le-\frac{1}{2}a_3\|x(t)\|^\kappa_{\Delta^r,\kappa}+b_3\|x(t)-x(t_k)\|^\kappa_{\Delta^r,\kappa}. \tag{3.20}$$

Because $\kappa>\max_{1\le i\le n}\{r_i\}$, it is deduced from Lemma 2.2 the existence of constant $\mu>0$ such that

$$\begin{aligned}b_3\|x(t)-x(t_k)\|^\kappa_{\Delta^r,\kappa}&=b_3\Big\|\int_{t_k}^t D^+x(s)\mathrm{d}s\Big\|^\kappa_{\Delta^r,\kappa}\\&\le b_3(d+T)^\mu\int_{t_k}^t\|D^+x(s)\|^\kappa_{\Delta^r,\kappa}\mathrm{d}s.\end{aligned} \tag{3.21}$$

Putting (3.21) and (3.20) together results in $\forall t\ge d+T$,

$$\begin{aligned}\dot V(x(t))&\le-\frac{1}{2}a_3\|x(t)\|^\kappa_{\Delta^r,\kappa}\\&\quad+b_3(d+T)^\mu\int_{t_k}^t\|D^+x(s)\|^\kappa_{\Delta^r,\kappa}\mathrm{d}s.\end{aligned} \tag{3.22}$$

For any $d,T>0$, there always exists an integer $j<k$ such that $t_k\in[t_j+d,t_{j+1}+d)$, where $t_j=jT$. Then, it follows from Lemma 2.4 that

$$\begin{aligned}&\int_{t_k}^t\|D^+x(s)\|^\kappa_{\Delta^r,\kappa}\mathrm{d}s\\&\le\int_{t_j+d}^{t_{j+1}+d}\|\dot x(s)\|^\kappa_{\Delta^r,\kappa}\mathrm{d}s+\int_{t_{j+1}+d}^{t_{j+2}+d}\|\dot x(s)\|^\kappa_{\Delta^r,\kappa}\mathrm{d}s\\&\quad+\cdots+\int_{t_{k-1}+d}^{t_k+d}\|\dot x(s)\|^\kappa_{\Delta^r,\kappa}\mathrm{d}s+\int_{t_k+d}^{t}\|\dot x(s)\|^\kappa_{\Delta^r,\kappa}\mathrm{d}s\\&\le 2b_4(d+T)\sup_{\theta\in[-2(d+T),0]}\|x(t+\theta)\|^\kappa_{\Delta^r,\kappa},\end{aligned} \tag{3.23}$$

where $b_4>0$ is a constant and is independent on $d,T$.

By (3.16), (3.22) and (3.23), one has

$$\begin{aligned}D^+V(x(t))&\le-a_4V(x(t))+b_5(d+T)^{\mu+1}\\&\quad\cdot\sup_{\theta\in[-2(d+T),0]}V(x(t+\theta)),\quad\forall t\ge T+d\end{aligned} \tag{3.24}$$

where $a_4,b_5$ are positive constants and independent on $d,T$.

Pick $\delta^*=(\frac{a_4}{b_5})^{\frac{1}{\mu+1}}$. In view of (3.24) and Lemma 2.5, there exists a constant $\lambda>0$ such that

$$V(x(t))\le\sup_{\theta\in[-d-T,d+T]}V(x(\theta))e^{-\lambda t},\ \forall d+T<\delta^* \tag{3.25}$$

Using (3.16) and (3.15), we arrive at

$$\sup_{\theta\in[-(d+T),d+T]}V(x(\theta))\le c_{10}\|\phi\|^\kappa_{\Delta^r,\kappa} \tag{3.26}$$

for some constant $c_{10}>0$.

Substituting (3.25) and (3.26) into (3.16) yields $\|x(t)\|_{\Delta^r,\kappa}\le(\frac{c_{10}}{a_1})^{\frac{1}{\kappa}}\|\phi\|_{\Delta^r,\kappa}e^{-\frac{\lambda}{\kappa}t}$. This, in turn, implies that the hybrid closed-loop system (1.1)-(3.8) is GAS. ∎

As a consequence of Theorem 3.3, the following global stabilization results can be obtained immediately.

*Corollary 3.4:* Under Assumptions 3.1 and 3.2, the nonlinear system (1.1) with $d=0$ is GAS by sampled-data state feedback. In particular, there exists a $T^*>0$, such that the sampled-data state feedback control law

$$u(t)=u(t_k)=\alpha(x(t_k)),\ \forall t\in[t_k,t_{k+1}), \tag{3.27}$$

with $t_k=kT,\ k=0,1,\ldots$, renders the system (1.1) with $d=0$ GAS, if the sampling period $T\in(0,T^*]$.

*Corollary 3.5:* Under Assumptions 3.1 and 3.2, there is a $d^*>0$, such that the memoryless state feedback controller $u(t)=\alpha(x(t))$ globally asymptotically stabilizes the nonlinear system (1.1) with input delay if $d\in(0,d^*]$.

## IV. Some Applications

We now apply Theorem 3.3 to derive some important sampled-data feedback control results for input delayed nonlinear systems in a lower-triangular or an upper-triangular form, without local exponential stabilizability (LES). In each case, an explicit formula is given for the design of sampled-data state feedback controllers.

One of the applications of Theorem 3.3 is devoted to a class of nonlinear systems with input delay described by

$$\begin{aligned}\dot x_i(t)&=x_{i+1}^{p_i}(t)+\phi_i(x_1(t),\cdots,x_i(t)),\ i=1,\cdots,n-1\\\dot x_n(t)&=u(t-d)+\phi_n(x_1(t),\cdots,x_n(t)),\end{aligned} \tag{4.28}$$

where $d > 0$ is the input delay, $p_1, \cdots, p_{n-1}$ are *positive odd* integers, and the functions $\phi_i$, $i = 1, \cdots, n$, are $C^0$ with $\phi_i(0, \cdots, 0) = 0$, and are homogeneous of degree $r_i$ with respect to the dilation $\Delta_\varepsilon^r(x) = (\varepsilon^{r_1} x_1, \cdots, \varepsilon^{r_n} x_n)$, with $r_1 = 1$ and $r_i = \frac{1}{p_1 p_2 \cdots p_{i-1}}$, $i = 2, \cdots, n$.

*Corollary 4.1:* There is a constant $\delta^* > 0$ such that the lower-triangular system (4.28) is GAS by sampled-data feedback if $d + T < \delta^*$, where $T > 0$ is the sampling period. In particular, a sampled-data controller is given by

$$\begin{aligned} u(t) = u(t_k) = &-L_1^{\lambda_{n+1}} \beta_n \Big( L_1^{-\lambda_n / r_n} x_n^{\frac{1}{r_n}}(t_k) + \cdots \\ &+ \beta_2 \big( L_1^{-\lambda_2 / r_2} x_2^{\frac{1}{r_2}}(t_k) + \beta_1 x_1(t_k) \big) \dots \Big)^{r_n}, \\ &t \in [t_k, t_{k+1}), \quad t_k = kT, \quad k = 0, 1, 2, \cdots, \end{aligned} \tag{4.29}$$

where $\lambda_1 = 0$, $\lambda_i = \frac{1 + \lambda_{i-1}}{p_{i-1}}$ for $i = 2, \cdots, n+1$, $p_n = 1$, $L_1 \geq 1$ and $\beta_1, \cdots, \beta_n$ are positive constants that can be determined explicitly.

*Proof:* By assumption, the function $\phi_i(x_1, \cdots, x_i)$ is homogeneous of degree $r_i$ with respect to the dilation $\Delta_\varepsilon^r(x) = (\varepsilon^{r_1} x_1, \cdots, \varepsilon^{r_n} x_n)$, for $i = 1, \cdots, n$. By Lemma 2.1-ii), there is a constant $c > 0$ such that $|\phi_i(x_1, \cdots, x_i)| \leq c \sum_{j=1}^{i} |x_j|^{\frac{r_i}{r_j}}$, $i = 1, \cdots, n$. With this in mind, we can apply the adding a power integrator (AAPI) technique [11], [21], [22], [30], [32], [31] to design a Hölder continuous controller of the form

$$\begin{aligned} u = L_1^{\lambda_{n+1}} v = &-L_1^{\lambda_{n+1}} \beta_n \Big( L_1^{-\lambda_n / r_n} x_n^{\frac{1}{r_n}} + \cdots \\ &+ \beta_2 \big( L_1^{-\lambda_2 / r_2} x_2^{\frac{1}{r_2}} + \beta_1 x_1 \big) \dots \Big)^{r_n}, \end{aligned} \tag{4.30}$$

globally stabilizing the nonlinear system (4.28) when $d = 0$.

In addition, it is also easy to check that for $f_i(x_1, \cdots, x_i) = x_{i+1}^{p_i} + \phi_i(x_1, \cdots, x_i)$, $i = 1, \cdots, n-1$, $f_n(x, u) = u + \phi_n(x)$, the vector field $f(x, u) = [f_1(\cdot) \cdots\ f_n(\cdot)]^T$ is homogeneous of degree $\tau = 0$ with respect to the dilation $\Delta_\varepsilon^{\tilde{r}}(x, u) = (\varepsilon^{r_1} x_1, \cdots, \varepsilon^{r_n} x_n, \varepsilon^{r_n} u)$, where $r_1 = 1$ and $r_i = \frac{1}{p_1 p_2 \cdots p_{i-1}}$, $i = 2, \cdots, n$. Thus, the lower-trioangular system (4.28) with $d = 0$ is homogeneous of degree zero with respect to the dilation $\Delta_\varepsilon^{\tilde{r}}(x, u)$. Finally, the designed controller (4.30) satisfies $u(\varepsilon^{r_1} x_1, \cdots, \varepsilon^{r_n} x_n) = \varepsilon^{r_n} u(x_1, \cdots, x_n)$, with $r_n = \frac{1}{p_1 p_2 \cdots p_{n-1}}$, and hence is homogeneous of degree $r_n$.

In conclusion, Assumptions 3.1 and 3.2 hold for the lower-triangular system (4.28) with $d = 0$. By Theorem 3.3, there is a constant $\delta^* > 0$ such that the nonlinear system (4.28) with input delay is GAS by the sampled-data feedback (4.29), as long as $d + T < \delta^*$. ∎

The following example illustrates the application of Corollary 4.1.

*Example 4.2:* Consider the input delayed system

$$\dot{x}_1(t) = x_2^3(t) + x_1(t), \ \dot{x}_2(t) = u(t - d) + 5 x_2(t) \tag{4.31}$$

which is of the form (4.28) with $p_1 = 3$, $\phi_1(x_1) = x_1$ and $\phi_2(x_1, x_2) = 5 x_2$.

Note that even when $d = 0$, the planar system (4.31) with strong nonlinearity is difficult to be controlled. Indeed, it is neither locally nor globally stabilizable by any smooth nonlinear feedback, as the uncontrollable mode of the linearization has a positive eigenvalue 1.

On the other hand, it is straightforward to check that the planar system (4.31) is homogeneous of degree zero with respect to the dilation $\Delta_\varepsilon^r(x) = (\varepsilon x_1, \varepsilon^{\frac{1}{3}} x_2)$. By Corollary 4.1, there is a sampled-data controller (4.29), i.e.,

$$u(t) = u(t_k) = -L_1^{\frac{4}{3}} \beta_2 \big( L_1^{-1} x_2^3(t_k) + \beta_1 x_1(t_k) \big)^{\frac{1}{3}}, \tag{4.32}$$

$\forall t \in [t_k, t_{k+1})$ with $t_k = kT$, $k = 0, 1, 2, \cdots$, rendering the planar system (4.31) GAS, provided that $d + T$ is limited. Applying the AAPI technique [11], [22], [20], [24], [32], we find explicitly a set of controller gains $L_1 = 2$ and $\beta_1 = 2, \beta_2 = 5$ in (4.32), which do the job.

The other application of Theorem 3.3 is devoted to a dual class of nonlinear systems (4.28) with input delay, namely, upper-triangular systems of the form

$$\begin{aligned} \dot{x}_1(t) &= x_2^{p_1}(t) + \phi_1(x_3(t), \cdots, x_n(t), u(t - d)) \\ &\vdots \\ \dot{x}_{n-1} &= x_n^{p_{n-1}}(t) + \phi_{n-1}(u(t - d)) \\ \dot{x}_n(t) &= u(t - d), \end{aligned} \tag{4.33}$$

where $d > 0$ is the input delay, $p_1, \cdots, p_{n-1}$ are *odd positive* integers, the functions $\phi_i$, $i = 1, \cdots, n$, are $C^0$ with $\phi_i(0) = 0$, and are homogeneous of degree $r_i$ with respect to the dilation $\Delta_\varepsilon^{\tilde{r}}(x, u) = (\varepsilon^{r_1} x_1, \cdots, \varepsilon^{r_n} x_n, \varepsilon^{r_n} u)$, with $r_1 = 1$ and $r_i = \frac{1}{p_1 p_2 \cdots p_{i-1}}$, $i = 2, \cdots, n$.

*Corollary 4.3:* There is constant $\delta^* > 0$ such that system (4.33) with input delay is GAS by sampled-data feedback if $d + T < \delta^*$, where $T > 0$ is the sampling period. In particular, a sampled-data controller is given by

$$\begin{aligned} u(t) = u(t_k) = &-L_2^{-\lambda_{n+1}} \tilde{\beta}_n \Big( L_2^{\frac{\lambda_n}{r_n}} x_n^{\frac{1}{r_n}}(t_k) + \cdots \\ &+ \tilde{\beta}_2 \big( L_2^{\frac{\lambda_2}{r_2}} x_2^{\frac{1}{r_2}}(t_k) + \tilde{\beta}_1 x_1(t_k) \big) \dots \Big)^{r_n} \\ &t \in [t_k, t_{k+1}), \quad t_k = kT, \quad k = 0, 1, 2, \cdots, \end{aligned} \tag{4.34}$$

where $\lambda_1 = 0$, $\lambda_i = \frac{1 + \lambda_{i-1}}{p_{i-1}}$ with $p_n = 1$, $L_2 \geq 1$ and $\tilde{\beta}_1, \cdots, \tilde{\beta}_n > 0$ are constants that can be designed explicitly.

*Proof:* By hypothesis, it is clear that the function $\phi_i(x_{i+2}, \cdots, x_n, u)$ is homogeneous of degree $r_i$ with respect to the dilation $\Delta_\varepsilon^{\tilde{r}}(x, u) = (\varepsilon^{r_1} x_1, \cdots, \varepsilon^{r_n} x_n, \varepsilon^{r_n} u)$, for $i = 1, \cdots, n$. Using Lemma 2.1-ii), one can show the existence of a $c > 0$ such that $|\phi_i(x_{i+2}, \cdots, x_n, u)| \leq c(\sum_{j=i+2}^{n} |x_j|^{\frac{r_i}{r_j}} + |u|^{\frac{r_i}{r_n}})$, $i = 1, \cdots, n$. Using the AAPI design method [22], [24], [31], [33], one can find a Hölder continuous, state feedback controller

$$\begin{aligned} u = L_2^{-\lambda_{n+1}} \tilde{v} = &-L_2^{-\lambda_{n+1}} \tilde{\beta}_n \big( L_2^{\frac{\lambda_n}{r_n}} x_n^{\frac{1}{r_n}} + \cdots \\ &+ \tilde{\beta}_2 \big( L_2^{\frac{\lambda_2}{r_2}} x_2^{\frac{1}{r_2}} + \tilde{\beta}_1 x_1 \big) \dots \big)^{r_n}, \end{aligned} \tag{4.35}$$

which GAS the nonlinear system (4.33) with $d = 0$.

Similar to the argument of Corollary 4.1, Corollary 4.3 follows from Theorem 3.3. ∎

*Example 4.4:* For the nonlinear system with input delay

$$\dot{x}_1(t) = x_2(t) + x_3^3(t),\ \dot{x}_2(t) = x_3^3(t) + u^3(t-d)$$
$$\dot{x}_3(t) = u(t-d), \tag{4.36}$$

it is of the form (4.33) with $p_1 = 1$ and $p_2 = 3$.

It is easy to verify that the nonlinear system (4.36) is homogeneous of degree zero with respect to the dilation $(\varepsilon x_1, \varepsilon x_2, \varepsilon^{\frac{1}{3}} x_3, \varepsilon^{\frac{1}{3}} u)$. By Corollary 4.3 and the AAPI technique [31], [33], we design the sampled-data controller

$$u(t) = u(t_k) = \frac{-\tilde{\beta}_3}{L_2^{\frac{5}{3}}}(L_2^2 x_3^3(t_k) + \tilde{\beta}_2(L_2 x_2(t_k) + \tilde{\beta}_1 x_1(t_k)))^{\frac{1}{3}}$$
$$t \in [t_k, t_{k+1}), \quad t_k = kT, \quad k = 0, 1, 2, \cdots,$$

with the gains $L_2 = 10$ and $\tilde{\beta}_1 = 0.3, \tilde{\beta}_2 = 3, \tilde{\beta}_3 = 7$, globally asymptotically stabilizing the nonlinear system (4.36), as long as $d + T$ is limited.

## V. Conclusion

In this paper, the problem of global asymptotic stabilization by sampled-daat state feedback has been addressed for possibly non-locally exponentially stabilizable or non-smoothly stabilizable systems with input delay. It was shown that sampled-data feedback stabilization is achievable under a fast sampling if the nonlinear system has certain homogeneity and input delay is limited. The proof was carried out by virtue of Halanay inequality, and the properties of homogeneity. As a consequence of this development, globally stabilizing sampled-data controllers were obtained for input-delayed lower-triangular and/or upper-triangular nonlinear systems with uncontrollable linearization. For the sake of space, the discussion on the relation between this paper and finite-time stabilization is omitted (beyond the scope), so is the simulation result of Example 4.2 or 4.4.

## References


[1] M. Di Ferdinando and P. Pepe, "Robustification of sample-and-hold stabilizers for control affine time-delay systems," *Automatica,* vol. 83, pp. 141-154, 2017.

[2] M. Di Ferdinando, P. Pepe and S. Di Gennaro, "On semi-global exponential stability under sampling for locally Lipschitz time-delay systems", *IEEE Trans. Auto. Contr.*, vol. 68, pp. 1508-1523, 2023.

[3] H. Du, C. J. Qian, and S. Li, Global stabilization of a class of uncertain upper-triangular systems under sampled-data control, *Int. J. of Robust and Nonlinear Contr.*, vol. 23, pp. 620-637, 2013.

[4] H. Du, C. J. Qian, S. Li and Z. Chu, Global sampled-data output feedback stabilization for a class of uncertain nonlinear systems, *Automatica,* Vol. 99, pp. 403-411, 2019.

[5] W. P. Dayawansa, Recent advances in the stabilization problem for low dimensional systems, *Plenary Lecture at the 2nd IFAC Nonlinear Control Systems Design Symposium,* Bordeaux, pp. 1-8, 1992.

[6] H. Hermes, "Homogeneous coordinates and continuous asymptotically stabilizing feedback controls," *Differential Equations.* New York, NY, USA: Dekker, vol. 127, pp. 249-260, 1991.

[7] I. Karafyllis and M. Krstic, "Nonlinear stabilization under sampled and delayed measurements, and with inputs subject to delay and zero-order hold," *IEEE Trans. Auto. Contr.,* vol. 57, 1141-1154, 2012.

[8] I. Karafyllis, M. Malisoff, F. Mazenc, and P. Pepe, "Stabilization of nonlinear delay systems: a tutorial on recent results," Recent Results on Nonlinear Time Delayed Systems, Advances in Delays and Dynamics, vol. 4, Springer, 2016.

[9] M. Kawski, Geometric homogeneity and applications to stabilization, *Proc of the 3rd IFAC Symp. on Nonlinear Control Systems*, Lake Tahoe, CA, pp. 164-169, 1995.

[10] W. Lin, "When is a nonlinear system semiglobally asymptotically stabilizable by digital feedback?" *IEEE Trans. on Automat. Contr.*, vol. 65, no. 11, pp. 4584-4599, 2020.

[11] W. Lin and C. Qian, "Adding one power integrator: a tool for global stabilization of high order lower-triangular systems," *Syst. Contr. Lett.,* vol. 39, pp. 339-351, 2000.

[12] W. Lin, W. Wei, and G. Ye, "Global stabilization of a class of nonminimum-phase nonlinear systems by sampled-data output feedback," *IEEE Trans. on Automat. Contr.*, vol. 61, pp. 3076-3082, 2016.

[13] W. Lin and J. W. Sun, "New results and examples in semiglobal asymptotic stabilization of nonaffine systems by sampled-data output feedback," *Systems and Control Letters*, vol. 148, 2021 (109457).

[14] X. Liu, W. Lin, C. Zhao, Y. Hu, "Sampled-data control of a class of time-delay nonlinear systems via memoryless feedback," *Systems and Control Letters*, vol. 157, 2021 (105048).

[15] X. Liu, W. Lin, C. Zhao, Y. Hu, "Digital control of a family of time-delay feedforward systems with sparse sampling," *Systems and Control Letters*, vol. 163, 2022 (105200).

[16] F. Mazenc, M. Malisoff, and T.N. Dinh, "Robustness of nonlinear systems with respect to delay and sampling of the controls," *Automatica*, vol. 49, pp. 1925-1931, 2013.

[17] F. Mazenc and D. Normand-Cyrot, "Reduction model approach for linear systems with sampled delayed inputs," *IEEE Trans. on Automat. Contr.*, vol. 58, no. 5, pp. 1263-1268, 2013.

[18] M. Mattioni, S. Monaco, and D. Normand-Cyrot, "Feedforwarding under sampling," *IEEE Trans. on Automat. Contr.* , vol. 64, no. 11, pp. 4668-4675, 2019.

[19] P. Pepe and E. Fridman, "On global exponential stability preservation under sampling for globally Lipschitz time-delay systems," *Automatica,* vol. 82, pp. 295-300, 2017.

[20] C. Qian and W. Lin, "Non-Lipschitz continuous stabilizers for nonlinear systems with uncontrollable unstable linearization," *Syst. Contr. Lett.* vol. 42, no. 3, pp. 185-200, 2001.

[21] C. Qian and W. Lin, "Almost disturbance decoupling for a chain of power integrators perturbed by a lower-triangular vector field", *Proc. of the 38th IEEE CDC*, Phoenix, AZ, pp. 2082-2087, Dec. 1999.

[22] C. Qian and W. Lin, "A continuous feedback approach to global strong stabilization of nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 46, no. 7, pp. 1061-1079, 2001.

[23] C. Qian and W. Lin, "Recursive observer design, homogeneous approximation, nonsmooth output feedback stabilization of nonlinear systems with unstabilizable/undetectable linearization," *IEEE Trans. Auto. Contr.*, vol. 51, no. 9, pp. 1457-1471, 2006.

[24] C. Qian, W. Lin and W. Zha, "Generalized homogeneous systems with applications to nonlinear control: A Survey," *Mathematical Control and Related Fields (AIMS)*, vol. 5, no. 3, pp. 585-611, 2015.

[25] L. Rosier, "Homogeneous Lyapunov function for homogeneous continuous vector fields," *Syst. Contr. Lett.*, vol. 19, pp. 467-473, 1992.

[26] D. Theodosis and J. Tsinias, Sufficient Lie algebraic conditions for sampled-data feedback stabilization, *Proc. 54th IEEE CDC*, Osaka, Japan, pp. 6490-6495, 2015.

[27] J. Tsinias, New results on sampled-data feedback stabilization for autonomous nonlinear systems, *Systems and Control Lett.,* vol. 61, pp. 1032-1040, 2012.

[28] Y. Wang and W. Lin, "Input delay tolerance of nonlinear systems under smooth feedback: A semiglobal control framework", *IEEE Trans. Auto. Contr.*, vol. 66, no. 1, pp. 146-161, 2021,

[29] Y. Wang and W. Lin, "Stability preservation of nonlinear systems with sampled-delayed input", *IEEE Contr. Syst. Lett.*, vol. 6, pp. 2761-2766, 2022.

[30] B. Yang and W. Lin, "Finite-time stabilization of nonsmoothly stabilizable systems", *IFAC Proc.*, vol. 38, no. 1, pp. 382-387, 2005

[31] X. Yu and W. Lin, "Global input delay tolerance of nonlinear systems under nonsmooth feedback: A homogeneous perspective," *IEEE Trans. Autom. Control*, vol. 68, no. 7, pp. 3992-4007, 2023.

[32] C. Zhao and W. Lin, "Homogeneity, forward completeness, and global stabilization of a family of time-delay nonlinear systems by memoryless non-Lipschitz continuous feedback," *IEEE Trans. Automat. Contr.*, vol. 67, no. 11, pp. 5916-5931, 2022.

[33] C. Zhao and W. Lin, "Homogeneous output feedback design for time-delay nonlinear integrators and beyond: An emulation", *Automatica*, vol 146, Dec. 2022. Article 110641.

[34] V. I. Zubov, *Mathematical methods for the study of automatic control systems*, Groningen: Noordhoff, 1964.