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Self-triggered Boundary Control of a Class of Reaction-Diffusion PDEs

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Abstract— **This paper provides a novel self-triggered boundary control (STBC) strategy for a class of reaction-diffusion PDEs with Robin actuation using infinite-dimensional backstepping boundary control. Our goal is to offer a solution for the continuous monitoring of triggering functions in conventional event-triggered control. We propose a method for converting a certain class of continuous-time dynamic event-triggers that require continuous monitoring to self-triggers that proactively compute the time of the next event at the current event time using the knowledge of the available system states and dynamics. We achieve this by designing a positively and uniformly lower-bounded function which, when evaluated at the current event time, outputs the waiting time until the next event. The control input is updated only at events indicated by the self-trigger and is applied in a zero-order hold fashion between two events. We establish the closed-loop system well-posedness under the proposed STBC approach. Furthermore, we prove that the global L^2-exponential convergence to zero under continuous-time event-triggered boundary control (CETBC) is preserved under the proposed STBC approach. We provide a simulation result that validates the theoretical claims.**

I. Introduction

In event-triggered control, the control input is updated only upon a specific event, determined by an appropriate triggering mechanism, rather than at fixed time intervals. This approach can be seen as a sampled-data control method that integrates feedback into both communication and control update tasks. Leveraging feedback, event-triggered control updates the control input aperiodically, only when necessary. This reduces communication and control updates while ensuring satisfactory closed-loop performance [1].

Event-triggered control consists of a feedback control law ensuring desired performance and an event-triggered mechanism dictating control input updates. It is essential to prevent *Zeno behavior*, where infinite control updates occur in finite time. This is usually achieved by designing an event mechanism with a guaranteed lower bound between event times known as *minimum dwell-time*. Over the past decade, significant findings on event-triggered control for systems driven by linear and nonlinear ODEs have emerged (see [2],[3]). This led to exploring strategies for PDE-governed systems [4]–[13]. Notably, [9] and [10] are most pertinent to this work, proposing event-triggered boundary control for a class of reaction-diffusion PDEs using dynamic event-triggers.

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A significant limitation of the mentioned event-triggered control for ODE and PDE plants is the continuous monitoring needed to detect events, unsuitable for digital implementation. These strategies are termed *continuous-time event-triggered control*. An alternative is *self-triggered control* [2], where the next event time is precomputed using prior data and plant dynamics knowledge. This approach retains the resource efficiency of continuous-time strategies. In self-triggered control, the control input updates aperiodically, calculating the next event's timing, making it ideal for digital software implementations. During the past few years, several interesting works devoted to self-triggered control of ODE systems have been published [14]–[18]. Despite these developments for ODE control problems, self-triggered control of PDE plants is still relatively nascent. To the best of our knowledge, the paper [19] is the only work that presents a self-triggered control approach for infinite-dimensional systems using semigroup theory.

This paper presents the first self-triggered boundary control (STBC) strategy for a class of reaction-diffusion PDEs with Robin actuation using infinite-dimensional backstepping approach. The proposed method transforms the continuous-time event-triggered boundary control (CETBC) strategy proposed in [9] to an STBC strategy. The designed STBC strategy consists of the construction of a positively and uniformly lower-bounded function that accepts several inputs involving the system states, which, when evaluated at an event time, outputs the waiting time until the next event. The design of the positive function requires upper and lower bounds of constituent variables of the underlying continuous-time event-trigger. Since the function is positively and uniformly lower-bounded, the closed-loop system is Zeno-free by design. Moreover, the closed-loop system well-posedness under the proposed STBC is established. Further, performance guarantees under CETBC are preserved under the proposed STBC in the sense that the closed-loop signals under both CETBC and STBC globally exponentially converges to zero in L^2-sense at comparable rates.

The paper is organized as follows. Section II summarizes the CETBC presented in [9]. We present the proposed STBC in Section III. A numerical example is provided in Section IV to illustrate the results, and the conclusion is provided in Section V.

Notation: $\mathbb{R}_+$ is the nonnegative real line whereas $\mathbb{N}$ is the set of natural numbers including zero. By $C^0(A;\Omega)$, we denote the class of continuous functions on $A \subseteq \mathbb{R}^n$, which takes values in $\Omega \subseteq \mathbb{R}$. By $C^k(A;\Omega)$, where $k \geq 1$, we denote the class of continuous functions on A, which takes values in Ω and has continuous derivatives of order k. $L^2(0,1)$ denotes the equivalence class of Lebesgue measurable functions $f:[0,1]\to\mathbb{R}$ such that $\|f\| = \left(\int_0^1 |f(x)|^2\right)^{1/2} < \infty$. Let $u:[0,1]\times\mathbb{R}_+\to\mathbb{R}$ be given. $u[t]$ denotes the profile of u at certain $t\geq 0$, *i.e.,* $\big(u[t]\big)(x) = u(x,t)$, for all $x\in[0,1]$. For an interval $J\subseteq\mathbb{R}_+$, the space $C^0\big(J;L^2(0,1)\big)$ is the space of continuous mappings $J\ni t\to u[t]\in L^2(0,1)$. $I_m(\cdot)$ and $J_m(\cdot)$ with m being an integer respectively denote the modified Bessel and (nonmodified) Bessel functions of the first kind.

II. Continuous-time Event-triggered Boundary Control

Let us consider the following 1-D reaction-diffusion sampled-data boundary control system with constant coefficients:

$$u_t(x,t) = \varepsilon u_{xx}(x,t) + \lambda u(x,t), \tag{1a}$$

$$u_x(0,t) = 0, \tag{1b}$$

$$u_x(1,t) + qu(1,t) = U_j, \tag{1c}$$

for $t\in[t_j,t_{j+1})$ with $\{t_j\}_{j\in\mathbb{N}}$ being an increasing sequence generated by a continuous-time event-trigger and the initial condition $u[0]\in L^2(0,1)$, where $\varepsilon,\lambda>0$, and U_j is the continuous-time event-triggered boundary control input held constant for $t\in[t_j,t_{j+1}), j\in\mathbb{N}$.

The well-posedness of the boundary controlled plant (1) with piecewise constant inputs in between two sampling instants can be established in the following proposition.

Proposition 1: For every $u[t_j]\in L^2(0,1)$, there exist a unique solution $u:[t_j,t_{j+1}]\times[0,1]\to\mathbb{R}$ between two time instants t_j and t_{j+1} such that $u\in C^0\big([t_j,t_{j+1}];L^2(0,1)\big)\cap C^1\big((t_j,t_{j+1})\times[0,1]\big)$ with $u[t]\in C^2([0,1])$ which satisfy (1b),(1c) for $t\in(t_j,t_{j+1}]$ and (1a) for $t\in(t_j,t_{j+1}], x\in(0,1)$.

Proof: This is a straightforward application of Theorem 4.11 in [20]. ■

Assumption 1: The parameters q,λ, and ε satisfy the following relation:

$$q > \frac{\lambda}{2\varepsilon} + \frac{1}{2}. \tag{2}$$

Remark 1: Assumption 1 is required to avoid the use of the signal $u(1,t)$ in the nominal control law for which it is impossible to obtain a useful bound on its rate of change. Furthermore, It is worth mentioning that an eigenfunction expansion of the solution of (1) with zero input shows that the system is unstable when $\lambda > \varepsilon\pi^2/4$, no matter what $q>0$ is.

In [9], the authors propose an observer-based continuous-time event-triggered boundary control strategy which ensures the global exponential convergence of the closed-loop system containing the plant and the observer to the equilibrium point. In this work, we consider its full-state feedback equivalence.

The continuous-time event-triggered boundary control strategy consists of two components [9]:

1) A event-triggered boundary control input U_j based on the infinite-dimensional backstepping technique

$$U_j = \int_0^1 k(y)u(y,t_j)dy, \tag{3}$$

for all $t\in[t_j,t_{j+1}), j\in\mathbb{N}$, where

$$k(y) = rK(1,y) + K_x(1,y), \tag{4}$$

with

$$K(x,y) = -\frac{\lambda}{\varepsilon}x\frac{I_1\big(\sqrt{\lambda(x^2-y^2)/\varepsilon}\big)}{\sqrt{\lambda(x^2-y^2)/\varepsilon}}, \tag{5}$$

for $0\leq y\leq x\leq 1$, and

$$r = q - \frac{\lambda}{2\varepsilon}. \tag{6}$$

2) A continuous-time event-trigger determining event-times

$$t_{j+1} = \inf\big\{t\in\mathbb{R}_+ | t>t_j, \Gamma(t)>0, j\in\mathbb{N}\big\}, \tag{7}$$

with $t_0=0$ where

$$\Gamma\big(d(t),m(t)\big) := \Gamma(t) = d^2(t) - \gamma m(t),\ \gamma>0. \tag{8}$$

Here

$$d(t) := \int_0^1 k(y)\big(u(y,t_j)-u(y,t)\big)dy, \tag{9}$$

for all $t\in[t_j,t_{j+1}), j\in\mathbb{N}$, and $m(t)$ satisfies the ODE

$$\dot{m}(t) = -\eta m(t) - \rho d^2(t) + \beta_1\|u[t]\|^2 + \beta_2|u(1,t)|^2, \tag{10}$$

for all $t\in(t_j,t_{j+1}), j\in\mathbb{N}$ with $m(t_0)=m(0)>0$ and $m(t_j^-)=m(t_j)=m(t_j^+)$ and $\eta,\rho,\beta_1,\beta_2>0$.

Below we outline the selection of event-trigger parameters $\gamma,\eta,\beta_1,\beta_2,\rho>0$ which ensures the Zeno-free behavior and the global exponential convergence of the closed-loop system (1)-(10) to zero in L^2-sense. The arguments for parameter selection closely follow those in [9]. Hence, we state the conditions on parameters without further details.

Remark 2: [Selection of event-trigger parameters] The parameters $\gamma,\eta>0$ are design parameters, and β_1,β_2 are chosen such that

$$\beta_1 = \frac{\alpha_1}{\gamma(1-\sigma)},\ \beta_2 = \frac{\alpha_2}{\gamma(1-\sigma)}, \tag{11}$$

where $\sigma\in(0,1)$ and

$$\alpha_1 = 3\int_0^1 \Big(\varepsilon k''(y) + \varepsilon k(1)k(y) + \lambda k(y)\Big)^2 dy, \tag{12}$$

$$\alpha_2 = 3\big(\varepsilon qk(1) + \varepsilon k'(1)\big)^2, \tag{13}$$

with $k(y)$ given by (4). The parameter $\rho>0$ is set as

$$\rho = \frac{\varepsilon\kappa B}{2}, \tag{14}$$

for $B,\kappa>0$ are chosen such that

$$B\left(\varepsilon\min\left\{r-\frac{1}{2},\frac{1}{2}\right\} - \frac{\varepsilon}{2\kappa}\right) - 2\beta_1\tilde{L}^2 - 2\beta_2 - 4\beta_2\int_0^1 L^2(1,y)dy > 0, \tag{15}$$

where

$$\tilde{L} = 1 + \Big(\int_0^1\int_0^x L^2(x,y)dydx\Big)^{1/2}, \tag{16}$$

with $L(x,y)$ given by

$$L(x,y) = -\frac{\lambda}{\varepsilon} x \frac{J_1\left(\sqrt{\lambda(x^2-y^2)/\varepsilon}\right)}{\sqrt{\lambda(x^2-y^2)/\varepsilon}}, \tag{17}$$

for $0 \le y \le x \le 1$. Note from Assumption 1 that $r > 1/2$, where r is given by (6).

We will summarize the main results of [9] in the following theorem.

Theorem 1: Consider the set of event-times $\{t_j\}_{j\in\mathbb{N}}$ generated by the event-triggered mechanism (7)-(10). Then, it holds that

$$\Gamma(t) \le 0 \text{ for all } t \in [t_j, t_{j+1}), j < j^* \in \mathbb{N}, \tag{18}$$

where $j^* = \inf\{i \in \mathbb{N} | t_i = \sup(\{t_j\}_{j\in\mathbb{N}})\}$. In consequence, with event-trigger parameters $\gamma,\eta,\beta_1,\beta_2,\rho > 0$ chosen as in Remark 2 and subject to the event-triggered boundary control law (3)-(6), the following results can be derived:

R1: The set of event-times $\{t_j\}_{j\in\mathbb{N}}$ generates an increasing sequence. Specifically, it holds that $t_{j+1} - t_j \ge \tau > 0$ where
$$\tau = \frac{1}{a}\ln\left(1 + \frac{\sigma a}{(1-\sigma)(a+\gamma\rho)}\right). \tag{19}$$
Here $\sigma \in (0,1)$ satisfies the relation (11) and
$$a = 1 + 3\varepsilon^2 k^2(1) + \eta > 0, \tag{20}$$
where $k(y)$ is given by (4).

R2: For every $u[0] \in L^2(0,1)$, there exist unique solution $u : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ such that $u \in C^0(\mathbb{R}_+; L^2(0,1) \cap C^1(I \times [0,1])$ with $u[t] \in C^2([0,1])$ which satisfy (1b),(1c),(3) for all $t > 0$ and (1a) for all $t > 0, x \in (0,1)$, where $I = \mathbb{R}_+ \backslash \{t_j \ge 0, j \in \mathbb{N}\}$.

R3: The dynamic variable $m(t)$ governed by (10) with $m(0) > 0$ satisfies $m(t) > 0$ for all $t > 0$.

R4: Subject to Assumption 1, the closed-loop system (1),(3)-(10) globally exponentially converges to zero in L^2-sense satisfying the following estimate
$$\|u[t]\| \le M e^{-\frac{b^*}{2}t}\sqrt{\|u[0]\|^2 + m(0)}, \tag{21}$$
where
$$M = \sqrt{\frac{2\tilde{L}^2}{B}\max\left\{\frac{B\tilde{K}^2}{2}, 1\right\}}, \tag{22}$$
and
$$b^* = \min\left\{\frac{2b}{B}, \eta\right\}. \tag{23}$$
Here $\tilde{L}$ is given by (16), $B > 0$ satisfies (15), $\tilde{K} = 1 + \left(\int_0^1\int_0^x K^2(x,y)dydx\right)^{1/2}$, and $b = \frac{\varepsilon B}{4} - \beta_1\tilde{L}^2 - 2\beta_2\int_0^1 L^2(1,y)dy > 0$.

III. Self-Triggered Boundary Control (STBC)

In this section, we describe a method for designing an STBC approach for the system (1) subject to Assumption 1 using the CETBC scheme (3)-(10). We achieve such a result by finding an upper bound for $d^2(t)$ and lower bound for $m(t)$, which are the constituent terms of the triggering function $\Gamma(t)$ given by (8), while keep using the pre-designed

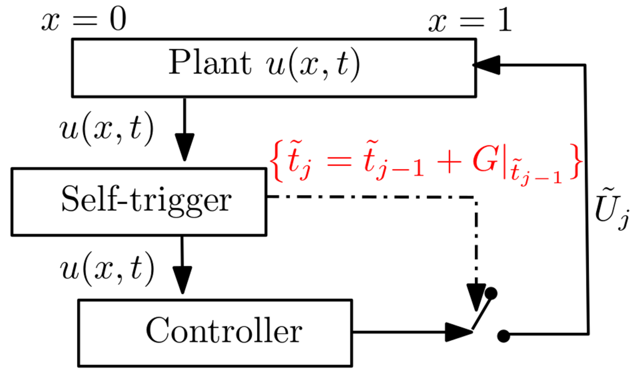

Fig. 1: Self-triggered closed-loop system.

feedback law (3)-(6). The closed-loop system under STBC is illustrated in Fig. 1.

Let the parameters $\eta,\gamma,\beta_1,\beta_2 > 0$ be chosen according to Remark 2. The envisioned STBC consists of two components:

1) An event-triggered boundary control input based on infinite-dimensional backstepping technique
$$\tilde{U}_j = \int_0^1 k(y)u(y,\tilde{t}_j)dy, \tag{24}$$
where $k(y)$ is given by (4)-(6), for all $t \in [\tilde{t}_j, \tilde{t}_{j+1}), j \in \mathbb{N}$. The boundary condition (1c) is modified as
$$u_x(1,t) + qu(1,t) = \tilde{U}_j. \tag{25}$$

2) A self-trigger determining the event-times
$$\tilde{t}_{j+1} = \tilde{t}_j + G\big(\|u[\tilde{t}_j]\|, m(\tilde{t}_j)\big), \tag{26}$$
with $\tilde{t}_0 = 0$ where $G(\cdot,\cdot) > 0$ is a positively lower-bounded function to be designed and $m(t)$ satisfies
$$\dot{m}(t) = -\eta m(t) - \rho d^2(t) + \beta_1\|u[t]\|^2 + \beta_2|u(1,t)|^2, \tag{27}$$
for all $t \in (\tilde{t}_j, \tilde{t}_{j+1}), j \in \mathbb{N}$ with $m(\tilde{t}_0) = m(0) > 0$ and $m(\tilde{t}_j^-) = m(\tilde{t}_j) = m(\tilde{t}_j^+)$ and $\eta,\rho,\beta_1,\beta_2 > 0$ chosen as in Remark 2. In (27), $d(t)$ is defined as
$$d(t) := \int_0^1 k(y)\big(u(y,\tilde{t}_j) - u(y,t)\big)dy, \tag{28}$$
where $k(y)$ is given by (4)-(6), for all $t \in [\tilde{t}_j, \tilde{t}_{j+1}), j \in \mathbb{N}$.

Note that unlike the continuous-time event-trigger (7)-(10) which has to be evaluated continuously, the self-trigger (26) determines the time of the next event using the available information at the current event time.

The well-posedness of the closed-loop system (1a),(1b),(24),(25) follows from Corollary 1 with the increasing sequence of event times given by $\{\tilde{t}_j\}_{j\in\mathbb{N}}$.

Corollary 1: Let there be a self-triggered boundary control approach (24)-(28) which generates an increasing set of event-times $\{\tilde{t}_j\}_{j\in\mathbb{N}}$. Then, for every $u[0] \in L^2(0,1)$, there exists a unique solution $u : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ such that $u \in C^0(\mathbb{R}_+; L^2(0,1) \cap C^1(\tilde{I} \times [0,1])$ with $u[t] \in C^2([0,1])$ which satisfy (1b),(25),(24) for all $t > 0$ and (1a) for all $t > 0, x \in (0,1)$, where $\tilde{I} = \mathbb{R}_+ \backslash \{\tilde{t}_j \geq 0, j \in \mathbb{N}\}$.

Our aim is to design the positively and uniformly lower-bounded function $G(\cdot,\cdot)$ to guarantee that $\Gamma(t)$ given by (8) remains non-positive along the solution of (1a),(1b),(24)-(28) for all $t \in [\tilde{t}_j, \tilde{t}_{j+1}), j \in \mathbb{N}$. If it is possible to ensure that $\Gamma(t) \leq 0$ for all $t \in [\tilde{t}_j, \tilde{t}_{j+1}), j \in \mathbb{N}$ by updating the control input according to (24) at times generated by the self-trigger (26)-(28), then results equivalent to *R3* and *R4* in Theorem 1 can be obtained respectively for the closed-loop system (1a),(1b),(24)-(28).

In the following lemma, we obtain an upper bound for $d^2(t)$ satisfying (28) and a lower bound for $m(t)$ satisfying (27), which are instrumental in designing the positively lower bounded function $G(\cdot,\cdot)$ in the self-trigger (26).

Lemma 1: Let there be an STBC approach (24)-(28) which generates an increasing set of event times $\{\tilde{t}_j\}_{j\in\mathbb{N}}$ with $\tilde{t}_j = 0$. Then, for the closed-loop system (1a),(1b),(24),(25), the error $d(t)$ given by (28), and $m(t)$ which satisfies (27), the following estimates hold

$$\|u[t]\|^2 \leq \Big(1 + \frac{\varepsilon^2\|k\|^2}{\lambda^2}\Big)\|u[\tilde{t}_j]\|^2 e^{2\lambda(t-\tilde{t}_j)}, \tag{29}$$

$$d^2(t) \leq H(\tilde{t}_j)e^{2\lambda(t-\tilde{t}_j)}, \tag{30}$$

and

$$m(t) \geq m(\tilde{t}_j)e^{-\eta(t-\tilde{t}_j)} - \frac{\rho H(\tilde{t}_j)}{2\lambda+\eta}e^{-\eta(t-\tilde{t}_j)}\Big(e^{(2\lambda+\eta)(t-\tilde{t}_j)} - 1\Big), \tag{31}$$

where

$$H(t) = 2\|k\|^2\Big(2 + \frac{\varepsilon^2\|k\|^2}{\lambda^2}\Big)\|u[t]\|^2, \tag{32}$$

with $k(y)$ given by (4)-(6), for all $t \in [\tilde{t}_j, \tilde{t}_{j+1}), j \in \mathbb{N}$.

Proof: Consider the positive definite function

$$V = \frac{1}{2}\int_0^1 u^2(x,t)dx. \tag{33}$$

Taking its time derivative along the solution of (1a),(1b),(25),(24), we can obtain that

$$\dot{V} = -\varepsilon q u^2(1,t) - \varepsilon\|u_x[t]\|^2 + \lambda\|u[t]\|^2 + \varepsilon u(1,t)\tilde{U}_j, \tag{34}$$

for $t \in (\tilde{t}_j, \tilde{t}_{j+1}), j \in \mathbb{N}$. Then, using Young's inequality and Cauchy Schwarz inequality, we can show that

$$\dot{V} \leq -\varepsilon q u^2(1,t) - \varepsilon\|u_x[t]\|^2 + \lambda\|u[t]\|^2 + \frac{\varepsilon h}{2}u^2(1,t) + \frac{\varepsilon}{2h}\tilde{U}_j^2, \tag{35}$$

in $t \in (\tilde{t}_j, \tilde{t}_{j+1}), j \in \mathbb{N}$ for some $h > 0$. Let us select h as follows:

$$h = \frac{\lambda}{2\varepsilon}. \tag{36}$$

Then, one can rewrite (35) as

$$\dot{V} \leq -\varepsilon\Big(q - \frac{\lambda}{4\varepsilon}\Big)u^2(1,t) - \varepsilon\|u_x[t]\|^2 + \lambda\|u[t]\|^2 + \frac{\varepsilon^2}{\lambda}\tilde{U}_j^2, \tag{37}$$

for $t \in (\tilde{t}_j, \tilde{t}_{j+1}), j \in \mathbb{N}$. Using Cauchy Schwarz inequality, and using (24) under the consideration of (33), the following holds:

$$\tilde{U}_j^2 \leq 2\|k\|^2 V(\tilde{t}_j). \tag{38}$$

Thus, recalling Assumption 1 from which it follows that $q > \lambda/2\varepsilon$, we can write (37) as

$$\dot{V} \leq 2\lambda V(t) + \frac{2\varepsilon^2\|k\|^2}{\lambda}V(\tilde{t}_j), \tag{39}$$

for $t \in (\tilde{t}_j, \tilde{t}_{j+1}), j \in \mathbb{N}$ from which we can obtain using the Comparison principle that

$$V(t) \leq e^{2\lambda(t-\tilde{t}_j)}V(\tilde{t}_j) + \frac{\varepsilon^2\|k\|^2}{\lambda^2}V(\tilde{t}_j)\Big(e^{2\lambda(t-\tilde{t}_j)} - 1\Big), \tag{40}$$

for $t \in [\tilde{t}_j, \tilde{t}_{j+1}), j \in \mathbb{N}$ from which we can obtain (29). Using Cauchy Schwarz inequality and Young's inequality on (28), we can show that

$$d^2(t) \leq 2\|k\|^2\|u[\tilde{t}_j]\|^2 + 2\|k\|^2\|u[t]\|^2. \tag{41}$$

Then, using (29) on (41), we can obtain (30). Considering the dynamics of $m(t)$ given by (27) and the relation (30), we can show

$$\dot{m}(t) \geq -\eta m(t) - \rho H(\tilde{t}_j)e^{2\lambda(t-t_j)}, \tag{42}$$

for $t \in (\tilde{t}_j, \tilde{t}_{j+1}), j \in \mathbb{N}$ from which we can obtain (31) using the Comparison principle. This completes the proof. ■

A. Design of the positive function $G(\cdot,\cdot)$

Let there be an STBC approach (24)-(28) which ensures the continuous-time event-trigger (7),(8) with $m(t)$ governed by (27) and $d(t)$ given by (28) satisfies $\Gamma(t) \leq 0$ for all $t \in [\tilde{t}_j, \tilde{t}_{j+1}), j \in \mathbb{N}$. This leads to $m(t) > 0$ for all $t > 0$ (see *R3* in Theorem 1). Assume an event has occurred at $t = \tilde{t}_j$. Then, $\Gamma(\tilde{t}_j) = -\gamma m(\tilde{t}_j) < 0$, and $\Gamma(t)$ will remain definitely negative until $t = \tilde{t}_j + \tau$, where τ is the minimal dwell-time given by (19). Assume that there exists a unique $t_j^* > \tilde{t}_j + \tau$ such that

$$H(\tilde{t}_j)e^{2\lambda(t_j^*-\tilde{t}_j)} = \gamma m(\tilde{t}_j)e^{-\eta(t_j^*-\tilde{t}_j)} - \frac{\gamma\rho H(\tilde{t}_j)}{2\lambda+\eta}e^{-\eta(t_j^*-\tilde{t}_j)}\Big(e^{(2\lambda+\eta)(t_j^*-\tilde{t}_j)} - 1\Big), \tag{43}$$

where the left-hand side is the upper-bound of $d^2(t_j^*)$ given by (30) and the right-hand side is the lower-bound of $\gamma m(t_j^*)$ given by (31). Note that the right-hand side of (30) is an increasing function of t whereas the right-hand side of (31) is a decreasing function of t. Thus, as we assume that these two functions become equal at a unique $t_j^* > \tilde{t}_j + \tau$, considering (30),(31), and (43), we can be certain that

$$d^2(t) \leq \gamma m(t), \tag{44}$$

or equivalently

$$\Gamma(t) \leq 0, \tag{45}$$

for $t \in [\tilde{t}_j, t_j^*)$. Thus, it is convenient to choose

$$\tilde{t}_{j+1} = t_j^* > \tilde{t}_j + \tau, \tag{46}$$

as the next time event. We formalize this idea in Lemma 2.

Lemma 2: Let us select the parameters $\gamma,\eta,\rho,\beta_1,\beta_2 > 0$ as outlined in Remark 2. Further, let us consider the STBC approach (24)-(28) with the positively and uniformly lower-bounded function $G(\cdot,\cdot)$ chosen as

$$G(\|u[\tilde{t}_j]\|,m(\tilde{t}_j)) = \max\left\{\tau, \frac{1}{2\lambda+\eta}\ln\left(\frac{\gamma m(\tilde{t}_j)+\frac{\gamma\rho H(\tilde{t}_j)}{2\lambda+\eta}}{H(\tilde{t}_j)+\frac{\gamma\rho H(\tilde{t}_j)}{2\lambda+\eta}}\right)\right\}, \quad (47)$$

which generates an increasing set of event-times $\{\tilde{t}_j\}_{j\in\mathbb{N}}$ with $\tilde{t}_0 = 0$. In (47), $\tau > 0$ is given by (19). Then, for $\Gamma(t)$ defined as (8) where $d(t)$ is given by (28) and $m(t)$ which satisfies (27) with $m(0) > 0$, it holds that $\Gamma(t) \leq 0$ for all $t \in [\tilde{t}_j,\tilde{t}_{j+1}), j \in \mathbb{N}$ and $m(t) > 0$ for all $t > 0$ along the solution of (1a),(1b),(24)-(28).

Proof: Let us assume that an event has occurred at $t = \tilde{t}_j$ and $m(\tilde{t}_j) > 0$. Then, $\Gamma(\tilde{t}_j) = -\gamma m(\tilde{t}_j) < 0$ and $\Gamma(t)$ will remain definitely negative until $t = \tilde{t}_j + \tau$, where τ is the minimal dwell-time given by (19). Let

$$\tilde{G}_j := \frac{1}{2\lambda+\eta}\ln\left(\frac{\gamma m(\tilde{t}_j)+\frac{\gamma\rho H(\tilde{t}_j)}{2\lambda+\eta}}{H(\tilde{t}_j)+\frac{\gamma\rho H(\tilde{t}_j)}{2\lambda+\eta}}\right). \quad (48)$$

Then, solving for t_j^* in (43), we can show that

$$t_j^* - \tilde{t}_j = \tilde{G}_j. \quad (49)$$

If $\tilde{G}_j > \tau$, then, we can select

$$\tilde{t}_{j+1} = \tilde{t}_j + \tilde{G}_j. \quad (50)$$

In this case, as we argued in Section III-A, it can be ensured that $\Gamma(t) \leq 0$ for $t \in [\tilde{t}_j,\tilde{t}_{j+1})$. If $\tilde{G}_j \leq \tau$, then we can select

$$\tilde{t}_{j+1} = \tilde{t}_j + \tau, \quad (51)$$

which will ensure that $\Gamma \leq 0$ for $t \in [\tilde{t}_j,\tilde{t}_{j+1})$ while preventing the occurrence of Zeno phenomenon. Thus, the designer is allowed to choose $G(\cdot,\cdot)$ as defined in (47).

As $\Gamma(t) \leq 0$ for $t \in [\tilde{t}_j,\tilde{t}_{j+1})$, we can write from (8) that $d^2(t) \leq \gamma m(t)$ for $t \in [\tilde{t}_j,\tilde{t}_{j+1})$. Then, considering the dynamics of $m(t)$ given by (27), we can write that $\dot{m}(t) \geq -(\eta+\gamma\rho)m(t)$ for $t \in (\tilde{t}_j,\tilde{t}_{j+1})$, which leads to $m(t) \geq e^{-(\eta+\gamma\rho)(t-\tilde{t}_j)}m(\tilde{t}_j) > 0$ for $t \in [\tilde{t}_j,\tilde{t}_{j+1})$. Considering the time continuity of $m(t)$, we have that $m(\tilde{t}_{j+1}^-) = m(\tilde{t}_{j+1}) > 0$. Then, after the control input has been updated at $t = \tilde{t}_{j+1}$, we have that $\Gamma(\tilde{t}_{j+1}) = -\gamma m(\tilde{t}_{j+1}) < 0$.

In a similar way, one can analyze the behavior of $\Gamma(t)$ and $m(t)$ in all $t \in [\tilde{t}_j,\tilde{t}_{j+1})$ for any $j \in \mathbb{N}$ starting from the first event at $\tilde{t}_0 = 0$ where $m(0) > 0$. Therefore, we can obtain that $\Gamma(t) \leq 0$ for all $t \in [\tilde{t}_j,\tilde{t}_{j+1}), j \in \mathbb{N}$ and $m(t) > 0$ for all $t > 0$ along the solution of (1a),(1b),(24)-(28). ■

Theorem 2: Consider the STBC approach given by (24)-(28), which generates an increasing set of increasing event-times $\{\tilde{t}_j\}_{j\in\mathbb{N}}$ with $\tilde{t}_0 = 0$. Then, subject to Assumption 1, the closed-loop system (1a),(1b),(24)-(28) has a unique solution in the sense of Corollary 1 and globally exponentially converges to zero in L^2-sense satisfying the estimate (21)-(23).

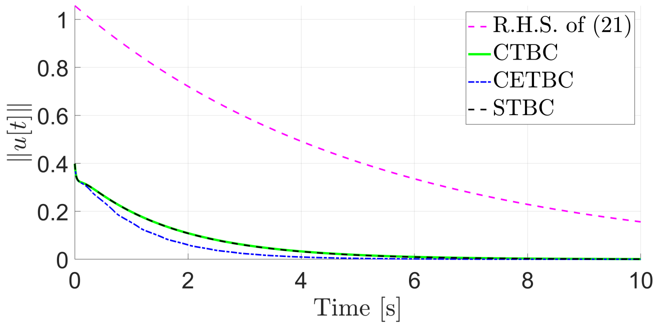

Fig. 2: Evolution of $\|u[t]\|$ under continuous-time boundary control (CTBC), CETBC, and STBC.

Proof: Since the STBC approach (24)-(28) generates an increasing sequence, the well-posedness of the boundary controlled plant directly follows from Corollary 1. As it has been shown that $\Gamma(t) \leq 0$ or all $t \in [\tilde{t}_j,\tilde{t}_{j+1}), j \in \mathbb{N}$ and $m(t) > 0$ for all $t > 0$ in Lemma 2, the global exponential convergence of the closed-loop system to zero satisfying the estimate (21)-(23) directly follows from Theorem 1. This completes the proof. ■

IV. Numerical Simulations

We consider a reaction-diffusion PDE with $\varepsilon = 1; \lambda = 2.5; q = 2$ and the the initial condition $u[0] = 10x^2(x-1)^2$. The parameters for the CETBC and STBC are chosen as follows: $m(0) = 10^{-4}, \gamma = 1, \eta = 1$, and $\sigma = 0.9$. It can be shown using (12) and (13) that $\alpha_1 = 34.67$ and $\alpha_2 = 54.05$. Therefore, from (11), we can obtain $\beta_1 = 346.65$ and $\beta_2 = 540.55$. Let us choose B and κ_1 as $B = 33078$ and $\kappa_1 = 5$ so that (15) is satisfied. Then, from (14), we can obtain $\rho = 165390$. The minimal dwell-time τ for the CETBC calculated using (19) is $5.4365e \times 10^{-5}$s. Since this is extremely small, we use $h = 0.00005$s to time discretize the plant dynamics under continuous-time boundary control, CETBC, and STBC using the implicit Euler scheme. Space discretization was done using a step size of $\Delta x = 0.05$.

Fig. 2 shows the response of the closed-loop system under continuous-time boundary control (CTBC), CETBC, and STBC whereas Fig. 3 shows the corresponding control inputs. It is clear that STBC preserves the convergence properties of CETBC since it obeys (21). Fig. 4 and 5 illustrate the dwell-times of CETBC and STBC, respectively. We can observe STBC triggers more frequent events than CETBC at the beginning. However, when the closed-loop system is approaching the equilibrium, events triggered by STBC become less frequent. Fig. 6 shows the evolution of $\Gamma(t)$ with time under CETBC and STBC, and we can observe that $\Gamma(t)$ remains nonpositive for all $t \geq 0$ as advertised.

V. Conclusion

In this paper, a novel self-triggered boundary control (STBC) strategy for a class of reaction-diffusion system with Robin boundary actuation has been proposed. The key idea of the developed method is the transformation of a class of continuous-time dynamic event-triggers which

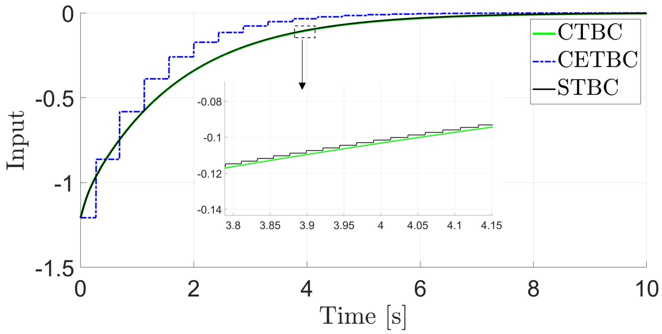

Fig. 3: Control input under continuous-time boundary control (CTBC), CETBC, and STBC.

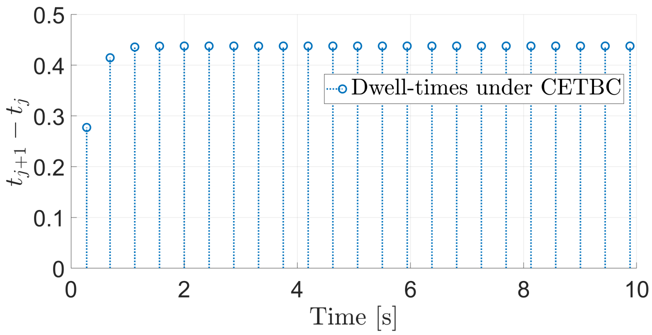

Fig. 4: Dwell-times under CETBC.

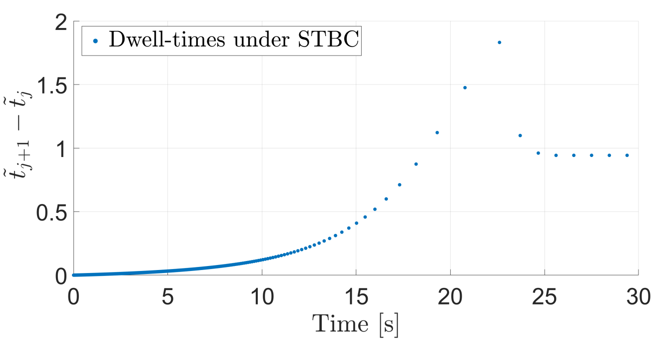

Fig. 5: Dwell-times under STBC. For better understanding, we have shown the graph upto 30s.

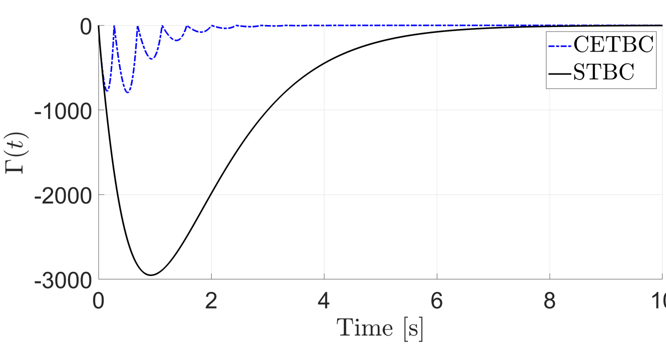

Fig. 6: Evolution of $\Gamma(t) = d^2(t) - \gamma m(t)$ under CETBC and STBC.

require continuous monitoring to self-triggers. Under STBC, we have designed a positively and uniformly lower-bounded function which, when evaluated at the time of an event, outputs the waiting time until the next event. We also have proved that the closed-loop system well-posedness and the global L^2-exponential convergence to zero under continuous-time event-triggered boundary control are preserved under the proposed STBC. The conducted numerical simulation has demonstrated the validity of the theoretical developments.

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