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A Submodular Energy Function Approach to Controlled Islanding with Provable Stability

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***Abstract*— Cascading failures occur when failures of one or more nodes in a network lead to failures in neighboring nodes that propagate through the remainder of the network. One approach to mitigate cascading failures is through controlled islanding, in which a subset of edges is deliberately removed in order to partition the network into disjoint and stable islands. In this paper, we propose a submodular optimization algorithm for selecting edges to remove in order to create islands with provable stability. In contrast to existing approaches that optimize over stability-related metrics such as network coherence, our approach maps standard Lyapunov stability conditions to the objective function of an optimization problem. We prove that this optimization problem is equivalent to minimizing a supermodular function subject to a matroid basis constraint. We propose a local search algorithm for selecting the islands with provable optimality bounds, and discuss special cases including signed linear consensus and nonlinear synchronization dynamics. We simulate our approach using linear consensus dynamics with negative edges and find that our proposed algorithms partition the network into a stable island and an unstable island.**

I. Introduction

Many large-scale infrastructures, including power grids, transportation networks, and communication systems are comprised of complex networks of coupled dynamical systems. Robustness and stability of such networks is critical, as instability can lead to outages, safety violations, and potential economic damage, equipment failures, and loss of life. Stability is particularly threatened by cascading failures, in which a disturbance causes an outage of one or more network nodes. These outages may destabilize neighboring nodes via the coupled dynamics, leading to failures that propagate through and destabilize the entire network. Cascading failures have led to economically costly blackouts in power systems including the 2023 Pakistan blackout [1] and 2021 Texas blackout [2].

Cascading failures occur when a disturbance to the system is so significant that the network states cannot be stabilized even by coordinated control efforts. An alternative approach, denoted as *controlled islanding*, is to deliberately remove a subset of network edges in order to partition the network into disjoint islands [3], [4]. The goal of this approach is to disconnect the remaining network from the nodes that are directly affected by the disturbance, and thus preserve stability of the residual network at the cost of reduced connectivity. An extensive literature has been developed on islanding for robustness and resilience of power systems [5], [6].

The stability of the islands will be determined by which network edges have been removed to form the islands. This set of edges must be chosen very quickly following a disturbance, as cascade failures may propagate through a network within seconds. Computing islands, however, is challenging for two key reasons. First, the problem of selecting a subset of edges to remove is inherently combinatorial in nature, and hence incurs exponential computational complexity unless additional computational structure can be found. Second, stability of each island must be verified, which may be difficult for complex networks with nonlinear dynamics.

Two fundamental approaches have been proposed in the literature to address these challenges. In the first (classical) approach, candidate choices of islands are precomputed by hand or through domain-specific rules of thumb. Stability is then analyzed through a combination of simulations and energy based methods such as the transient energy function method [7]. This approach results in verifiable stability but is computationally intractable, especially in when islands must be computed on a short timescale in response to a disturbance. The second approach selects islands by optimizing over metrics that are correlated with stability such as generator coherence [8]–[10]. Optimization methods include mixed integer programming [11]–[14], submodular optimization [15]–[17], and spectral clustering [18]. While this approach is more computationally efficient, stability of the islands must still be verified through simulation. An approach that combines the stability guarantees of energy-based methods with the computational efficiency of heuristic approaches would enable scalable islanding computation with verifiable stability, however, at present no such approach is available in the literature.

This paper proposes a submodular optimization approach to controlled islanding with provable stability. The idea of our approach is to formulate a combinatorial optimization problem whose objective function is derived from a candidate Lyapunov function for the system. We then prove that this optimization problem is equivalent to minimizing a supermodular function under a matroid basis constraint, which can be approximated in polynomial time. We make the following specific contributions:

- We formulate the problem of optimizing a trade-off between the stability of the islands and the cost of

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network outages in unstable islands. We construct an objective function that captures Lyapunov stability of the islands. This formulation captures the fact that it may not be possible to stabilize all of the islands, and hence the goal is to minimize the cost of instability.

- We prove that islanding is equivalent to optimizing over the bases of a matroid, in particular, a graphic matroid defined on an augmented network graph. We then show that the objective function is equivalent to a nonincreasing supermodular function when restricted to the matroid bases.
- We propose a local search algorithm with provable $1/2$-optimality for solving the problem. We further analyze special cases of linear and generator swing equation dynamics.
- We simulate our approach in a case study of signed consensus dynamics using the IEEE 30- 57- and 118-bus power systems for the network topology. Even when 30% of the edges have negative weights, our approach is able to partition the system into one stable and one unstable island.

The paper is organized as follows. Section II presents related work. Section III presents the system and islanding models as well as background on submodularity and matroids. Section IV formulates the problem and presents our proposed submodular optimization approach to islanding. Section V presents two special cases of our framework. Section VI presents simulation results. Section VII concludes the paper.

II. Related Work

Power system stability is critical to satisfy customers' power demands [19], [20]. Following disturbances such as natural disasters [2] and cyber attacks [21], power systems may be destablized and incur cascading failures. Controlled islanding has been demonstrated to be an effective approach to mitigate cascading failures and improve the resilience of power systems [6], [22].

Various techniques have been proposed for islanding computations of power systems. In [8]–[10], slow coherency-based approaches were adopted. These work first categorized the generators based on their behavior following disturbances. Then the power system was partitioned such that incoherent generators were disconnected. Two step spectral clustering-based approach was later proposed in [18] to compute islandings. This approach first grouped the generators based on their dynamics and post-disturbance behaviors. In the second step, the islands were computed by incorporating metrics such as power flow disruption or load-generation imbalance. Computing islands to jointly minimize power flow disruption, load-generation imbalance, and/ or island stability was formulated as mixed-integer linear programs (MILPs) in [11]–[14]. However, MILPs are NP-hard and can be computationally challenging to solve. Furthermore, these MILP-based solution approaches could not provide any verifiable stability guarantee.

To improve the scalability of islanding computations, ordered binary decision diagram (OBDD)-based methods were used in [23], [24], where the power systems were simplified. However, these approaches could not provide optimality guarantees on the islanding. Submodularity-based approaches were developed in [15]–[17], where the authors proved that power flow disruption and generator coherency were supermodular, leading to an efficient islanding computation with $1/2$-optimality guarantees. These approaches, however, could not guarantee the satisfaction of stability constraints when formulating islands. In this paper, we consider the Lyapunov stability of islands and develop an algorithm for controlled islanding with a stability guarantee.

III. Preliminaries

This section presents the class of systems that we consider and our model of how islanding impacts the system dynamics. We then give background on submodularity and matroids.

A. System and Islanding Model

We consider a network of n nodes with node set $V = \{1,\ldots,n\}$ and edge set $E \subseteq V \times V$. The graph $G=(V,E)$ is undirected. We let $N(i) = \{j : (i,j) \in E\}$ denote the neighbor set of node i. Each node i has a state $x_i(t) \in \mathbb{R}^{p_i}$ with dynamics

$$\dot{x}_i(t) = f_i(x_i(t)) + \sum_{j \in N(i)} f_{ij}(x_i(t), x_j(t)). \qquad (1)$$

We let $M = \sum_{i=1}^{n} p_i$ denote the dimension of the state of the entire network. A valid *islanding* of the network consists of a partition $I_1,\ldots,I_m$ with $I_r = (V_r, E_r)$, $V = \bigcup_{r=1}^{m} V_r$, $V_r \cap V_s = \emptyset$ for $r \neq s$, and $E_r = E \cap (V_r \times V_r)$. We assume that, when islanding occurs, the state $x_i(t)$ of node i does not change, but the dynamics $\dot{x}_i(t)$ instantaneously changes to

$$\dot{x}_i(t) = f_i(x_i(t)) + \sum_{j \in N(i) \cap V_r} f_{ij}(x_i(t), x_j(t)), \qquad (2)$$

where r is the unique index with $i \in V_r$.

B. Background on Submodularity and Matroids

For any finite set Z, a function $f : 2^Z \rightarrow \mathbb{R}$ is submodular if, for any $S \subseteq T \subseteq Z$ and $v \notin T$, we have

$$f(S \cup \{v\}) - f(S) \geq f(T \cup \{v\}) - f(T).$$

A function $f : 2^Z \rightarrow \mathbb{R}$ is monotone nonincreasing (resp. nondecreasing) if, for any S, T with $S \subseteq T$, we have $f(S) \geq f(T)$ (resp. $f(S) \leq f(T)$). Nonnegative weighted sums of submodular (resp. supermodular) functions are submodular (resp. supermodular). Furthermore, if $f(S)$ is nonincreasing and supermodular, then $\max\{f(S), c\}$ is nonincreasing and supermodular for any constant c. We next define the concept of a matroid.

Definition 1: A tuple $\mathcal{M} = (Z, \mathcal{I})$, where Z is a finite set and $\mathcal{I}$ is a collection of subsets of Z, is a matroid if: (i) $\emptyset \in \mathcal{I}$, (ii) $A \subseteq B$ and $B \in \mathcal{I}$ implies that $A \in \mathcal{I}$, and (iii) if $A, B \in \mathcal{I}$ and $|A| < |B|$, then there exists $v \in B \setminus A$ with $(A \cup \{v\}) \in \mathcal{I}$.

For any matroid $\mathcal{M} = (Z, \mathcal{I})$, the collection $\mathcal{I}$ is denoted as the independent sets of $\mathcal{M}$. A maximal independent set of a matroid is a basis, with the set of bases of matroid $\mathcal{M}$ denoted as $\mathcal{B}(\mathcal{M})$.

Definition 2: Let $\mathcal{M} = (Z, \mathcal{I})$ be a matroid. The rank function of $\mathcal{M}$ is defined as

$$\rho_{\mathcal{M}}(S) = \max \{|R| : R \subseteq S, R \in \mathcal{I}\}.$$

By inspection, $S \in \mathcal{I}$ iff $\rho_{\mathcal{M}}(S) = |S|$. Furthermore, it can be shown that $\rho_{\mathcal{M}}(S)$ is nondecreasing and submodular in S. Graphic matroids are defined by the following lemma.

Lemma 1: Let $G = (V, E)$ be a graph. Define $\mathcal{I}$ as the collection of subsets of edges that do not contain a cycle. Then $\mathcal{M} = (E, \mathcal{I})$ is a matroid.

The basis sets of a graphic matroid correspond to spanning trees of the graph G.

IV. Problem Formulation

In what follows, we formulate the optimal stable islanding problem and then present our framework for computing stable islands.

A. Problem Statement

The goal of the system is to satisfy a stability constraint defined as follows. We let Y denote a set of initial states (i.e., the state of the system when islanding occurs), and say that node i is stable under islanding $I_1, \ldots, I_m$ if $x(t_0) \in Y$ implies that $\lim_{t \rightarrow \infty} x_i(t) = 0$ under the dynamics (2).

We next define the cost of islanding. We say that an island I_r is unstable if there exists $i \in V_r$ such that $x_i(t)$ is not stable, while I_r is stable if all nodes $i \in V_r$ are stable. We let $\mathcal{U} \subseteq \{1, \ldots, m\}$ denote the indices of the unstable islands. For node i, we let $c_i \geq 0$ denote the cost of instability. For example, in a power system c_i may represent the cost of shedding a load or shutting down a generator to prevent damage to the system. We let $Z = \bigcup_{r \in \mathcal{U}} V_r$ denote the set of nodes that belong to unstable islands.

Problem 1: Given a graph $G = (V, E)$ and node dynamics (1), compute islands $I_1, \ldots, I_m$ such that $\sum_{i \in Z} c_i$ is minimized.

B. Islanding Framework

The following gives a sufficient condition for stability of an island.

Proposition 1: Suppose that there is a continuously differentiable positive definite function $W : \mathbb{R}^M \rightarrow \mathbb{R}$ and $\beta \geq 0$ such that $Y \subseteq \Omega_\beta \triangleq \{x : W(x) \leq \beta\}$ and

$$\frac{\partial W}{\partial x_i} \left(f_i(x_i) + \sum_{j \in N(i) \cap V_r} f_{ij}(x_i, x_j) \right) < 0 \quad (3)$$

for all $i \in V_r$ and $x \in \Omega_\beta$ with $x_i \neq 0$. Then the island I_r is stable.

Proof: Suppose, without loss of generality, that $V_r = \{1, \ldots, n'\}$ for some $n' \in \{1, \ldots, n\}$. Let $M' = \sum_{i=1}^{n'} p_i$, and let $\hat{x}(t_0)$ denote the state of the network nodes that are not in island r. Consider the system with state $\overline{x}(t) \in \mathbb{R}^M$ and dynamics (2). Let $\overline{W}(\overline{x})$ be defined by $\overline{W}(\overline{x}) = W(\overline{x}, \hat{x}(t_0))$. If $W(\overline{x}, \hat{x}(t_0)) \leq \beta$, we then have

$$\begin{aligned} \dot{\overline{W}} &= \sum_{i=1}^{n'} \left[\frac{\partial \overline{W}}{\partial x_i}(\overline{x}) \left(f_i(x_i) + \sum_{j \in N(i) \cap V_r} f_{ij}(x_i, x_j) \right) \right] \\ &= \sum_{i=1}^{n'} \left[\frac{\partial W}{\partial x_i}(\overline{x}, \hat{x}(t_0)) \left(f_i(x_i) + \sum_{j \in N(i) \cap V_r} f_{ij}(x_i, x_j) \right) \right] \\ &\leq 0, \end{aligned}$$

implying that $W(\overline{x}(t), \hat{x}(t_0)) \leq \beta$ for all t and hence the set $\{\overline{x} : \overline{W}(\overline{x}) \leq \beta\}$ is positive invariant. Furthermore, we have that $\dot{\overline{W}}$ is zero unless $x_i = 0$ for all $i \in V_r$, implying that $\overline{x}(t) \rightarrow 0$ and hence the island is stable. ∎

Based on Proposition 1, we can formulate Problem 1 as an optimization problem

$$\begin{array}{ll} \underset{I_1, \ldots, I_m}{\text{minimize}} & \sum_{i \in Z} c_i \\ \text{s.t.} & \int_{\{x : W(x) \leq \beta\}} \sum_{r \notin \mathcal{U}} \sum_{i \in V_r} \{\Phi_i(x; I)\}_+ \, dx = 0 \\ & I_1, \ldots, I_m \text{ is a valid islanding} \end{array} \quad (4)$$

where

$$\Phi_i(x; I) \triangleq \frac{\partial W}{\partial x_i} \left(f_i(x_i) + \sum_{j \in N(i) \cap V_r} f_{ij}(x_i, x_j) \right),$$

I represents the islands $I_1, \ldots, I_m$, and m is a variable representing the number of islands. Eq. (4) represents choosing a set of islands such that the cost of unstable nodes is minimized, subject to a constraint that the conditions of Proposition 1 are satisfied. We can relax the constraint of (4) to the objective function as

$$\begin{array}{ll} \underset{I_1, \ldots, I_m}{\text{minimize}} & \alpha \int_{\Omega_\beta} \sum_{r \notin \mathcal{U}} \sum_{i \in V_r} \{\Phi_i(x; I)\}_+ \, dx + \sum_{i \in Z} c_i \\ \text{s.t.} & I_1, \ldots, I_m \text{ is a valid islanding} \end{array} \quad (5)$$

where $\Omega_\beta = \{x : W(x) \leq \beta\}$ and $\alpha > 0$ is a parameter. The value of α can be tuned to prioritize stability. Solving (5) involves searching over all possible choices of $I_1, \ldots, I_m$, which may be exponential in the network size. In order to mitigate this complexity, we will develop an equivalent submodular representation of the problem. As a first step, we define an augmented graph $\overline{G} = (\overline{V}, \overline{E})$ as $\overline{V} = V \cup \{s_0, d_0\}$ and $\overline{E} = E \cup \{(s_0, d_0)\} \cup \{(s_0, i) : i \in V\} \cup \{(d_0, i) : i \in V\}$. A mapping from an islanding strategy to a spanning tree on $\overline{G}$ is given by the following lemma.

Lemma 2: Let $(I_1, \ldots, I_m)$ be an islanding with unstable islands $\mathcal{U}$. For each $r \in \{1, \ldots, m\}$, select one node $v_r \in V_r$ and a spanning tree $A_r \subseteq E_r$. Then

$$\overline{A} = \{(s_0, v_r) : r \notin \mathcal{U}\} \cup \{(s_0, d_0)\} \cup \{(d_0, v_r) : r \in \mathcal{U}\} \cup \bigcup_{r=1}^{m} A_r$$

defines a spanning tree on $\overline{G}$.

Proof: We show that there is exactly one path from s_0 to each other node in the network. We have that $\{(s_0, d_0)\}$ is the unique path from s_0 to d_0 when the network has been partitioned by a valid islanding strategy. If node $v \in V_r$ with $r \in \mathcal{U}$, then the path is given by $\{(s_0, d_0), (d_0, v_r)\}$ concatenated with the path from v_r to v in A_r. If $v \in V_r$ with $r \notin \mathcal{U}$, then the path is given by $\{(s_0, v_r)\}$ concatenated with the path from v_r to v in A_r. ■

Next, we describe a procedure for mapping a spanning tree on $\overline{G}$ to an islanding strategy and set of unstable islands $\mathcal{U}$. Let $\overline{A}$ be a spanning tree on $\overline{G}$. The set of edges $\overline{A} \cap (E \cup \{(d_0, i) : i \in V\})$ defines a set of islands, where i and j are in the same island if they are connected in the graph with edge set $\overline{A} \cap E$. For each stable island r, there is a unique node v_r such that $(v_r, s_0) \in \overline{A}$; we denote this node as the reference node of the island. All nodes that have reference node d_0 are labeled as failed nodes, i.e., nodes in Z.

The above description implies that a valid islanding strategy corresponds to a spanning tree on $\overline{G}$, or equivalently, a basis of $\overline{\mathcal{M}} \triangleq \mathcal{M}(\overline{G})$, the graphic matroid defined on $\overline{G}$. Therefore, we can use $F(A)$ to denote the objective function of (5), where A denotes a spanning tree on $\overline{G}$.

We now define two functions that we will use to derive a submodular formulation of (5). First, for a spanning tree A of $\overline{G}$, define the function $\chi_{ij}(A)$ as

$$\chi_{ij}(A) = \begin{cases} 1, & j \text{ belongs to island with reference } i \\ 0, & \text{else} \end{cases}$$

We let $i = 0$ represent the island with reference d_0. For any spanning tree A, we have

$$\sum_{i=0}^{n} \chi_{ij}(A) = 1.$$

We then define the function $\phi_{i,i',j,j'}(A)$ as

$$\phi_{i,i',j,j'}(A) = \begin{cases} 1, & j \text{ belongs to island with reference } i \\ & j' \text{ belongs to island with reference } i' \\ 0, & \text{else} \end{cases}$$

with

$$\sum_{i=0}^{n} \sum_{i'=0}^{n} \phi_{i,i',j,j'}(A) = 1.$$

Using the equivalence between valid islanding strategies and bases of $\mathcal{B}(\overline{\mathcal{M}})$, Eq. (5) can be expressed as the matroid optimization problem

$$\begin{array}{ll} \text{minimize} & \alpha \int_{\Omega_\beta} \sum_{i=1}^{n} \sum_{j=1}^{n} \{\Phi_{ij}(x; A)\}_+ \, dx \\ & \qquad + \sum_{i=1}^{n} c_i \chi_{0i}(A) \\ \text{s.t.} & A \in \mathcal{B}(\overline{\mathcal{M}}) \end{array} \quad (6)$$

where

$$\Phi_{ij}(x; A) = \frac{\partial W}{\partial x_j} \left(\chi_{ij}(A) f_j(x_j) + \sum_{j' \in N(j)} \phi_{i,i,j,j'}(A) f_{jj'}(x_j, x_{j'}) \right).$$

While (6) has a matroid constraint, the objective function is not submodular. In what follows, we prove that (6) is equivalent to a submodular optimization problem. We first define a new graph $\overline{G}_{ij} = (\overline{V}_{ij}, \overline{E}_{ij})$ as $\overline{V}_{ij} = \overline{V}$ and

$$\overline{E}_{ij} = \overline{E} \cup \{(i, j)\}.$$

We let $\mathcal{M}_{ij}$ denote the graphic matroid on $\overline{G}_{ij}$. Define $R = \{(s_0, i) : i \in V\}$. We have the following preliminary result.

Lemma 3: If A is independent in $\overline{\mathcal{M}}$ then $A \setminus R$ and $A \setminus (R \setminus \{(s_0, i)\})$ are independent in $\mathcal{M}_{ij}$.

Proof: By construction, A is a subset of $\overline{E}_{ij}$. Since A is independent in $\overline{\mathcal{M}}$, A does not contain a cycle and hence is independent in $\mathcal{M}_{ij}$. ■

We next describe $\chi_{ij}(A)$ using the constructions $\overline{\mathcal{M}}$ and $\mathcal{M}_{ij}$.

Lemma 4: For any $A \in \mathcal{B}(\overline{\mathcal{M}})$,

$$\begin{aligned} \chi_{ij}(A) = \max\big\{2 - (&\rho_{\mathcal{M}_{ij}}(A \setminus (R \setminus \{(s_0, i)\}) \cup \{(s_0, i)\}) \\ &-\rho_{\mathcal{M}_{ij}}(A \setminus (R \setminus \{(s_0, i)\}))) \\ -(&\rho_{\mathcal{M}_{ij}}(A \setminus R \cup \{(i, j)\}) - \rho_{\mathcal{M}_{ij}}(A \setminus R)), 1\big\} - 1 \end{aligned} \quad (7)$$

Proof: We have

$$\rho_{\mathcal{M}_{ij}}(A \setminus R \cup \{(i, j)\}) = \rho_{\mathcal{M}_{ij}}(A \setminus R) \quad (8)$$

$$\rho_{\mathcal{M}_{ij}}(A \setminus (R \setminus \{(s_0, i)\} \cup \{(s_0, i)\})) = \rho_{\mathcal{M}_{ij}}(A \setminus (R \setminus \{(s_0, i)\})) \quad (9)$$

if and only if adding (i, j) to $A \setminus R$ does not increase the rank of $A \setminus R$ and adding (s_0, i) to $A \setminus (R \setminus \{(s_0, i)\})$ does not increase the rank of $A \setminus (R \setminus \{(s_0, i)\})$. The former condition occurs if adding (i, j) creates a cycle, implying that i and j are already connected in $A \setminus R$, or equivalently, if i and j are in the same island. The latter condition holds if (s_0, i) is already in $A \setminus (R \setminus \{(s_0, i)\})$, i.e., if i is an anchor node in the islanding defined by A. Hence (8) and (9) hold together if and only if j is in the island anchored at i, which is equivalent to $\chi_{ij}(A) = 1$, completing the proof. ■

Based on Lemma 4, we define the function $\overline{\chi}_{ij}(A)$ by

$$\begin{aligned} \overline{\chi}_{ij}(A) = \max\{&2n + 4 - |A \cap R| - |A \cap (R \setminus \{(s_0, i)\})| \\ &-\rho_{\mathcal{M}_{ij}}(A \setminus (R \setminus \{(s_0, i)\}) \cup \{(s_0, i)\}) \\ &-\rho_{\mathcal{M}_{ij}}(A \setminus R \cup \{(i, j)\}), 1\big\} - 1 \end{aligned} \quad (10)$$

The following result establishes equivalence of χ_{ij} and $\overline{\chi}_{ij}$ and examines the structure of $\overline{\chi}_{ij}$.

Lemma 5: For all $A \in \mathcal{B}(\overline{\mathcal{M}})$, $\chi_{ij}(A) = \overline{\chi}_{ij}(A)$. The function $\overline{\chi}_{ij}(A)$ is nonincreasing and supermodular in A.

Proof: We start with the equivalent definition of $\chi_{ij}(A)$ from (7). We have

$$\begin{aligned} &\rho_{\mathcal{M}_{ij}}(A \setminus (R \setminus \{(s_0, i)\}) \cup \{(s_0, i)\}) \\ &\qquad - \rho_{\mathcal{M}_{ij}}(A \setminus (R \setminus \{(s_0, i)\})) \\ =&\rho_{\mathcal{M}_{ij}}(A \setminus (R \setminus \{(s_0, i)\}) \cup \{(s_0, i)\}) \\ &\qquad - |A \setminus (R \setminus \{(s_0, i)\})| \\ &\rho_{\mathcal{M}_{ij}}(A \setminus R \cup \{(i, j)\}) - \rho_{\mathcal{M}_{ij}}(A \setminus R) \\ =&\rho_{\mathcal{M}_{ij}}(A \setminus R \cup \{(i, j)\}) - |A \setminus R| \end{aligned}$$

since, by Lemma 3, $(A \setminus (R \setminus \{(s_0, i)\})) \in \mathcal{M}_{ij}$ and $(A \setminus R) \in \mathcal{M}_{ij}$ and hence $\rho_{\mathcal{M}_{ij}}(A \setminus (R \setminus \{(s_0, i)\})) = |A \setminus (R \setminus \{(s_0, i)\})|$ and $\rho_{\mathcal{M}_{ij}}(A \setminus R) = |A \setminus R|$. Furthermore,

$$|A \setminus (R \setminus \{(s_0, i)\})| = |A| - |A \cap (R \setminus \{(s_0, i)\})|,$$

$$|A \setminus R| = |A| - |A \cap R|.$$

Since A is the set of edges of a spanning tree of $\overline{G}$ and $\overline{G}$ is a connected graph with $(n+2)$ vertices, we have $|A| = n+1$. Substituting this into (7) completes the first part of the proof.

We have that $\rho_{\mathcal{M}_{ij}}$ and the cardinality function are increasing and submodular, and hence

$$2n + 4 - |A \cap R| - |A \cap (R \setminus \{(s_0, i)\})| \\ -\rho_{\mathcal{M}_{ij}}(A \setminus (R \setminus \{(s_0, i)\}) \cup \{(s_0, i)\}) - \rho_{\mathcal{M}_{ij}}(A \setminus R \cup \{(i,j)\})$$

is nonincreasing and supermodular in A. The monotonicity and supermodularity of $\overline{\chi}_{ij}(A)$ then follows from the fact that the maximum of a nonincreasing supermodular function and a constant is also nonincreasing and supermodular. ■

We next turn to the function $\overline{\phi}_{i,i',j,j'}(A)$. We observe that

$$\phi_{i,i',j,j'}(A) = \{\chi_{ij}(A) + \chi_{i'j'}(A) - 1\}_+ \qquad (11)$$

We then define

$$\overline{\phi}_{i,i',j,j'}(A) = \{\overline{\chi}_{ij}(A) + \overline{\chi}_{i'j'}(A) - 1\}_+.$$

We have the following result.

Lemma 6: For all $A \in \mathcal{B}(\overline{\mathcal{M}})$,

$$\overline{\phi}_{i,i',j,j'}(A) = \phi_{i,i',j,j'}(A).$$

The function $\overline{\phi}_{i,i',j,j'}(A)$ is nonincreasing and supermodular in A.

Proof: The equivalence of $\phi_{i,i',j,j'}$ and $\overline{\phi}_{i,i',j,j'}$ follows from Lemma 5. The monotonicity and supermodularity of $\overline{\phi}_{i,i',j,j'}$ follows from monotonicity and supermodularity of $\overline{\chi}_{ij}$ and the fact that, if f is supermodular and nonincreasing, then $\{f\}_+$ is supermodular and nonincreasing (Section III-B). ■

We finally describe the key step in an equivalent submodular formulation for (6). First, define

$$\omega_j(A;x) = \begin{cases} \sum_{i=1}^n \overline{\chi}_{ij}(A), & \frac{\partial W}{\partial x_j} f_j(x_j) \geq 0 \\ 1 - \overline{\chi}_{0j}(A), & \frac{\partial W}{\partial x_j} f_j(x_j) < 0 \end{cases}$$

and

$$\psi_{j,j'}(A;x) \\ = \begin{cases} \sum_{i=1}^n \overline{\phi}_{i,i,j,j'}(A), & \frac{\partial W}{\partial x_j} f_{jj'}(x_j, x_{j'}) \geq 0 \\ 1 - \sum_{i \neq i'} \overline{\phi}_{i,i',j,j'}(A), & \frac{\partial W}{\partial x_j} f_{jj'}(x_j, x_{j'}) < 0 \end{cases}$$

Finally, we let

$$\overline{F}(A) \\ = \alpha \int_{\{x : W(x) \leq \beta\}} \left\{ \sum_{j=1}^n \Psi_j(x, A) \right\}_+ dx + \sum_{j=1}^n c_j \overline{\chi}_{0j}(A).$$

where

$$\Psi_j(x, A) \triangleq \frac{\partial W}{\partial x_j} f_j(x_j) \omega_j(A;x) \\ + \sum_{j' \in N(j)} \frac{\partial W}{\partial x_j} f_{jj'}(x_j, x_{j'}) \psi_{j,j'}(A;x)$$

The following result establishes a submodular approach to Problem 1.

Theorem 1: For all $A \in \mathcal{B}(\overline{\mathcal{M}})$, $\overline{F}(A)$ is equal to the objective function of (5). The function $\overline{F}(A)$ is nonincreasing and supermodular in A.

Proof: The approach of the proof is to show that

$$\Psi_j(x, A) = \sum_{i=1}^n \{\Phi_{ij}(x, A)\}_+.$$

If $j \notin Z$, then

$$\sum_{i=1}^n \overline{\chi}_{ij} = 1 - \overline{\chi}_{0j} = 1,$$

and hence

$$\frac{\partial W}{\partial x_j} f_j(x_j) = \frac{\partial W}{\partial x_j} f_j(x_j) \omega_j(A;x).$$

Conversely, if $j \in Z$, then $\omega_j(A;x) = 0$. Similarly, we have $\frac{\partial W}{\partial x_j} f_{jj'}(x_j, x_{j'}) \psi_{j,j'}(A;x)$ is equal to $f_{jj'}(x_j, x_{j'})$ if j and j' are in the same stable island and 0 otherwise. Finally, we have that the second term of $\overline{F}(A)$ is equal to $\sum_{i \in Z} c_i$ by definition of $\overline{\chi}_{0j}$.

Turning to supermodularity, we have that $\Psi_j(x, A)$ is a nonincreasing supermodular function of A, and hence $\left\{ \sum_{j=1}^n \Psi_j(x, A) \right\}_+$ is a nonincreasing supermodular function of A and $\overline{F}(A)$ is a nonnegative weighted sum of supermodular functions. ■

Theorem 1 suggests a submodular optimization approach to controlled islanding. However, since $\overline{F}(A)$ contains an integral, it is infeasible to directly compute and optimize over this objective function. As an alternative, we can optimize over a set of sample points $\Omega = \{x^1, \ldots, x^N\} \subseteq \{x : W(x) \leq \beta\}$ via a relaxed objective function

$$\hat{F}(A) = \frac{\alpha}{N} \sum_{s=1}^N \left\{ \sum_{j=1}^n \Psi_j(x^s, A) \right\}_+ + \sum_{j=1}^n c_j \overline{\chi}_{0j}(A).$$

The function $\hat{F}(A)$ is also nonincreasing and supermodular in A.

The following result describes an optimality bound for the inner loop of this algorithm.

Theorem 2: Suppose that

$$F(A) \leq F(A \setminus \{(j_1, j_1')\} \cup \{(j_2, j_2')\}$$

for all $(j_1, j_1'), (j_2, j_2')$ with $(A \setminus \{(j_1, j_1')\} \cup \{(j_2, j_2')\}) \in \mathcal{B}(\overline{\mathcal{M}})$. Then, for any $A' \in \mathcal{B}(\overline{\mathcal{M}})$, $F(A) \geq \frac{1}{2} F(A')$.

Proof: The proof follows from the fact that $F(A)$ is equivalent to a nonincreasing supermodular function by Theorem 1 and the $\frac{1}{2}$-optimality bound for monotone submodular

Algorithm 1 Local search algorithm for controlled islanding

```
1: Ω ← N randomly chosen points with W(x) ≤ β
2: while 1 do
3:     Initialize A to define an islanding strategy
4:     while 1 do
5:         (j_1, j'_1), (j_2, j'_2) ← arg min{F̂(A \ {(j_1, j'_1)} ∪ {(j_2, j'_2)}) : (A \ {(j_1, j'_1)} ∪ {(j_2, j'_2)}) ∈ B(M̄)}
6:         if F̂(A) > (1 + ε)F̂(A \ {(j_1, j'_1)} ∪ {(j_2, j'_2)}) then
7:             A ← A \ {(j_1, j'_1)} ∪ {(j_2, j'_2)}
8:         else
9:             Break
10:        end if
11:    end while
12:    x ← x satisfying W(x) ≤ β, ∂W/∂x_i (f(x_i) + Σ_{j∈N(i)∩V_r} f_ij(x_i, x_j)) > 0 for some i ∉ Z
13:    if x == ∅ then return A
14:    else
15:        Ω ← Ω ∪ {x}
16:    end if
17: end while
```

optimization with a matroid basis constraint given in [25]. ■

Based on the nonincreasing and supermodular properties of $\hat{F}(A)$, we develop Algorithm 1 to compute the islanding strategy. We first randomly choose N sample points with $W(x) \leq \beta$ to initialize the set Ω. Then we select an arbitrary spanning tree as the initial islanding strategy. Next, at each iteration, we select an edge (j_2, j_2'), that connects two disjoint islands and an edge (j_1, j_1') to remove so that $(A \setminus \{(j_1, j_1')\} \cup \{(j_2, j_2')\}) \in \mathcal{B}(\overline{\mathcal{M}})$. We select a small positive parameter $\epsilon \in (0, 1)$. If $\hat{F}(A) > (1+\epsilon)\hat{F}(A \setminus \{(j_1, j_1')\} \cup \{(j_2, j_2')\})$, Algorithm 1 updates A based on this change in edges. If no improvement can be found, Algorithm 1 generates a new sample $x \in \{x : W(x) \leq \beta\}$. If x is a counter-example that satisfies the condition in line 12, Algorithm 1 adds this counter-example to the initial set Ω, and reconstructs a new spanning tree. Otherwise, Algorithm 1 outputs the current spanning tree. Due to the nonincreasing and supermodular properties of $\hat{F}(A)$, line 4 to line 11 in Algorithm 1 will terminate within polynomial time [16].

C. Islanding for Stable Synchronization

In many domains, instead of or in addition to ensuring internal stability of each agent, the goal is to ensure that the agent states are synchronized. We consider Lyapunov functions of the form $W(x) = \sum_{(i,j)\in E} W_{ij}(z_{ij})$, where $z_{ij} = x_i - x_j$ and W_{ij} is positive definite, so that the Lyapunov function is zero if and only if all of the nodes within each island have the same state values. The derivative of the Lyapunov function is then given by

$$\begin{aligned}\dot{W}(x) &= \sum_{(i,j)\in E} \frac{\partial W_{ij}}{\partial z_{ij}}(\dot{x}_i - \dot{x}_j) \\ &= \sum_{(i,j)\in E} \frac{\partial W_{ij}}{\partial z_{ij}} \Bigg[f_i(x_i) - f_j(x_j) + \sum_{j'\in N(i)} f_{ij'}(x_i, x_{j'}) \\ &\quad - \sum_{i'\in N(j)} f_{i'j}(x_{i'}, x_j) \Bigg]\end{aligned}$$

As a preliminary, we define function $\phi(A;S)$ for $S \subseteq V \times V$ as

$$\phi(A;S) = \begin{cases} 1, & i \text{ is in island with reference } i' \ \forall (i,i') \in S \\ 0, & \text{else} \end{cases}$$

We can then rewrite the expression for $\dot{W}(x)$ as

$$\begin{aligned}&\dot{W}(x) \\ &= \sum_{(i,j)\in E} \Bigg[\sum_{k=1}^{n} \phi(A; \{(i,k),(j,k)\}) \frac{\partial W_{ij}}{\partial z_{ij}} (f_i(x_i) - f_j(x_j)) \\ &\quad + \sum_{j'\in N(i)} \sum_{k=1}^{n} \phi(A; \{(i,k),(j,k),(j',k)\}) \frac{\partial W_{ij}}{\partial z_{ij}} f_{ij'}(x_i, x_{j'}) \\ &\quad - \sum_{i'\in N(j)} \sum_{k=1}^{n} \phi(A; \{(i,k),(i',k),(j,k)\}) \\ &\qquad \cdot \frac{\partial W_{ij}}{\partial z_{ij}} f_{i'j}(x_{i'}, x_j) \Bigg]\end{aligned}$$

Lemma 7: For any A and S,

$$\phi(A;S) = \left\{ \sum_{(i,i')\in S} \chi_{i,i'}(A) - |S| + 1 \right\}_+ .$$

The following corollary leads to a submodular formulation for choosing an optimal islanding strategy for stable synchronization.

Corollary 1: There exists a nonincreasing supermodular function $\overline{\phi}(A;S)$ such that, for all $A \in \mathcal{B}(\overline{\mathcal{M}})$, $\overline{\phi}(A;S) = \phi(A;S)$.

We can then attempt to ensure stability by solving the optimization problem

$$\begin{array}{ll} \text{minimize} & \frac{\alpha}{N} \sum_{x\in\Omega} \{\dot{W}(x;A)\}_+ + \sum_{j=1}^{n} c_j \overline{\chi}_{0j}(A) \\ \text{s.t.} & A \in \mathcal{B}(\overline{\mathcal{M}}) \end{array} \tag{12}$$

By Corollary 1, we have that the objective function of (12) is equivalent to an nonincreasing supermodular function. Hence a similar procedure to Algorithm 1 can be used to select stable islands.

V. Verification of Island Stability

In this section, we consider two special cases of networked system dynamics and discuss how to verify the stability of the islands or find counterexamples. We first discuss linear consensus with negative edge weights, and then power systems governed by the generator swing equation.

A. Linear Consensus Dynamics

We consider linear systems with dynamics given by $f_i(x_i) = 0$ for all i and $f_{ij}(x_i, x_j) = \Gamma_{ij}(x_j(t) - x_i(t))$, with $\Gamma_{ij} = \Gamma_{ji}$. It is well-known that these dynamics are asymptotically stable if $\Gamma_{ij} \geq 0$ for all $(i, j) \in E$, however, the system may be unstable if a subset of edges have $\Gamma_{ij} < 0$.

The state dynamics are given by

$$\dot{x}(t) = -Lx(t),$$

where L is a matrix with

$$L_{ij} = \begin{cases} -\Gamma_{ij}, & (i,j) \in E \\ \sum_{l \in N(i)} \Gamma_{li}, & i = j \\ 0, & \text{else} \end{cases}$$

Hence, a set of islands is stable if and only if the resulting matrix L is positive semidefinite, which can be checked in polynomial time in n. We choose $W(x) = x^T x$ for this case, and note that β can be chosen arbitrarily since the system is linear.

B. Power Systems

We consider a network of generators governed by the equation

$$\dot{\theta}_i(t) = \omega_i - \sum_{j \in N(i)} \Gamma_{ij} \sin(\theta_i(t) - \theta_j(t)) \quad (13)$$

It is a known result that, if there is a positive invariant set Λ such that $|\theta_i - \theta_j| \leq \frac{\pi}{2}$ for all $(i, j) \in E$ and $\theta \in \Lambda$, then (13) converges to a stable fixed point. Hence, our aim is to find a β such that $\{x : W(x) \leq \beta\}$ is positive invariant.

One energy function for systems with dynamics (13) is given by

$$W(\theta) = \sum_{(i,j) \in E} (1 - \cos(\theta_i - \theta_j)) \quad (14)$$

The following result gives a sufficient condition for stability based on the energy function (14).

Proposition 2: For each stable island r, define matrix $\Theta \in \mathbb{R}^{|E_r| \times |E_r|}$ by

$$\Theta_{ee'} = \begin{cases} \frac{\Gamma_e + \Gamma_{e'}}{2}, & e, e' \text{ incident to same node} \\ 0, & \text{else} \end{cases}$$

Suppose that, for all islands indexed $r \in \{1, \ldots, m\} \setminus \mathcal{U}$, $\{\theta : W_r(\theta) \leq \gamma\} \subseteq \{||B^T \theta|| \leq \frac{\pi}{2}\}$ and there does not exist $z \in \mathbb{R}^{|E_r|}$ satisfying

$$\omega^T Bz - z^T \Theta z > 0 \quad (15)$$

$$||z||_\infty \leq 1 \quad (16)$$

$$\sum_{(i,j) \in E} (1 - \sqrt{1 - z_{ij}^2}) = \gamma \quad (17)$$

Then the set $\{x : W_r(x) \leq \gamma\}$ is positive invariant, where B denotes the incidence matrix of island r.

Proof: We have that the set $\{\theta : W_r(\theta) \leq \gamma\}$ is positive invariant if, for any θ with $W_r(\theta) = \gamma$, we have $\dot{W}_r(\theta) \leq 0$. Suppose that this does not hold, and hence there exists θ with $\dot{W}_r(\theta) > 0$ and $W_r(\theta) = \gamma$.

Systems	Number of edges with negative Γ_{ij}	Number of nodes in stable islands	Number of nodes in unstable islands
30-node	12	7	23
57-node	31	25	32
118-node	71	34	84

TABLE I: Islanding strategies given by Algorithm 1 for 30-, 57-, and 118-node systems

Define $z_{ij} = \sin(\theta_i - \theta_j)$ and $z = \sin B^T \theta$. We then have

$$\begin{aligned} \dot{W}(\theta) &= \frac{\partial W}{\partial \theta} \dot{\theta} = \sum_{i=1}^{n} \frac{\partial W}{\partial \theta_i} \dot{\theta}_i \\ &= \frac{1}{2} \sum_{i=1}^{n} \left[\left(\sum_{j \in N(i)} \sin(\theta_i - \theta_j) \right) \left(\omega_i - \sum_{l \in N(i)} \Gamma_{il} \sin(\theta_i - \theta_l) \right) \right] \\ &= \frac{1}{2} \sum_{i=1}^{n} \left[\omega_i \sum_{j \in N(i)} \sin(\theta_i - \theta_j) - \sum_{l \in N(i)} \sum_{j \in N(i)} \sin(\theta_i - \theta_j) \sin(\theta_i - \theta_l) \right] \\ &= \frac{1}{2} \omega^T Bz - \frac{1}{2} z^T \Theta z. \end{aligned}$$

Furthermore, we must have $||z||_\infty \leq 1$. Finally, we have $z = W_r(\theta) = \sum_{(i,j) \in E_r} 1 - \sqrt{1 - z_{ij}^2}$. Hence, if the island is unstable, then there exists z satisfying (15)–(17), completing the proof. ■

VI. Simulation

In this section, we present the setup and results of our case study. The experiments are implemented using MATLAB R2020a on a laptop with Intel(R) Core(TM) i9-10885H CPU with 2.40GHz processor and 32GB memory. We simulate linear consensus dynamics with signed edge weights, which is introduced in Section V. We implement Algorithm 1 on the 30-, 57-, and 118-node systems whose topologies can be found in [26]. For each node i, the cost of instability c_i was chosen independently and uniformly at random in the interval $(10, 20)$.

In Algorithm 1, we set $\alpha = 1000$ and $N = 100$ for all the systems. The values of Γ_{ij} are uniformly distributed in the interval $(0, 5)$. The number of edges in the three test cases are 41, 78, and 179. We select 12, 31, and 71 edges in 30-, 57-, and 118-node systems to have negative weights.

We first present the islanding strategy for the 30-node system in Fig. 1. The unstable system is partitioned into two islands. Island 1 is stable and is marked within the blue box. The red box marks the unstable island 2.

Table I presents the islanding strategies for 30-, 57- , and 118-node systems under Algorithm 1. The systems are unstable before islanding. After implementing islanding strategies, the systems are partitioned into disjointed islands,

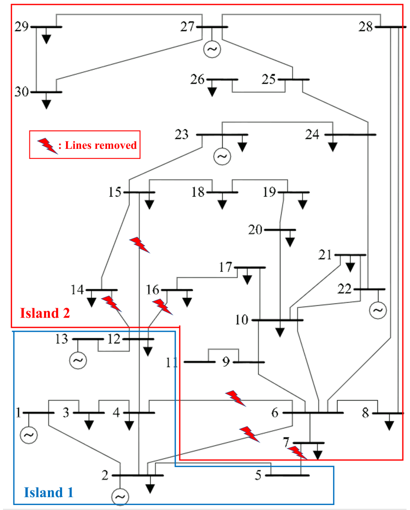

Fig. 1: The islanding strategy given by Algorithm 1 for the 30-node system. Island 1 is stable, and island 2 is unstable.

in which some islands are stable. In the 30-node system, there are 7 nodes on the stable island and 23 nodes on the unstable island. The total cost of the nodes on the unstable island is 348.56. The running time of the algorithm is 5.35 seconds. In the 57-node system, there are 25 nodes on the stable island and 32 nodes on the unstable island. The total cost of the nodes on the unstable island is 471.76. The running time of the algorithm is 50.71 seconds. In the 118-node system, there are 34 nodes on the stable island, and the total number of nodes on the two unstable islands is 84. The total cost of the nodes on the unstable island is 738.76. The running time of the algorithm is 52.97 seconds.

VII. Conclusions

This paper considered the problem of controlled islanding of networked systems, in which edges are deliberately removed from a network in order to partition the network into stable sub-networks, denoted as islands. We proposed an optimization approach, in which Lyapunov conditions for stability of each island were mapped to an objective function to be minimized. We proved that this optimization problem is equivalent to minimizing a monotone supermodular function with a matroid basis constraint. Based on this result, we developed a polynomial-time local-search algorithm with a provable $1/2$ optimality bound. We evaluated our approach using the IEEE 30-, 57- and 118-node test cases with signed consensus dynamics.

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