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# Ensemble Control of a Large Population of Stochastic Oscillators: Periodic–Feedback Control Approach

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*Abstract*— **In this paper, we address the problem of steering the distribution of oscillators all receiving the same control input to a given desired distribution. In a large population limit, the distribution of oscillators can be described by a probability density. Then, our problem can be seen as an ensemble control problem with a constraint on the steady-state density. In particular, we consider the case where oscillators are subjected to stochastic noise. One of the difficulties of this problem is that due to the stochasticity, it is generally impossible to design a control law under which oscillators converge to a target density exactly. To avoid this issue, we first give an alternative target density that is close enough to the original target. The modified target is carefully designed via a periodic input so that the distribution of oscillators can converge to it by an appropriate control strategy. Next, we construct a controller that decreases the Kullback–Leibler divergence between the distribution of oscillators and the modified target combining a periodic input and feedback control. We exhibit some convergence results for our proposed method. The effectiveness of the proposed method is demonstrated by a numerical example.**

## I. Introduction

A population of oscillators is useful for modeling various phenomena such as in neuronal ensembles, pedestrian crowds, and firefly swarms. For example in neuroscience, many diseases such as Parkinson's disease, Alzheimer's disease, and sleep disorders can be explained as pathological behavior of oscillator populations [1]–[3]. Treating these diseases with external stimuli via medicine or electrical signals can be modeled as controlling the population of oscillators as desired by an external input so that they behave normally. One of the difficulties of this control problem is that oscillators receive the same input because it is impossible to apply different inputs to each cell.

This kind of control problem has been studied as an ensemble control problem. In [4], an ensemble of systems is represented by a parameterized family of control systems driven by the same control, and the associated controllability condition is derived. In [5], multi-agent coordination by broadcasting the same signal to all agents is considered, and it is revealed that randomness of a local controller of each agent is essential for achieving given motion-coordination tasks. As a dual problem of ensemble control, [6], [7] studied state estimation problems of ensembles that are expressed by probability distributions and characterized the observability of ensembles.

The framework of ensemble control has attracted attention in the study of oscillators [8], [9]. In [10], the control problem of deterministic oscillators is formulated as an ensemble control problem, which describes the distribution of oscillators as a probability density function. Then the authors proposed a control law that decreases the $L^2$ distance between the density of oscillators and a given target density. Moreover, [10] revealed that Fourier coefficients of the so-called phase sensitivity function of oscillators play a crucial role in the convergence of the distribution of oscillators to a target under the proposed controller.

In the above deterministic case, once the distribution of oscillators is transferred to a desired density, then the distribution keeps the desired shape without any control. However, when taking into account random fluctuations of oscillators, the situation is quite different. Especially when oscillators are driven by Wiener processes, there is not, in general, an input that keeps the distribution of the oscillators equal to a target density. Moreover, even if the distribution is steered close to a target, the stochastic oscillators mix to the uniform distribution without control. Thus, we need to use an appropriate control input continually to keep the oscillators close to the target density. For the ensemble control of stochastic oscillators, [11] considered the approximated dynamics of their distribution obtained by the averaging method. Then, the authors formulated an optimization problem which yields a periodic input that makes the stationary distribution of the approximated dynamics close to a desired density. However, the transient response of oscillators cannot be taken into account here because the optimization is only based on the stationary distribution.

To circumvent the issue for stochastic oscillators, in this paper, we take a two-stage process: first we design an alternative target density that is close enough to the original target. Second, we develop a control law that steers oscillators to the modified target rather than the original one. As a result, the distribution of oscillators is transferred close to the original target. The key point here is how to choose a modified target that enables to ensure oscillators converge to it by an appropriate controller. That is, the above two stages are closely connected. Then, we reveal that such an alternative target can be given via a periodic feedforward input obtained by the previous work [11]. Next, we propose a controller that decreases the Kullback–Leibler (KL) divergence between the distribution of oscillators and a modified target combining a periodic input and feedback control. Moreover, we derive the

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convergence properties of the proposed controller. Lastly, our work can also be seen as improving the transient behavior of the previous method [11].

*Organization:* The remainder of this paper is organized as follows. In Section II, we briefly introduce an oscillator model and explain the difficulty of ensemble control of oscillators due to stochastic noise. In Section III, we design a modified target by an appropriate feedforward input. In Section IV, we propose a controller to steer oscillators to a modified target based on the KL divergence, and then derive its convergence properties. In Section V, a numerical example illustrates the effectiveness of the proposed method. Some concluding remarks are given in Section VI.

*Notation:* Let $\mathbb{R}$ denote the set of real numbers. The unit circle is denoted by $S^1$. The inner product of real-valued functions $f, g$ on $S^1$ is defined by $\langle f, g\rangle := \int_{S^1} f(\theta)g(\theta)\mathrm{d}\theta$ provided that it is finite. The $L^2$-norm and the $L^1$-norm of $f$ are defined by $\|f\|_2 := \sqrt{\langle f, f\rangle}, \|f\|_1 := \int_{S^1} |f(\theta)|\mathrm{d}\theta$, respectively. The set of all $k$-times continuously differentiable functions on $S^1$ is denoted by $C^k(S^1)$. Denote $\partial f(\theta)/\partial\theta, \partial^2 f(\theta)/\partial\theta^2$ by $\partial_\theta f(\theta), \partial_\theta^2 f(\theta)$, respectively. When no confusion can arise, we will omit arguments of functions.

## II. Oscillator Model in a Large Population Limit

In this section, we briefly introduce an oscillator model in a large population limit. To this end, we first consider identical, uncoupled $N$ oscillators following the phase model driven by an external input [12], [13]:

$$\mathrm{d}\theta_i(t) = (\omega + Z(\theta_i(t))u(t))\,\mathrm{d}t + \sqrt{2D}\mathrm{d}W_i(t), \quad i \in \{1, \dots, N\}, \tag{1}$$

where $\theta_i(t) \in S^1$ denotes the phase of the $i$-th oscillator, $\omega \in \mathbb{R}$ denotes the natural frequency, $u(t) \in \mathbb{R}$ denotes a control input, and $\{W_i\}_i$ denotes independent standard Wiener processes wrapped around $S^1$. The noise intensity is denoted by $D > 0$. The function $Z : S^1 \to \mathbb{R}$ is called the phase sensitivity function, which quantifies the linear response of the phase to an input $u$. We emphasize that all the oscillators are driven by the same input $u$. Without the input and the noise ($u \equiv 0, D = 0$), the oscillators rotate with the constant angular velocity $\omega$. The model (1) appears for example in neuroscience [8], [10], where it is difficult to apply different control inputs to each oscillator.

In a large population limit $N \to \infty$, the distribution of oscillators following (1) can be represented by a probability density function $\rho : [0, \infty) \times S^1 \to \mathbb{R}$ (see Fig. 1) satisfying the Fokker–Planck equation:

$$\begin{aligned}\partial_t\rho(t,\theta) &= -\partial_\theta\left[(\omega + Z(\theta)u(t))\rho(t,\theta)\right] + D\partial_\theta^2\rho(t,\theta)\\ &=: L_{u(t)}\rho(t,\theta), \end{aligned}\tag{2}$$

$$\rho(0,\theta) = \rho_0(\theta). \tag{3}$$

Here, the situation where all the oscillators receive the same input means that $u$ is not allowed to depend on $\theta$. Throughout this paper, we assume the existence of a unique solution to (2). Then, our goal is to steer the distribution of oscillators

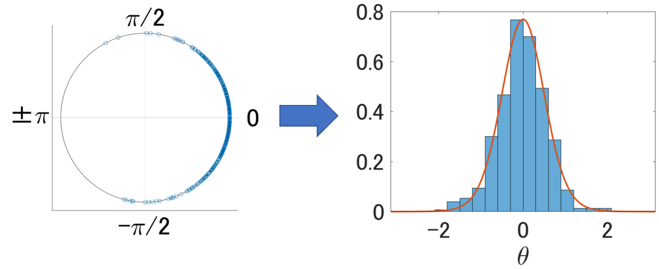



Fig. 1: Distribution of an infinitely large number of oscillators can be represented as a density function (solid, red). Each oscillator is regarded as an independent sample drawn from the density. As the number of oscillators tends to infinity, their empirical distribution (blue histogram) converges to the density.

$\rho$ to a given target distribution. In [10], [14], the target distribution $\rho_f$ is given by

$$\begin{aligned}\partial_t\rho_f(t,\theta) &= -\omega\partial_\theta\rho_f(t,\theta),\\ \rho_f(0,\theta) &= \rho_{f,0}(\theta),\end{aligned}\tag{4}$$

where the solution $\rho_f$ rotates on $S^1$ at the angular velocity $\omega$ maintaining the shape of the initial density $\rho_{f,0}$. Especially in the deterministic case ($D = 0$), [10] designed a population-level feedback control law that decreases the $L^2$ distance between $\rho$ and $\rho_f$. Here, "population-level" means that the distribution $\rho(t,\cdot)$ is available for determining $u(t)$. Moreover, it is revealed that if all Fourier coefficients of $Z$ are non-zero, then $\rho$ converges to $\rho_f$ under the proposed control law.

However, when $D > 0$, the convergence to $\rho_f$ cannot be achieved in general. This can be seen by considering the evolution of the error $\Delta := \rho - \rho_f$:

$$\partial_t\Delta = -\omega\partial_\theta\Delta - \partial_\theta\left(Z(\theta)\rho(t,\theta)\right)u(t) + D\partial_\theta^2\rho(t,\theta). \tag{5}$$

That is, in order to make $\Delta = 0$ an equilibrium of (5), $u$ must satisfy

$$\partial_\theta(Z(\theta)\rho_f(t,\theta))u(t) = D\partial_\theta^2\rho_f(t,\theta),\ \forall t \ge t'$$

for some $t' \ge 0$. However, this condition cannot be fulfilled by $u$ which does not depend on $\theta$ except for special cases such as the uniform distribution $\rho_f \equiv 1/(2\pi)$. Even when $D > 0$, the control law in [10] decreases the $L^2$ distance between $\rho$ and $\rho_f$ to some extent. Indeed, the time derivative of the squared $L^2$-norm $V(\Delta(t,\cdot)) := \frac{1}{2}\|\Delta(t,\cdot)\|_2^2$ along (5) is given by

$$\begin{aligned}\frac{\mathrm{d}V(\Delta)}{\mathrm{d}t} &= \langle\Delta, -\omega\partial_\theta\Delta - \partial_\theta(Z\rho)u + D\partial_\theta^2\rho\rangle\\ &= \langle\partial_\theta\Delta, Z\rho\rangle u + D\langle\Delta, \partial_\theta^2\rho\rangle,\end{aligned}\tag{6}$$

where the argument $t$ is omitted. When $D = 0$, the control law

$$u(t) = -\langle\partial_\theta\Delta(t,\cdot), Z\rho(t,\cdot)\rangle \tag{7}$$

proposed in [10] monotonically decreases $V$. In the presence of the diffusion term ($D > 0$), while $D\left\langle\Delta, \partial_\theta^2\rho\right\rangle \le$

$\langle \partial_\theta \Delta, Z\rho\rangle^2$ holds, the proposed control law (7) decreases $V$. The resulting deviation of $\rho$ from $\rho_f$ may be small. However, it is difficult to estimate the deviation beforehand.

Instead, in this work, we first provide a modified target distribution fluctuating around $\rho_f$, and then design the control law steering oscillators to the modified target exactly. An advantage of this approach is that if the deviation between the modified and original targets is small, the resulting stationary deviation between $\rho$ and the original target $\rho_f$ is ensured to be small. In the next section, we explain how to design such a modified target. Lastly, we mention the previous work [14], which proposed the control law

$$u(t) = -\langle \partial_\theta \Delta, Z\rho\rangle - \frac{D\langle \Delta, \partial_\theta^2 \rho\rangle}{\langle \partial_\theta \Delta, Z\rho\rangle}, \tag{8}$$

whose second term cancels the second term of (6). The authors assert that this control law decreases the $L^2$ distance between $\rho$ and $\rho_f$ until $\rho$ becomes equal to $\rho_f$. However, this is obviously not the case in general since there is not an input under which $\rho$ converges to $\rho_f$ as already observed in (5). The $L^2$ distance converges to a strictly positive constant under the assumption $\langle \partial_\theta \Delta(t,\cdot), Z\rho(t,\cdot)\rangle \neq 0, \forall t$, which ensures the well-definedness of the control law (8).

## III. Design of a Modified Target Distribution

In the previous section, we observed that the distribution of stochastic oscillators cannot be stabilized to the target density given by (4). Then, it is reasonable to use an alternative target density which is close to $\rho_f$ and to which the distribution of oscillators can be kept equal by an appropriate control. In view of this, we consider a density $\rho_d$ evolving as

$$\begin{aligned} \partial_t \rho_d(t,\theta) &= -\partial_\theta \left[(\omega + Z(\theta)u_{\mathrm{FF}}(t))\rho_d(t,\theta)\right] + D\partial_\theta^2 \rho_d(t,\theta) \\ &= L_{u_{\mathrm{FF}}(t)}\rho_d(t,\theta), \end{aligned} \tag{9}$$

$$\rho_d(0,\theta) = \rho_{d,0}(\theta), \tag{10}$$

where $u_{\mathrm{FF}}$ is a given feedforward control input. If $\rho(t',\cdot) = \rho_d(t',\cdot)$ holds for some $t'$, then the equality $\rho(t,\cdot) = \rho_d(t,\cdot)$ is maintained for $t \geq t'$ under the control $u(t) = u_{\mathrm{FF}}(t)$. Thus, if $u_{\mathrm{FF}}$ that makes $\rho_d$ close to $\rho_f$ is available before designing $u$, $\rho_d$ is a promising candidate for a modified target density.

In what follows, focusing on $2\pi/\omega$-periodic inputs, we explain how to obtain such $u_{\mathrm{FF}}$ based on the previous work [11]. Applying the coordinate transformation $\theta \mapsto \theta - \omega t$ with $t \mapsto t$ to (9) yields

$$\begin{aligned} \partial_t \rho_d(t,\theta) = &-\partial_\theta \left[(Z(\theta+\omega t)u_{\mathrm{FF}}(t))\rho_d(t,\theta)\right] \\ &+ D\partial_\theta^2 \rho_d(t,\theta). \end{aligned} \tag{11}$$

In addition, by the averaging method, we obtain

$$\partial_t \bar\rho_d(t,\theta) = -\partial_\theta \left[\Gamma(\theta, u_{\mathrm{FF}})\bar\rho_d(t,\theta)\right] + D\partial_\theta^2 \bar\rho_d(t,\theta), \tag{12}$$

$$\Gamma(\theta, u_{\mathrm{FF}}) := \frac{\omega}{2\pi}\int_0^{2\pi/\omega} Z(\theta+\omega t)u_{\mathrm{FF}}(t)\mathrm{d}t,$$

which approximates (11) by averaging the periodic drift coefficient $Z(\theta+\omega t)u_{\mathrm{FF}}(t)$. It is known that when $u_{\mathrm{FF}}$ is small, (11) is well approximated by (12) [15]. The Fokker–Planck equation (12) is time-invariant and has the stationary distribution

$$\bar\rho_{\mathrm{st}}(\theta) := \frac{1}{C}\int_\theta^{\theta+2\pi} \exp\left(-\frac{\int_\theta^\psi \Gamma(\phi, u_{\mathrm{FF}})\mathrm{d}\phi}{D}\right)\mathrm{d}\psi, \tag{13}$$

where $C > 0$ is the normalizing constant. In order to find $u_{\mathrm{FF}}$ that makes $\bar\rho_{\mathrm{st}}$ close to the desired shape $\rho_{f,0}$, we consider the following optimization problem:

$$\underset{u_{\mathrm{FF}}}{\text{minimize}} \quad \|\bar\rho_{\mathrm{st}} - \rho_{f,0}\|_2^2 + \sigma\|u_{\mathrm{FF}}(\cdot/\omega)\|_2^2 \tag{14}$$

$$\text{s.t.} \quad c(u_{\mathrm{FF}}) \leq 0, \tag{15}$$

where $\sigma > 0$, $c$ is a real-valued function, and (15) is a constraint to keep $u_{\mathrm{FF}}$ small. This problem can then be solved numerically by approximating it to a finite dimensional problem by a finite difference method and truncating Fourier series. We must note that in [11], there is no formal analysis to show convergence of $\rho$ under $u = u_{\mathrm{FF}}$. Our result to be shown next includes such a result, that is, $\rho$ converges to the designed target $\rho_d$.

Now, we formulate the problem to be addressed in the remainder of the paper as follows.

*Problem 1:* Given a feedforward input $u_{\mathrm{FF}}$, find a control input $u$ that achieves $\rho(t,\cdot) \to \rho_d(t,\cdot)$ as $t \to \infty$ in some sense. ◁

## IV. Design of a Controller and Convergence Analysis

In this section, given a feedforward (typically periodic) input $u_{\mathrm{FF}}$, we develop a control law steering the distribution of oscillators to the modified target density $\rho_d$. In [10], [14], a control law is designed by using the $L^2$ distance as a Lyapunov functional. On the other hand, it is known that the KL divergence is useful for the convergence analysis of Fokker–Planck equations to their stationary distributions [16]. Inspired by this, we construct a control law based on the KL divergence rather than the $L^2$ distance, and moreover, we reveal its convergence properties to the modified target.

### A. Design of a Controller

The KL divergence between probability densities $\rho_1, \rho_2$ on $S^1$ is defined by

$$H_{\rho_2}[\rho_1] := \int_{S^1} \rho_1(\theta)\log\frac{\rho_1(\theta)}{\rho_2(\theta)}\mathrm{d}\theta = \left\langle \rho_1, \log\frac{\rho_1}{\rho_2}\right\rangle,$$

provided that it is finite. The KL divergence is nonnegative and takes the value 0 if and only if $\rho_1 = \rho_2$. Therefore, we design a controller which decreases the KL divergence between the distribution of oscillators $\rho$ and the modified target $\rho_d$. In what follows, we assume that $\rho_d(t,\cdot)$ is strictly positive for any $t > 0$ to guarantee the finiteness of $H_{\rho_d(t,\cdot)}[\rho(t,\cdot)]$.

A lengthy calculation shows that the derivative of $H_{\rho_d(t,\cdot)}[\rho(t,\cdot)]$ along the trajectories of (2), (9) is given by

$$\frac{\mathrm{d}H_{\rho_d}[\rho]}{\mathrm{d}t} = -\left\langle \frac{\rho}{\rho_d}, \partial_\theta(Z\rho_d)\right\rangle(u - u_{\mathrm{FF}}) - DJ_{\rho_d}[\rho], \tag{16}$$

where the time $t$ is omitted for notational simplicity, and

$$J_{\rho_d}[\rho] := \int_{S^1} \rho(\theta) \left( \partial_\theta \log \frac{\rho(\theta)}{\rho_d(\theta)} \right)^2 \mathrm{d}\theta$$

is called the relative Fisher information of $\rho$ with respect to $\rho_d$ [17]. The relative Fisher information is a nonnegative functional and takes the value 0 if and only if $\rho = \rho_d$. Based on (16), we obtain the following result.

*Proposition 1:* Assume that $\rho_d(t, \cdot)$ following (9) is strictly positive for any $t > 0$, and $u_{\mathrm{FB}} : [0, \infty) \to \mathbb{R}$ satisfies

$$\left\langle \frac{\rho(t,\cdot)}{\rho_d(t,\cdot)}, \partial_\theta (Z \rho_d(t,\cdot)) \right\rangle u_{\mathrm{FB}}(t) \geq 0, \ \forall t > 0. \tag{17}$$

Then, under $u = u_{\mathrm{FF}} + u_{\mathrm{FB}}$, $J_{\rho_d(t,\cdot)}[\rho(t,\cdot)]$ converges to 0 as $t \to \infty$ for any initial densities $\rho_0, \rho_{d,0}$ and any $u_{\mathrm{FF}}$. ◁

By the above result, under a control input $u = u_{\mathrm{FF}} + u_{\mathrm{FB}}$ with $u_{\mathrm{FB}}$ satisfying (17), the density $\rho$ converges to the modified target $\rho_d$ in the sense that $J_{\rho_d}[\rho] \to 0$. Note that $u_{\mathrm{FF}}$ need not be periodic. The condition (17) is satisfied for example by the population-level feedback control

$$\begin{aligned} u_{\mathrm{FB}}(t) &= \boldsymbol{u}_{\rho_d}[\rho(t,\cdot)] \\ &:= k \left\langle \frac{\rho(t,\cdot)}{\rho_d(t,\cdot)}, \partial_\theta (Z \rho_d(t,\cdot)) \right\rangle, \ k > 0. \end{aligned} \tag{18}$$

Moreover, when considering a constraint on the amplitude of the input $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$, the following control law also decreases $J_{\rho_d}[\rho]$ to zero because (17) is a condition on the sign of $u_{\mathrm{FB}}$.

$$u(t) = \mathrm{sat}_{\underline{u}(t)}^{\overline{u}(t)} \left( u_{\mathrm{FF}}(t) + \boldsymbol{u}_{\rho_d}[\rho(t,\cdot)] \right), \tag{19}$$

$$\mathrm{sat}_a^b(u) := \begin{cases} a, & u < a, \\ u, & a \leq u \leq b, \\ b, & u > b, \end{cases} \tag{20}$$

$$\underline{u}(t) \leq u_{\mathrm{FF}}(t) \leq \overline{u}(t), \ \forall t > 0. \tag{21}$$

In summary, we propose to use (19) combining the feedforward input $u_{\mathrm{FF}}$ and the feedback control $u_{\mathrm{FB}} = \boldsymbol{u}_{\rho_d}$.

### B. Convergence Analysis

Next, we present the convergence result for the proposed method in the sense of the KL divergence rather than the relative Fisher information and derive its consequences. A key ingredient for the analysis is the logarithmic Sobolev inequality.

*Definition 1:* A density function $\rho_2 \in C^1(S^1)$ is said to satisfy the logarithmic Sobolev inequality (LSI($\lambda$)) with $\lambda > 0$ if for any density $\rho_1 \in C^1(S^1)$, it holds

$$J_{\rho_2}[\rho_1] \geq 2\lambda H_{\rho_2}[\rho_1]. \tag{22}$$

◁

If for some $\lambda > 0$, the modified target $\rho_d(t, \cdot)$ satisfies LSI($\lambda$) for any $t > 0$, then $J_{\rho_d}[\rho] \to 0$ means $H_{\rho_d}[\rho] \to 0$. Indeed, under the uniform boundedness and positivity of $\rho_d$, we can show that $\rho_d$ satisfies LSI($\lambda$) for some $\lambda$. Due to space limitations, we omit the proof.

Now, we are ready to state the main result of this paper. Let $W_2(\rho_1, \rho_2)$ be the 2-Wasserstein distance between densities $\rho_1, \rho_2$ on $S^1$ endowed with a distance [17]. The proof is based on the LSI (22), the Csiszár–Kullback–Pinsker inequality [17], [18], and Talagrand's inequality [19, Theorem 1], and is omitted.

*Theorem 1:* Suppose that $\rho_d(t, \cdot)$ following (9) is uniformly bounded in $t$ and satisfies $\inf_{t,\theta} \{\rho_d(t,\theta)\} > 0$. Then, under a control input $u = u_{\mathrm{FF}} + u_{\mathrm{FB}}$ whose $u_{\mathrm{FB}}$ satisfies (17), the following hold for any initial densities $\rho_0, \rho_{d,0}$ and any $u_{\mathrm{FF}}$:

1) $H_{\rho_d(t,\cdot)}[\rho(t,\cdot)]$ converges exponentially to 0 as $t \to \infty$.
2) $\rho$ converges exponentially to $\rho_d$ in the $L^1$-norm as $t \to \infty$.
3) $W_2(\rho(t,\cdot), \rho_d(t,\cdot))$ converges exponentially to 0 as $t \to \infty$. ◁

The exponential convergence rate of $\rho$ depends on the noise intensity $D$ and the parameter $\lambda$ of the LSI (22) used in the proof of Theorem 1. For larger $D$, the convergence becomes faster. In addition, the smaller the up-down swings of $\rho_d$, the larger $\lambda$ and the faster $\rho$ converges.

## V. Numerical Example

In this section, we illustrate the effectiveness of the proposed method by a numerical example. We consider the FitzHugh–Nagumo model used to describe the action potential of a neuron [11], [20], [21]:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x - ax^3 - y + u, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \eta(x + b), \end{aligned}$$

where we set $a = 1/3, b = 1/4, \eta = 1/4$. Then by using the phase reduction method (see e.g., [22, Subsection 3.4]), we obtain $\omega = 0.4034$ and the corresponding $Z$ as shown in Fig. 2. Set the noise intensity as $D = 0.01$. As a desired shape of the distribution of oscillators, we employ a wrapped Cauchy distribution on $S^1$ [23]:

$$w(\theta, \mu, \gamma) := \frac{1}{2\pi} \frac{\sinh(\gamma)}{\cosh(\gamma) - \cos(\theta - \mu)}, \tag{23}$$

where $\mu \in S^1$ is a location parameter, and $\gamma > 0$ is a scale parameter. Fig. 3(a) illustrates two cases $w(\theta, 0, 1)$ and $w(3\theta, 0, 1)$. As can be seen, $\rho_{f,0}(\theta) = w(3\theta, 0, 1)$ divides oscillators into three clusters. For designing a periodic input $u_{\mathrm{FF}}$, we solved the optimization problem (14) with $\sigma = 0.4$ without the constraint (15). Fig. 3(b) shows that the shape of the resulting modified target $\rho_d$ is close to $\rho_{f,0}$. The initial densities are set to $\rho_0(\theta) = w(\theta, \pi, 0.5)$, $\rho_{d,0} = \rho_{f,0} = w(3\theta, 0, 1)$.

Then, we apply the proposed control law (19) with the obtained $\rho_d$ to (2). In Figs. 4(a), (b), (e), (f), we plot snapshots of $\rho$, the KL divergence between $\rho, \rho_d$, and the control input $u$ for different $k$ without an input constraint, i.e., $\overline{u}(t) \equiv \infty, \underline{u}(t) \equiv -\infty$. As can be seen, larger $k$ leads to the faster convergence of $\rho$. Note that $k = 0$ corresponds to the case where only a periodic feedforward input $u_{\mathrm{FF}}$

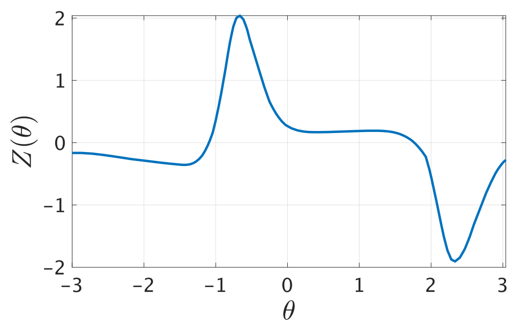


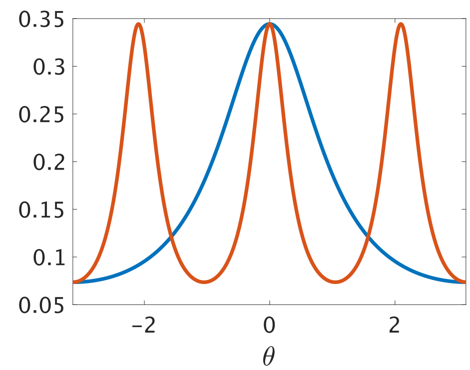

Fig. 2: Phase sensitivity function $Z(\theta)$ for the FitzHugh–Nagumo model.

is present, and the condition (17) for Proposition 1 and Theorem 1 is satisfied even for $k=0$. Hence, $\rho$ converges to $\rho_d$ irrespective of the initial densities $\rho_0, \rho_{d,0}$ by using only the periodic feedforward input $u = u_{\mathrm{FF}}$; as explained earlier, this control input is based on the previous work [11], but convergence to the desired density has not been established there. In addition to the convergence, the proposed controller with $k > 0$ improves the transient response of $\rho$ compared to [11].

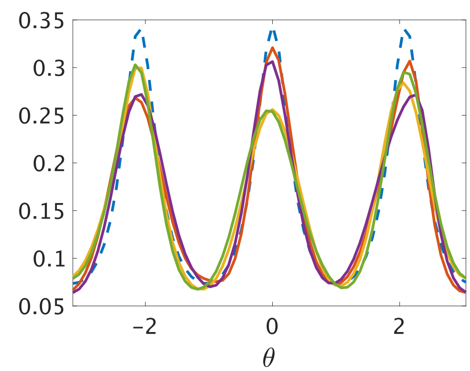
Next, in Figs. 4(c), (d), we also show the result obtained by the proposed controller (19) with the input constraint $\overline{u}(t) \equiv 0.1, \underline{u}(t) \equiv -0.1$. In this case, the convergence of $H_{\rho_d}[\rho]$ hardly deteriorates compared to the case without the constraint and improves upon the previous work [11] ($k=0$).

## VI. Conclusions

In this paper, we developed a control method composed of a periodic feedforward input and a feedback control to transfer the distribution of stochastic oscillators close to a given target. The periodic input plays a role in designing an appropriate alternative target instead of the given target. The feedback control accelerates the convergence of oscillators to the alternative target. We revealed the convergence properties of the proposed method based on the KL divergence. Numerical simulations demonstrated the effectiveness of the proposed method. An important future direction for the current work is to generalize our idea to coupled oscillators.




(a)




(b)

Fig. 3: (a) Wrapped Cauchy distributions $w(\theta, 0, 1)$ (blue), $w(3\theta, 0, 1)$ (red). (b) Original target shape $\rho_{f,0}(\theta) = w(3\theta, 0, 1)$ (dashed) and snapshots of the modified target $\rho_d(t, \theta)$ at every $1/4$ period (solid) under the coordinate transformation $\theta \mapsto \theta - \omega t$.

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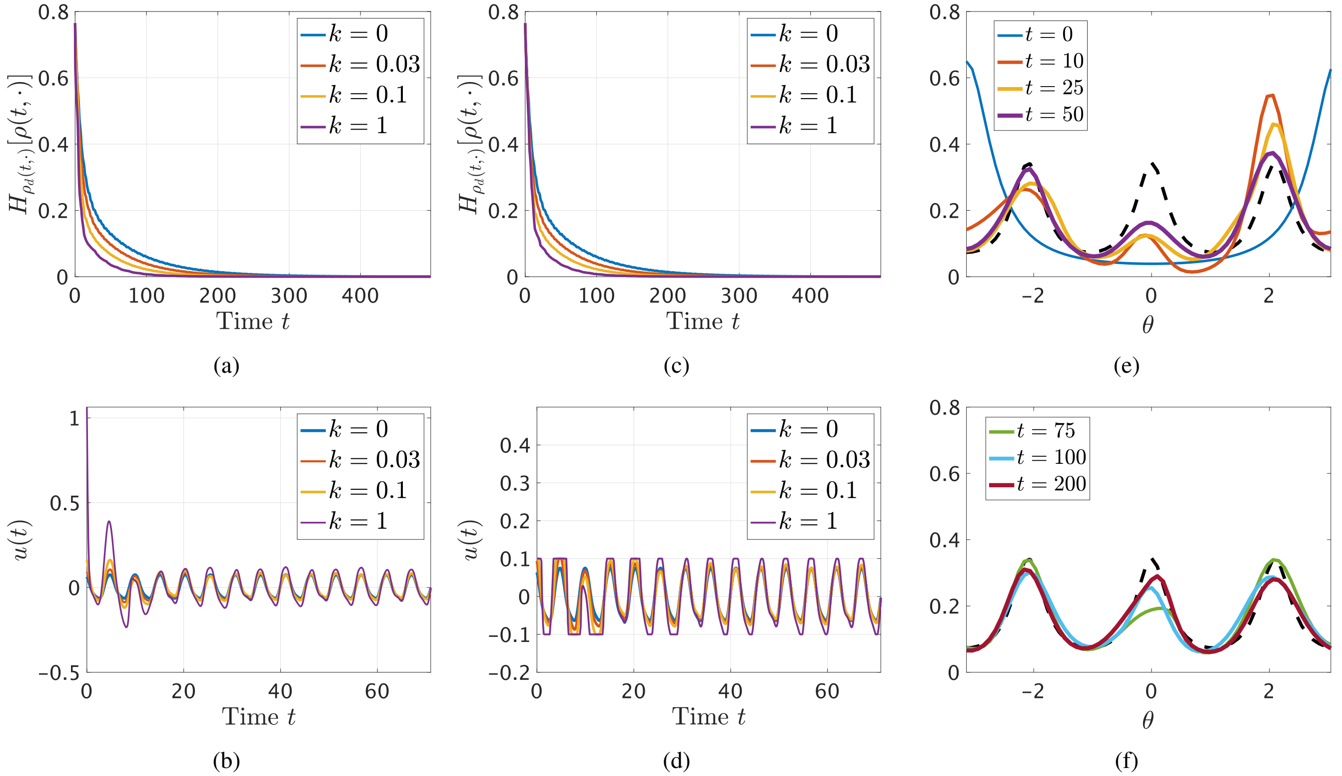


Fig. 4: (a,c) KL divergence between the distribution of oscillators $\rho$ and the modified target $\rho_d$ (a) without and (c) with the constraint $\overline{u}(t) \equiv 0.1, \underline{u}(t) \equiv -0.1$, respectively. (b,d) Control input $u$ given by (19) (b) without and (d) with the constraint. (e,f) Snapshots of $\rho$ without the constraint (solid) and the original target $\rho_f$ (dashed) under the coordinate transformation $\theta \mapsto \theta - \omega t$.


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