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Closed-Loop Identification of Luré Systems

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Abstract— **This paper presents an extension of the so-called "Hansen scheme" for turning closed-loop system identification into open-loop-like identification to a class of discrete-time nonlinear systems with sector-bounded nonlinearities in the state equation. In order to deploy the Hansen scheme, it is necessary to know the existence of a dual Youla-Kucera parametrization of all plants controlled by an observer-based controller. We deduce the existence of such a parametrization based on the solution of a pair of Linear Matrix Inequalities, combined with some differential boundedness arguments. The dual Youla-Kucera parameter may be identified in a number of different ways; in the paper, two examples are presented.**

Index Terms— **Closed-loop Identification, Youla-Kucera Parametrization, Coprime Factorization, Dynamic Mode Decomposition**

I. Introduction

Identification of system models in closed loop is typically much more difficult than identification of systems in open loop. The reason for this is two-fold; firstly, when the plant to be identified is placed in closed loop with a controller, the measurement noise (which is usually assumed uncorrelated with the input) is fed back via output measurements to the controller, which means that the input cannot be assumed uncorrelated with the noise. Secondly, a controller tends to operate within a certain bandwidth, which also complicates the identification process as the input is often not persistently exciting.

Various methods for transforming the closed-loop identification problem into identification of (linear) models in an open-loop setting on the basis of closed-loop data have been proposed [1], [2]. These methods, which are sometimes referred to as the "Hansen scheme" [3], commonly rely on the fact that the set of all system models that can be stabilized by a given controller, can be parametrized using an arbitrary stable system in a particular feedback structure with a nominal system and the controller. The controller and nominal system model are separated into pairs of stable factors, one of which is inverted, and a signal generated as the output from one of these factors is used as input to the other factor as well as to the parameter system. The parameter system is known as the *dual Youla-Kucera* parameter [4], [5]. See the survey papers [6] and [7] for an overview of both theoretical and practical developments throughout the last four decades.

In the Hansen scheme, the Youla-Kucera parametrization is exploited to compute certain auxiliary signals representing the in- and output to and from the dual Youla-Kucera parameter, and the output from the parameter turns out to be unobservable from the input. Since the signals can then also be assumed to be uncorrelated with the measurement noise, the identification of the dual Youla-Kucera parameter is an open-loop-like problem.

Various attempts at extending the Hansen scheme to nonlinear systems have been made; first, in [8], the nominal plant and controller were assumed to be linear, but the dual Youla-Kucera parameter was allowed to be nonlinear and time-varying. Subsequently, in [9], [10], [11], it was shown that the set of all nonlinear plants stabilized by some, possibly nonlinear, controller can be parametrized by a stable (again, possibly nonlinear) operator. These extensions rely on the concept of *differentially coprime* fractional representations of the nonlinear system in order to generate the input to the Youla-Kucera parameter.

Unfortunately, in general it is not easy to find appropriate coprime factors. All input-affine systems in principle have right coprime factorizations; however, many such systems do *not* have left coprime factorizations, because their associated operators do not obey distributivity [12]. Results have been presented on Youla-Kucera parameterizations of plants and/or controllers for various classes of nonlinear systems, such as [13], [12], [14], and [15], but none of these works seem to address the identification aspect (and generally tend to be abstract/non-constructive).

In this paper we propose a Hansen scheme for Luré-type systems, which are linear systems with a sector-bounded memoryless nonlinearity added to the state dynamics. We deduce the existence of a dual Youla-Kucera parametrization of all plants controlled by an observer-based controller based on the solution of a pair of Linear Matrix Inequalities, combined with some relatively mild differential boundedness arguments. The identification scheme subsequently becomes a straightforward extension of the linear case. The presented scheme is constructive in the sense that we provide explicit state space formulae for all the operators involved, which we consider an improvement on previous efforts.

The outline of the rest of the paper is as follows. Section II recalls some preliminary results that will be employed throughout the rest of the paper. Section III then presents the main results of the paper. Section IV illustrates the result with a numerical simulation, and Section V rounds off the paper.

II. Preliminaries

This section states some notation and assumptions we will employ throughout the rest of the paper. Most of the notation is standard; ℓ_2^p denotes the normed vector space of square summable sequences taking values in $\mathbb{R}^p$, with

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associated norm $\|y\|^2 = \sum_{t=0}^{\infty} y_t^T y_t < \infty$, where t is an index denoting the sample number in the sequence y; 0 and I denote zero and identity matrices of appropriate dimensions. A function $\beta_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\beta_1(0) = 0$. A function $\beta_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if for each fixed t, $\beta_2(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed χ, $\beta_2(\chi, t)$ is nonincreasing and tends to 0 as $t \to \infty$. For easy distinction, nonlinear operators are written in calligraphic script, while linear operators and matrices are written with ordinary capital letters. Also, subscript t will often be suppressed in the notation to save space.

A. System operators with sector-bounded dynamics

We consider discrete-time, causal, nonlinear operators $\mathcal{G} : \ell_2^m \to \ell_2^p$ mapping an input signal $u_t \in \mathbb{R}^m$ to an output signal $y_t \in \mathbb{R}^p$ specified by an input-affine state space realization of the form

$$\mathcal{G}: \quad x_{t+1} = Ax_t + \phi(x_t) + Bu_t \qquad (1)$$
$$y_t = Cx_t + Du_t \qquad (2)$$

where $t \in \mathbb{Z}_+$ denotes sample number and $x_t \in \mathbb{R}^n$ is the state at sample t. The nonlinearity $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth component-wise sector-bounded nonlinear function satisfying the sector-bound inequality $\|\phi(\xi)\| \le \alpha\|\xi\|$ for all $\xi \in \mathbb{R}^n$ for some (known) $\alpha > 0$, as well as the component-wise inequalities $(\phi_i(\xi_i)/\alpha + \xi_i)(\phi_i(\xi_i)/\alpha - \xi_i) \le 0$ for all $\xi_i \in \mathbb{R}, 1 \le i \le n$. Finally, A, B, C, and D are real constant matrices of appropriate dimensions, and the triplet (A, B, C) is assumed to be stabilizable and detectable.

It is easy to verify that operators of the class (1)–(2) are closed under the standard interconnection operations. Consider for instance two operators $\mathcal{G}_1 : \ell_2^m \to \ell_2^q$ and $\mathcal{G}_2 : \ell_2^q \to \ell_2^p$ with state space realizations given by

$$\mathcal{G}_1: \quad x_{t+1} = A_1 x + \phi_1(x) + B_1 u$$
$$y_t = C_1 x + D_1 u$$
$$\mathcal{G}_2: \quad \chi_{t+1} = A_2 \chi + \phi_2(\chi) + B_2 w$$
$$v_t = C_2 \chi + D_2 w$$

where all the variables on the RHS are at sample t. Then the *series* connection of $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted $\mathcal{G}_2\mathcal{G}_1$, is the operator

$$\begin{bmatrix} x_{t+1} \\ \chi_{t+1} \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix} \begin{bmatrix} x \\ \chi \end{bmatrix} + \begin{bmatrix} \phi_1(x) \\ \phi_2(\chi) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix} u$$
$$v_t = \begin{bmatrix} D_2C_1 & C_2 \end{bmatrix} \begin{bmatrix} x \\ \chi \end{bmatrix} + D_2D_1u.$$

Similarly, the *parallel* connection of $\mathcal{G}_1$ and $\mathcal{G}_2$, $\mathcal{G}_1 + \mathcal{G}_2$, is the operator

$$\begin{bmatrix} x_{t+1} \\ \chi_{t+1} \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x \\ \chi \end{bmatrix} + \begin{bmatrix} \phi_1(x) \\ \phi_2(\chi) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$
$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x \\ \chi \end{bmatrix} + (D_2 + D_1)u$$

In the above expressions, it is important to note that the concatenated nonlinear function $[\phi_1(x)^T \;\; \phi_2(\chi)^T]^T$ also satisfies the required sector-bound inequalities.

Perhaps a less obvious property is that, like their linear counterparts, operators with state space realizations (1)–(2) have left and right inverses *if and only if* the dimensions of their inputs and outputs are equal and their direct feedthrough matrix D is invertible:

Lemma 1: Consider the system $\mathcal{G} : \ell_2^p \to \ell_2^p$ with state space realization (1)–(2) and assume D is invertible. Then $\mathcal{G}^{-1}$ is given by

$$\chi_{t+1} = A\chi + \phi(\chi) - BD^{-1}C\chi + BD^{-1}y \qquad (3)$$
$$u_t = -D^{-1}C\chi + D^{-1}y. \qquad (4)$$

Proof: Define $\xi = x - \chi$; combining (1)–(2) with (3)–(4) then yields

$$\xi_{t+1} = Ax + \phi(x) + Bu - A\chi - \phi(\chi) + BD^{-1}C\chi - BD^{-1}y$$
$$= A\xi + \phi(x) - \phi(\chi) - BD^{-1}C\xi$$
$$v_t = -D^{-1}C\chi + D^{-1}(Cx + Du)$$
$$= D^{-1}C\xi + u$$

Hence, if $x_0 = \chi_0$ we must have $\xi_t \equiv 0$ and thus $v_t = u_t$ for all $t \ge 0$. ■

B. Right coprime factorizations

It was demonstrated in [12] that if one can find feedback laws $u = F(\hat{x})\hat{x}$ and $\epsilon = L(\hat{x})(y - C\hat{x})$ that stabilize the autonomous systems $x_{t+1} = (A + \phi(x) + BF(x))x$ and $\hat{x}_{t+1} = (A+\phi(\hat{x})+L(\hat{x})C)\hat{x}$, respectively, then systems of the general form (1)–(2) can be factorized as $\mathcal{G} = \mathcal{N}\mathcal{M}^{-1}$ with

$$\mathcal{M} : x_{t+1} = (A + \phi(x) + BF(x))x + Bv \qquad (5)$$
$$u_t = F(x)x + v, \qquad (6)$$
$$\mathcal{N} : \chi_{t+1} = (A + \phi(\chi) + BF(\chi))\chi + Bv \qquad (7)$$
$$y_t = (C + DF(\chi))\chi + Dv. \qquad (8)$$

Furthermore, the operators

$$\mathcal{V} : \hat{x}_{t+1} = (A + \phi(\hat{x}) + BF(\hat{x}))\hat{x} - L(\hat{x})w, \qquad (9)$$
$$y_t = (C + DF(\hat{x}))\hat{x} + w \qquad (10)$$
$$\mathcal{U} : \hat{\chi}_{t+1} = (A + \phi(\hat{\chi}) + BF(\hat{\chi}))\hat{\chi} - L(\hat{\chi})w, \qquad (11)$$
$$u_t = F(\hat{\chi})\hat{\chi} \qquad (12)$$

can be combined to form an observer-based controller

$$\mathcal{K} = \mathcal{U}\mathcal{V}^{-1} \qquad (13)$$

that stabilizes $\mathcal{G}$; these are commonly called *right coprime factorizations* [10].

C. Stability

The next question is then to find the necessary feedback laws to implement the plant/controller factors (5)–(12).

As mentioned, a system of the form (1)–(2) can be seen as a type of Luré system. Because the nonlinear part of the drift term is sector-bounded, it is possible to compute a static feedback gain matrix that stabilizes the system provided the sector bound is sufficiently small:

Theorem 1: Consider the system (1)–(2) with $D = 0$ and $\phi(x) = B_\varphi \varphi(C_q x)$ satisfying

$$\|\varphi(\xi)\| \leq \alpha\|\xi\| \text{ and } \|\varphi(\xi) - \varphi(\hat{\xi})\| \leq \mu\|\xi - \hat{\xi}\| \tag{14}$$

for known constants $\alpha > 0, \mu > 0$, along with the state observer

$$\begin{aligned} \hat{x}_{t+1} &= A\hat{x} - L(y - \hat{y}) + B_\varphi \varphi(C_q \hat{x}) + BF\hat{x} && (15)\\ \hat{y}_t &= C\hat{x}. && (16) \end{aligned}$$

If there exist feasible solutions to the matrix inequalities

$$\min_{Q,W} \gamma \quad \text{s.t.} \tag{17}$$

$$0 < Q = Q^T$$

$$0 > \begin{bmatrix} -Q & 0 & QA^T + W^T B^T & QC_q^T \\ 0 & -I & B_\varphi^T & 0 \\ AQ + BW & B_\varphi & -Q & 0 \\ C_q Q & 0 & 0 & -\gamma I \end{bmatrix}$$

$$\min_{P,Y} \lambda \quad \text{s.t.} \tag{18}$$

$$0 < P = P^T$$

$$0 > \begin{bmatrix} -\lambda P & 0 & A^T + C^T Y^T & C_q^T \\ 0 & -I & B_\varphi^T P & 0 \\ PA + YC & PB_\varphi & -P & 0 \\ C_q & 0 & 0 & -\mu^{-2} I \end{bmatrix}$$

then the closed-loop system described by Eqns. (1), (2), (15), and (16) with $F = WQ^{-1}$, $L = P^{-1}Y$, and $u_t = F\hat{x}_t$, is Bounded-Input-Bounded-Output (BIBO) stable for any φ such that $\gamma \leq \alpha^{-2}$.

Proof: See [16]. ■

Here, BIBO stability means that the controlled system (1)–(2) with $u_t = F\hat{x}_t$ admits a quadratic Lyapunov function $V(x_t) = x_t^T Q x_t$ satisfying $V(x_{t+1}) < V(x_t)$ for all t if $x_t = \hat{x}_t$, while the state estimation error driven by the observer dynamics (15) is bounded by a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathcal{K}$ function $\rho(\cdot)$:

$$\|x_t - \hat{x}_t\| \leq \beta(\|x_0 - \hat{x}_0\|, t) + \rho(\delta_0) \quad \forall t \geq 0$$

where δ_0 is the peak magnitude of some vanishing disturbance; for further details, see [16].

Remark 1 Problem (17) is a Linear Matrix Inequality (LMI) and can thus be solved efficiently. Problem (18) is strictly speaking not an LMI due to the presence of the product λP; however, one may simply fix consecutive values of λ and check for feasibility, which is again straightforward to handle. ◁

Remark 2 The original result in [16] includes an extra observer gain in the argument of φ in (15) to eliminate any estimation errors caused by the nonlinearity, thereby achieving global asymptotic stability of the controller-observer interconnection for bounded and vanishing disturbances. We do not include this extra term in the sequel, as we only require BIBO stability in our coming arguments for existence of left coprime factors. ◁

Remark 3 Finally, it is noted that it is of course possible to have $\phi(x) = \varphi(x)$ by choosing $B_\phi = I, C_q = I$ in the LMIs. ◁

III. Closed-loop System Identification

With all preliminaries in place, we are now ready to present the main contribution of the paper.

A. Left coprime factorization

As previously mentioned, the Hansen scheme is a strategy for turning a closed-loop identification problem, which is usually hard, into identification of a dual Youla-Kucera parameter—an operator of the same structure as (1)–(2), embedded within a right/left coprime factorization of the plant model in closed loop with a controller, which can also be factorized. The method relies crucially on the existence of a Bezout Identity, which links together left and right coprime plant/controller factorizations.

However, it must be noted that it is not at all common for nonlinear systems to have left factorizations. This is because nonlinear systems do not in general obey distributivity; that is, for some $\mathcal{P}_1$ and $(\mathcal{P}_2 + \mathcal{P}_3)$, the cascade $\mathcal{P}_1(\mathcal{P}_2 + \mathcal{P}_3)$ will *not* equal $\mathcal{P}_1\mathcal{P}_2 + \mathcal{P}_1\mathcal{P}_3$.

Fortunately, some systems of the class considered here do indeed possess left factorizations:

Proposition 1: Consider a system $\mathcal{G}$ of the form (1)–(2) and suppose there exist matrices Q, P, W, and Y along with scalars $0 < \lambda < 1$ and $0 < \gamma < \alpha^{-2}$ satisfying (17)–(18). Let the associated observer-based controller be given by (13).

Then the operators

$$\begin{aligned} \tilde{\mathcal{M}} : x_{t+1} &= (A + LC)x + \phi(x) + Lv, && (19)\\ y_t &= Cx + v && (20)\\ \tilde{\mathcal{N}} : x_{t+1} &= (A + LC)x + \phi(x) + (B + LD)u && (21)\\ v_t &= Cx + Du && (22)\\ \tilde{\mathcal{U}} : \hat{x}_{t+1} &= (A + LC)\hat{x} + \phi(\hat{x}) + Ly, && (23)\\ w_t &= -F\hat{x} && (24)\\ \tilde{\mathcal{V}} : \hat{x}_{t+1} &= (A + LC)\hat{x} + \phi(\hat{x}) - (B + LD)u && (25)\\ w_t &= F(\hat{x})\hat{x} + u. && (26) \end{aligned}$$

constitute a right coprime factorization $\mathcal{G} = \tilde{\mathcal{M}}^{-1}\tilde{\mathcal{N}}$, $\mathcal{K} = \tilde{\mathcal{V}}^{-1}\tilde{\mathcal{U}}$ that satisfy the Bezout identity

$$\begin{bmatrix} \tilde{\mathcal{V}} & -\tilde{\mathcal{U}} \\ -\tilde{\mathcal{N}} & \tilde{\mathcal{M}} \end{bmatrix} \begin{bmatrix} \mathcal{M} & \mathcal{U} \\ \mathcal{N} & \mathcal{V} \end{bmatrix} = I \tag{27}$$

and all systems of the form (1)–(2) stabilized by $\mathcal{K}$ may be written on the form

$$\begin{aligned} \mathcal{G}_\mathcal{S} &= (\mathcal{N} + \mathcal{V}\mathcal{S})(\mathcal{M} + \mathcal{U}\mathcal{S})^{-1} \\ &= (\tilde{\mathcal{M}} + \mathcal{S}\tilde{\mathcal{U}})^{-1}(\tilde{\mathcal{N}} + \mathcal{S}\tilde{\mathcal{V}}). && (28) \end{aligned}$$

where $\mathcal{S}$ is a dual Youla-Kucera parameter of the form (1)–(2).

Proof: We base our claim on the following observations:

1) The state and observer state equations are well defined and bounded for bounded initial conditions, and the input/output maps of each operator is linear;
2) The controlled state equation $x_{t+1} = (A + BF)x + \phi(x)$ is exponentially stable; and

3) The observer dynamics $\hat{x}_{t+1} = (A + LC)\hat{x} + \phi(\hat{x})$ is bounded by a class $\mathcal{KL}$ function.

These observations imply that all of the operators $\mathcal{M}, \mathcal{N}, \mathcal{U}, \mathcal{V}, \tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \tilde{\mathcal{U}}$, and $\tilde{\mathcal{V}}$ are *differentially bounded*, i.e., there exist constants $\varepsilon_1, \varepsilon_2$ such that for any pair of signals $w_1, w_2 \in \ell_2$ one has

$$\|w_1 - w_2\| < \varepsilon_1 \Rightarrow \|\mathcal{F}w_1 - \mathcal{F}w_2\| < \varepsilon_2$$

where $\mathcal{F}$ may be any the aforementioned operators.

Then we note that the fact that $x_{t+1} = (A + BF)x + \phi(x)$ is exponentially stable is equivalent to the state equation

$$x_{t+1} = (\bar{A}(x) + BF)x, \quad \bar{A}(x) = A + \phi(x)\mathrm{diag}\{x\}^{-1}$$

being exponentially stable.

Theorem 3 in [12] then guarantees the existence of the desired Bezout identity along with the existence of the Youla-Kucera parameterization with $\mathcal{S}$ as parameter. ■

This shows that if the observer/control design problems (17), (18) are feasible, Youla-Kucera-like parametrizations can be found for the class of systems considered here, characterizing all systems stabilized by the controller (13).

B. Hansen Scheme

Consider next the problem of open-loop identification of a system $\mathcal{G}_\theta$ parameterized by some unknown parameter vector θ as depicted in Figure 1 (top left). An input u may be applied to the system, and corresponding output measurements y affected by noise ν_y are obtained. These measurements are related to the input and noise through

$$y_t = \mathcal{G}_\theta u_t + \nu_{y,t}$$

and an unbiased estimate of $\mathcal{G}_\theta$ can in principle be obtained as long as u_t and $\nu_{y,t}$ are uncorrelated and u is sufficiently rich. Unfortunately, in a closed-loop setting u_t is not uncorrelated with $\nu_{y,t}$ because the noise is fed back through the controller, as also shown in Figure 1 (top right). Furthermore, the dynamics of the controller tend to limit the richness of the excitation in closed-loop operation. To alleviate this problem, we employ the dual Youla-Kucera factorization to recast the closed-loop system identification problem into an open-loop-like problem.

Assume a controller $\mathcal{K}$ that stabilizes a nominal plant $\mathcal{G}$ has been found, as outlined in the previous two sections. Then the set of all plants stabilized by $\mathcal{K}$ can be represented as shown in the bottom block diagram of Figure 1. Here, $\nu = (\tilde{\mathcal{M}} + \mathcal{S}\tilde{\mathcal{U}})\nu_y$ is the measurement noise that would normally affect the measurements y at sample time t, relocated in the block diagram to affect the output of the Youla-Kucera parameter $\mathcal{S}$ instead, and r_1 and r_2 are external excitation signals.

By manipulating the block diagram and using the Bezout identity (27), it is possible to check that the block diagram expresses the relation

$$y_t = (\mathcal{N} + \mathcal{V}\mathcal{S})(\mathcal{M} + \mathcal{U}\mathcal{S})^{-1}u_t + \nu_{y,t}$$

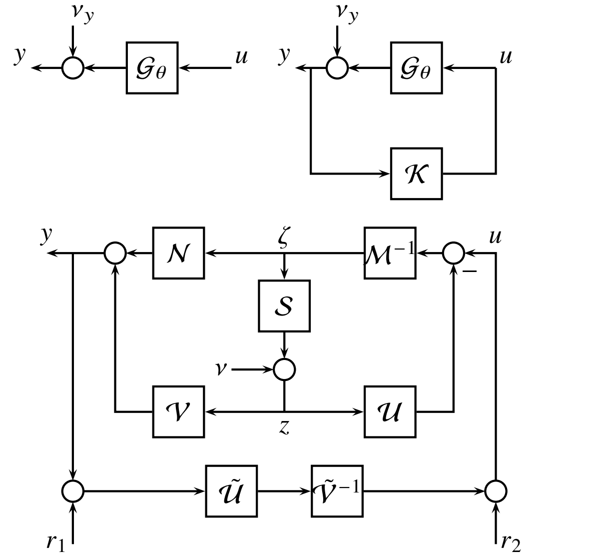

Fig. 1. Top left: Open-loop identification. Top right: closed-loop identification. Bottom: Hansen scheme; $\mathcal{G} = \mathcal{N}\mathcal{M}^{-1}$ and $\mathcal{K} = \tilde{\mathcal{V}}^{-1}\tilde{\mathcal{U}}$ are nominal model and controller factorizations, respectively, while $\mathcal{S}$ is an unknown Dual Youla-Kucera parameter.

where the dual Youla-Kucera parameter $\mathcal{S}$ appears as the only unknown. Thus, the unknown parameter vector θ must be embedded in $\mathcal{S}$ only.

Furthermore, it is possible to deduce, again using (27), that

$$\zeta = \tilde{\mathcal{U}}r_1 + \tilde{\mathcal{V}}r_2 \tag{29}$$

$$z = \tilde{\mathcal{M}}y - \tilde{\mathcal{N}}u \tag{30}$$

$$z = \mathcal{S}\zeta + \nu \tag{31}$$

As can be seen, ζ and z are available from measurements and inputs filtered through stable operators. Furthermore, as long as ν_y is independent of r_1 and r_2, then ζ is independent of ν as well; and if ν_y is zero-mean, then so is ν. Additionally, $\mathcal{S}$ is known to be stable due to the dual Youla-Kucera theory (cf. Section III-A). Thus, it can be seen that although u and y are measured in closed-loop, the identification of $\mathcal{S}$ becomes equivalent to an open-loop identification problem.

IV. Numerical Example

To demonstrate the proposed scheme, we choose the following modified version of the example system in [16]:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.2702 & -0.0124 & 0.2703 & 0 \\ 0 & 0 & 0 & 1 \\ 0.1075 & 0 & 0.0743 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0.216 & 0 \\ 0 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad B_\varphi = \begin{bmatrix} 0 & 0 \\ 0.2703 & 0 \\ 0 & 0 \\ -0.1075 & 0.0332 \end{bmatrix}$$

$$C_q = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = 0$$

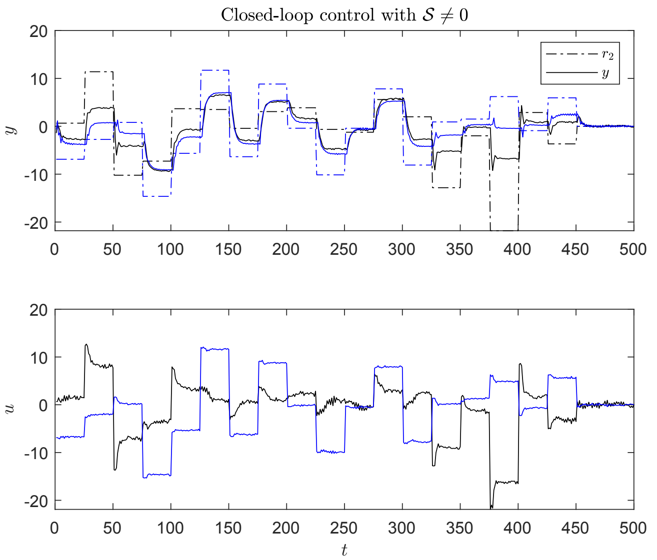

Fig. 2. *Measurement data for system identification. Top: y (—), r_2 (- -); bottom: u.*

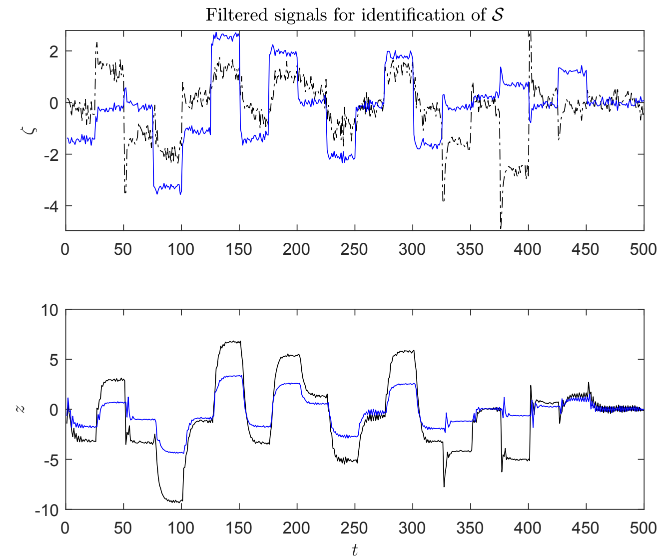

Fig. 3. *Filtered data for system identification. Top: ζ; bottom: z*

and $\varphi_i(\cdot) = \tanh(\cdot), 1 \leq i \leq 2$. For this system, it is possible to solve the design LMIs to yield the stabilizing state feedback and observer gains

$$F = \begin{bmatrix} 1.2509 & 0.0574 & -1.2514 & -1.3 \times 10^{-6} \\ -0.5375 & -4 \times 10^{-7} & -0.3715 & -1.0 \times 10^{-6} \end{bmatrix}$$

$$L = \begin{bmatrix} 0.0078 & 2.7 \times 10^{-8} \\ 0.2701 & -0.2703 \\ -0.0003 & 4.2 \times 10^{-8} \\ -0.1075 & -0.0743 \end{bmatrix}.$$

The LMIs were solved for $\lambda = 0.75, \mu = 2.5$ using the popular YALMIP front-end for the MOSEK solver in Matlab [17], [18]. The dual Youla-Kucera parameter $\mathcal{S}$ was chosen semi-randomly as

$$z_{t+1} = \begin{bmatrix} 0.2277 & 0.2641 \\ 0.5104 & 0.4180 \end{bmatrix} z + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} \varphi(z) + \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix} \zeta$$

while the inputs r_1 and r_2 were chosen as a random Gaussian sequence with mean 0 and spread 0.5 and a series of steps with random amplitude in the interval $[-2, 2]$, respectively. Since r_2 enters as a standard reference, it was chosen to prescale the signal with the steady-state gain of the linear part of $\mathcal{G}$ to provide a visual illustration of the controller's tracking performance. The input-output data is shown in Figure 2; as can be seen, the controller performs about as well as can be expected without integral action and being perturbed by the nonlinearity $\phi(\cdot)$.

The signals r_1, r_2, u, and y are then filtered according to (29) and (30), respectively. The filtered signals used for identification of S are shown in Figure 3.

Here, primarily for simplicity, it is chosen to identify the nonlinear state space model for $\mathcal{S}$ using Dynamic Mode Decomposition (DMD), see e.g., [19]. Although crude, the DMD method has the advantage of being easy to implement and is suitable for the model structure considered here. For comparison, we also employ Matlab's nonlinear grey-box estimation toolbox `nlgreyest` to identify $\mathcal{S}$.

Sample pairs of $(z_t, \zeta_t), t = 1, \dots N = 500$ are obtained from the filtering processes (29)–(30) and the model order is chosen as $n_{\mathcal{S}} = 2$. We define the matrices

$$\begin{aligned} Z^+ &= \begin{bmatrix} z_2 & z_3 & \dots & z_N \end{bmatrix} \\ Z &= \begin{bmatrix} z_1 & z_2 & \dots & z_{N-1} \end{bmatrix} \\ \Phi &= \begin{bmatrix} \phi(z_1) & \phi(z_2) & \dots & \phi(z_{N-1}) \end{bmatrix} \\ \Gamma &= \begin{bmatrix} \zeta_1 & \zeta_2 & \dots & \zeta_{N-1} \end{bmatrix} \end{aligned}$$

Equation (31) can then be written as

$$Z^+ = A_{\mathcal{S}} Z + B_{\phi,\mathcal{S}} \Phi + B_{\mathcal{S}} \Gamma + \nu = \Theta \Lambda + \nu$$

where Λ contains the observations and nonlinear function evaluations, and Θ contains the parameters to be estimated. A truncated Singular Value Decomposition (SVD) is performed on the observation matrix, resulting in $\Lambda \approx \bar{U}\bar{\Sigma}\bar{V}^T$ where $\bar{\cdot}$ represents rank-r truncation. The least-squares estimate of the parameter matrices can then be found as

$$\hat{\Theta} = Z^+ \bar{V} \bar{\Sigma}^{-1} \bar{U}. \tag{32}$$

Figure 4 shows a comparison of the output response of the estimated $\hat{\mathcal{S}}$ (with rank 5 truncation) and actual $\mathcal{S}$ to a series of random steps (top trace). The figure also shows the corresponding estimation performance for `nlgreyest` (bottom trace).

As can be seen, the estimated dual Youla-Kucera parameter is not identified perfectly by DMD; while the dynamics are mostly captured reasonably, the steady-state gains are clearly not very precise, which may be attributed to poor estimation of B_{φ}; `nlgreyest`, on the other hand, performs considerably better.

When placed back into (28), however, the DMD estimate is found to perform quite well; as seen in Figure 5, the

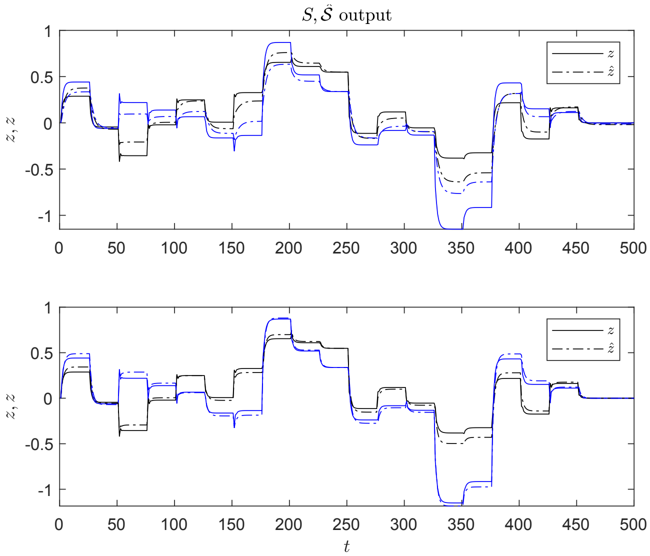

Fig. 4. *Comparison between $\mathcal{S}$ (—) and $\hat{\mathcal{S}}$ (- -).*

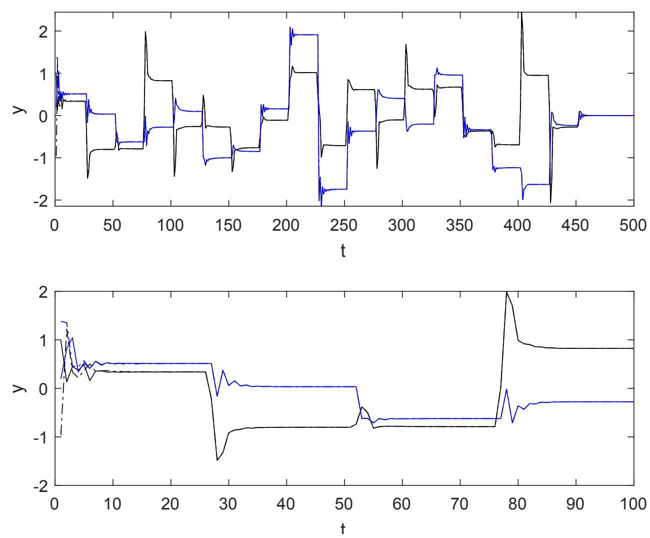

Fig. 5. *Simulation with $y_t = (\mathcal{N} + \mathcal{V}\mathcal{S})(\mathcal{M} + \mathcal{U}\mathcal{S})^{-1}u_t$, $\hat{y}_t = (\mathcal{N} + \mathcal{V}\hat{\mathcal{S}})(\mathcal{M} + \mathcal{U}\hat{\mathcal{S}})^{-1}u_t$. Top: y_t (—), $\hat{y}_t$ (- -) bottom: zoom of the first 100 samples.*

outputs (y and $\hat{y}$) with $\mathcal{S}$ and $\hat{\mathcal{S}}$ inserted in the Youla-Kucera factorization are very nearly identical except in the beginning of the simulation, which is due to differences in initial conditions.

V. Comments and Conclusion

This paper presented an open-loop-like identification scheme for discrete-time Luré-type systems. We deduced the existence of a dual Youla-Kucera parameterization of all plants controlled by an observer-based controller based on the solution of a pair of Linear Matrix Inequalities, combined with some relatively mild differential boundedness arguments. The presented scheme is constructive in the sense that we provide explicit state space formulae for all the operators involved, which we consider an improvement on previous efforts.

Two different identification schemes were tested on the dual Youla-Kucera parameter, and it was found that the more sophisticated nonlinear grey-box estimation method performs better than DMD. However, this conclusion may not necessarily be true in cases where the model structure deviates from the 'true' structure of $\mathcal{S}$, which is typically not known in practical applications.

Future work involves determining optimal identification methods for the dual Youla-Kucera parameter, as well as incorporation of less conservative (nonlinear) control and observer gain design methods.

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