2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Reverse Kron Reduction of Three-phase Radial Network

Steven H. Low
CMS, EE, Caltech, Pasadena CA
slow@caltech.edu

***Abstract*— We consider the problem of identifying the admittance matrix of a three-phase radial network from voltage and current measurements at a subset of nodes. These measurements are used to estimate a virtual network represented by the Kron reduction (Schur complement) of the full admittance matrix. We focus on recovering exactly the full admittance matrix from its Kron reduction, i.e., computing the inverse of Schur complement. The key idea is to decompose Kron reduction into a sequence of iterations that maintains an invariance structure, and exploit this structure to reverse each step of the iterative Kron reduction.**

I. Introduction

Modeling distribution networks below substations is increasingly important as distributed energy resources proliferate on these systems. Today these networks are sparsely monitored at best, with few μPMUs (micro-Phasor Measurement Units) beyond the SCADA (Supervisory Control and Data Acquisition) system at substations and smart meters at utility customers. As a result, the utility company often does not have an accurate model of its network as it evolves either due to faults, repairs or upgrades. This limits their ability to analyze power flows and optimize their planning and operations. This has motivated a large number of papers to identify the topology, line admittances, or switch status of distribution grids. A recent tutorial [1] explains different approaches in the literature and contains an extensive list of references. Various methods have been proposed for these identification problems using AMI data (voltage magnitudes, real and reactive power injections) or PMU data (voltage and current phasors, real and reactive line flows), measured at all or a subset of nodes, with only passive measurement or also active probing, for single-phase or unbalanced three-phase networks, in radial or mesh topologies (see Table 1 of [1]). Most of the literature focuses on identification problems for single-phase networks or assumes measurements are available at every node in the network. This paper studies the identification of the admittance matrix (topology and line admittances) for unbalanced three-phase radial networks from voltage and current measurements at a subset of nodes. This is more realistic for distribution systems.

A. Summary

The phasors of the nodal voltages V and current injections I are related linearly, $I = YV$, by the complex symmetric admittance matrix Y. When only a subset of the voltage and current phasors (V_1, I_1) are measured, they satisfy $I_1 = \bar{Y}V_1$ where $\bar{Y}$ is called the Kron reduced admittance matrix. It describes a virtual network topology consisting of only measured nodes and the "effective" line admittances connecting these nodes. To identify the full matrix Y from the partial measurement (V_1, I_1), we first estimate the Kron reduced admittance matrix $\bar{Y}$ from (V_1, I_1). This corresponds to identification from full measurements and several methods in the literature can be applied, e.g., [1], [2–4]. Then we identify the unique Y given $\bar{Y}$, the focus of this paper.

The mathematical problem is as follows. Consider any complex symmetric matrix $Y =: \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$ with a non-singular Y_{22} where Y_{11} describes the connectivity between measured nodes, Y_{22} describes the connectivity between hidden nodes, and Y_{12} describes the connectivity between measured and hidden nodes. The Kron reduced admittance matrix $\bar{Y}$ is the Schur complement of Y_{22} of Y. This defines a mapping from Y to $\bar{Y}$ given by

$$\bar{Y} \;:=\; f(Y) \;:=\; Y_{11} - Y_{12}Y_{22}^{-1}Y_{21}$$

When does the inverse $f^{-1}(\bar{Y})$ exist and how to compute it when it does? In this paper we show that $f^{-1}(\bar{Y})$ exists when the graph underlying Y is a tree (and when some other conditions hold), by describing an explicit construction of Y from $\bar{Y}$.

Our construction method extends the method in [5] for single-phase radial networks to an unbalanced three-phase setting. For a single-phase radial network, the series line admittances are always nonzero. This allows certain structural properties important for the identification of Y to be preserved under Kron reduction. In a multi-phase network however line admittances are 3×3 matrices for a three-wire model. It is often unclear when Kron reduction exists or whether these structural properties are still preserved under Kron reduction. Following the same idea in [5], we have developed in [19] two new results to overcome the difficulties in the three-phase setting. Specifically the construction method in [19] consists of three main algorithms:

1) Algorithm 1: Reduces the overall identification problem into the special case of identifying the admittance matrix of a single maximal clique consisting of measured leaf nodes connected by a tree of hidden nodes. This procedure is the same for both single-phase and unbalanced three-phase networks [5].
2) Algorithm 2: Identifies a maximal clique in an unbalanced three-phase setting. Consider iterative Kron

This research was supported by NSF through grants ECCS 1931662. We thank Stanojev Ognjen of ETH, Lucien Werner and Yiheng Xie of Caltech for helpful discussions.

reduction where a single hidden node is reduced in each iteration. The novel idea is to characterize an invariant structure that is preserved in one-step Kron reduction and use it to derive its one-step inverse, yielding an iterative reverse Kron reduction.

3) Algorithm 3: Identifies a new hidden node to be added in each iteration of Algorithm 2 to the set of identified nodes. This new hidden node is adjacent to a subset of sibling nodes that have been identified in previous iterations. A sibling grouping property of [5] that underlies Algorithm 3 is generalized from single-phase to three-phase setting under a (restrictive) uniform line assumption.

Due to the page limit, we will only formulate the network identification problem in this paper and summarize Algorithm 2 that solves the special case and forms the core of the overall construction method.

B. Prior work

We leave the discussion of the large literature on single-phase identification problems to the comprehensive tutorial [1]. We leave out papers on topology and line parameter identification using active probing (e.g. [6]) and cover only methods that use passive measurements. We also leave out a large literature on line parameter estimation given network topology and cover only methods that involve topology estimation. We now summarize several papers on the identification of unbalanced three-phase networks, with an emphasis on papers that explicitly exploit the radial structure of the network

Operational radial networks. Consider a given mesh network in which tie switches and sectionalizing switches are configured so that the operational network at any time consists of a forest of nonoverlapping trees that span all nodes. Often the switch status may not be known accurately due to frequent reconfigurations or manual changes in distribution systems. The problem of estimating the switch status and hence the operational topology is called topology detection. It is studied in [7, 8], both of which extend the statistical method of [9] from a single-phase to an unbalanced three-phase setting. The key idea is to use a linear or linearized model that related voltages (phasors or magnitudes) V to nodal power injections (p, q) or current injections I. Then assuming the nodal injections are statistical independent, their second moments induce second moments on V that have the sibling grouping property in [10, Lemma 4]. A recursive grouping algorithm similar to that in [10] is then used to identify successively each node's unique parent, starting from leaf nodes, and thus the operational tree. In [11], the topology detection problem is formulated instead as a mixed integer linear program based on the three-phase linear DistFlow model of [12]. The operational topology is taken to be a minimizer of normalized l_1 norm of observation error. All these works assume the topology and line admittances are known, while they are what we aim to identify.

Radial topology identification. Given a set of network nodes, topology identification is the problem of identifying the lines connecting these nodes. Reference [13] identifies the radial network topology (and bus phase labels) by extending the graphical-model method of [14] from a single-phase to an unbalanced three-phase setting with unknown phase identities. A key assumption of [14] is that the injection current phasors at different nodes are statistically independent. This leads to the important conclusion that the joint distribution $p(V) := p(V_1, \ldots, V_n)$ of the voltage phasors on a radial network is what is called a probability distribution of first-order tree dependence in [15] or a latent tree graphical model in [10]. Moreover, if we label the edges (j, k) of the complete graph with n nodes by the mutual information $I(V_j, V_k)$, then the latent tree is a maximum-weight spanning tree [15]. This leads to the identification algorithm of [14] for single-phase radial networks and its extension in [13] for unbalanced three-phase radial networks. Unlike the problem we study, topology identification does not estimate line parameters.

Radial admittance matrix identification. The problem of admittance matrix (or impedance matrix) identification is the problem of determining topology and line admittances (or impedances) from measurements. This is the problem studied in [2], as well as in this paper, when measurements are available only at a subset of the nodes. Both papers first use measurements to estimate the Kron reduced admittance (or impedance) matrix. To identify from the Kron reduction the hidden nodes where measurements are not available, [2] uses the recursive grouping algorithm of [10, Section 4] to identify the impedance matrix for single-phase radial networks. We instead develop a different method to identify the admittance matrix Y.

Mesh networks. Besides radial networks, identification problems are studied in [3, 4, 16, 17] for unbalanced three-phase mesh networks. All these methods apply to both single or three-phase systems, radial or mesh. Our method explicitly exploits the radial structure of distribution grids.

C. Organization and notation

We formulate in Section II the network identification problem studied in this paper. To identify a single maximal clique, we characterize in Section III an invariant structure that is preserved under iterative Kron reduction and use it to derive a reverse iterative Kron reduction. We conclude in Section IV.

We write a column vector in $\mathbb{C}^n$ either as $x = (x_1, \ldots, x_n)$ or $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. We use $\mathbf{1}_k$ to denote the column vector of k 1s and $\mathbf{I}_k$ to denote the identity matrix of size k. Then $\mathbf{1}_j \otimes \mathbf{I}_k$ is the $jk \times k$ matrix of j identity matrices each of size k stacked vertically. When the dimension is clear from the context we often write $\mathbf{1}$ and $\mathbf{I}$ for $\mathbf{1}_k$ and $\mathbf{I}_k$ respectively. If $A_1, \cdots, A_j$ are j matrices each of $k \times k$, then $\operatorname{diag}(A_1, \cdots, A_j)$ is the $jk \times jk$ block-diagonal matrix with $A_1, \cdots, A_j$ as its diagonal blocks. Given an admittance matrix A, $G(A)$ denotes its underlying graph. We will often

use A to refer to either the admittance matrix, its underlying graph $G(A)$, or the set of nodes in $G(A)$, when the meaning should be clear from the context.

II. Network identification problem

In this section we formulate our network identification problem.

A. Admittance matrix Y

Consider a network $G := (N, E)$ with n nodes where $N := \{1, \dots, N\}$ is the set of nodes and $E \subseteq N \times N$ is the set of lines. We use N to denote both the set and the number of nodes; the meaning should be clear from the context. We assume the network G is three-phased, where each line is characterized by a 3×3 series admittance matrix $y_{jk} \in \mathbb{C}^{3\times 3}$ and the admittance matrix Y of G is a $3N \times 3N$ matrix. We assume shunt admittances are zero. We will refer to the rows (columns) $3j-2$, $3j-1$, $3j$ that are associated with node j as the jth row block (jth column block). We use $Y[j,k] \in \mathbb{C}^{3\times 3}$ to denote the 3×3 submatrix consisting of the jth row block and the kth column block. Then the admittance matrix Y is defined by $Y[j,k] = -y_{jk} \neq 0$ if $(j,k) \in E$, $Y[j,j] = \sum_{k:(j,k)\in E} y_{jk}$, and $Y[j,k] = 0$ otherwise.

We make the following assumption on the network G and its line admittance matrices y_{jk}.

Assumption 1 (Line admittances)**.** *We assume the network G is radial (i.e., with tree topology) and connected. For all lines $(j,k) \in E$, we assume that shunt admittances are zero and the series admittance matrices y_{jk} satisfy:*

1) *$y_{jk} = y_{kj} \in \mathbb{C}^{3\times 3}$ so that $Y[j,k] = Y[k,j]$.*
2) *y_{jk} are symmetric so that $Y^{\mathsf{T}} = Y$.*
3) *$\mathrm{Re}(y_{jk}) \succ 0$, i.e., y_{jk} is positive definite, where $\mathrm{Re}(y_{jk})$ is the real part of the line admittance matrix y_{jk}.* □

Assumption 1 leads to useful properties, from [18], summarized in the next lemma which is fundamental to three-phase identification.

Lemma 1 ([18])**.** *Suppose Assumption 1 holds.*

1) *For any line $(j,k) \in E$, y_{jk}^{-1} exists and is complex symmetric. Moreover $\mathrm{Re}\left(y_{jk}^{-1}\right) \succ 0$.*
2) *For any strict subset $A \subsetneq N$, Y_A^{-1} exists and is complex symmetric. Moreover both $\mathrm{Re}(Y_A) \succ 0$ and $\mathrm{Re}\left(Y_A^{-1}\right) \succ 0$.*
3) *$(Y_A/B_{22})^{-1}$ exists and is complex symmetric. Moreover both $\mathrm{Re}(Y_A/B_{22}) \succ 0$ and $\mathrm{Re}\left((Y_A/B_{22})^{-1}\right) \succ 0$.*

Remark 1 (Assumption 1)**.** *The importance of Assumption 1 is that it allows us to take inverse of any principal submatrix of the admittance matrix Y, justifying arbitrary and successive Kron reductions. It also allows certain structural properties to be preserved under Kron reduction, which underlies our results in Sections III.* □

B. Hidden nodes and Kron reduction $\bar{Y}$

We assume there are two types of nodes, i.e., $N =: M \cup H$. Nodes $j \in M$ are called *measured nodes* whose three-phase nodal voltage and current injection phasors $(V_j, I_j) \in \mathbb{C}^6$ are measured. Nodes $j \in H$ are called *hidden nodes* whose nodal voltages and currents are not measured. We abuse notation and use M and H to denote both the sets and the numbers of measured and hidden nodes, so $N = M + H$. The nodes are labeled such that the first M nodes are measured and the last H buses are hidden. We partition the admittance matrix into four sub-matrices accordingly:

$$Y =: \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \tag{1}$$

where $Y_{11} \in \mathbb{C}^{3M\times 3M}$ specifies the connectivity between measured nodes, $Y_{22} \in \mathbb{C}^{3H\times 3H}$ specifies the connectivity between hidden nodes, Y_{12} specifies the connectivity between measured and hidden nodes, and $Y_{21} = Y_{12}^{\mathsf{T}}$ under Assumption 1. Partition the voltage and current phasors accordingly: (V_1, I_1) correspond to the voltages and currents of measured nodes and (V_2, I_2) those of hidden nodes. If the current injections I_2 at hidden nodes are zero, then the network model is:

$$\begin{bmatrix} I_1 \\ 0 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Under Assumption 1, Lemma 1 implies that Y_{22} is non-singular and hence we can eliminate V_2 by computing the Schur complement Y/Y_{22} of Y_{22} of Y. We denote the Schur complement by $\bar{Y}$:

$$\bar{Y} \;:=\; Y/Y_{22} \;:=\; Y_{11} - Y_{12}Y_{22}^{-1}Y_{12}^{\top} \tag{2a}$$

and call $\bar{Y}$ a *Kron-reduced admittance matrix* or a *Kron reduction of Y* because $\bar{Y}$ relates the voltages and currents at the measured nodes:

$$I_1 = \bar{Y} V_1 \tag{2b}$$

The matrix $\bar{Y}$ is the admittance matrix of a virtual network consisting of only measured nodes in which two measured nodes are adjacent $G(\bar{Y})$ if and only if there is a path consisting of only hidden nodes that connect them in the original graph $G(Y)$. As mentioned above, the identification of $\bar{Y}$ from the measurements (V_1, I_1) is the same as identification from full measurements and several methods in the literature can be used, e.g., [1], [2–4]. We assume this has been done and focus on identifying Y from a given Kron reduction $\bar{Y}$.

The following assumption is necessary for reversing Kron reduction in Section III-C.

Assumption 2 (Hidden nodes)**.** 1) *Hidden nodes have zero injections $I_i = 0 \in \mathbb{C}^3$, $\forall i \in H$.*
2) *Every hidden node has node degree at least 3.* □

C. Special case: single maximal clique

The network identification problem we study is: given a Kron-reduced admittance matrix $\bar{Y}$, our goal is to construct the original admittance matrix Y that satisfies (2a) under

Assumptions 1 and 2 (as well as Assumption 3 in Section III-C).

Due to page limit, we will consider only a special case in this paper where the network graph $G(Y)$ consists of (degree-1) leaf nodes, all of which are measured, connected through a tree of hidden nodes. Its Kron reduction $G(\bar{Y})$ is a single *maximal clique* consisting of only the measured nodes arranged in a complete graph. It is shown in the full version of this paper [19] that the general identification problem can be reduced to solving this special case.

III. Identification of maximal clique

In Section III-A we define the network to be identified in this paper and describe an iterative procedure that computes its Kron reduction by reducing one hidden node in each step, starting from a tree of measured and hidden nodes and terminating in a single maximal clique of only measured nodes. The key is to track a certain permuted admittance matrix $\hat{A}^l$ of the graph in each step as well as its principal submatrix C^l that represents the clique subgraph. In Section III-B we derive an invariant structure of this sequence of matrices $(\hat{A}^l, C^l)$ as the original graph is being iteratively Kron reduced. In Section III-C we show how to reverse each iteration using this invariant structure and present the identification algorithm for a single maximal clique.

A. Basic idea: reversible one-step Kron reduction

1) Single maximal clique: Consider a connected three-phase radial network consisting of M degree-1 measured nodes connected by H non-leaf hidden nodes each with degree at least 2 (or at least 3 under Assumption 2). We partitions the set M of measured nodes into *internal measured nodes* that are not adjacent to any hidden nodes and *boundary measured nodes* that are adjacent to some hidden nodes. Similarly partition the set H of hidden nodes into *internal hidden nodes* that are not adjacent to any measured nodes and *boundary hidden nodes* that are adjacent to some measured nodes. Let $H_b \leq H$ denote the number of boundary hidden nodes. There are no internal measured nodes in this network. Let Y denote its admittance matrix and $\bar{Y}$ its Kron reduction where all the hidden nodes have been Kron reduced. Since the Kron reduced network $G(\bar{Y})$ is a clique where every measured node is adjacent to every other measured node, $\bar{Y}$ is an admittance matrix whose 3×3 entry blocks are all nonzero. We will often use "maximal clique" to refer either to the tree underlying Y or the clique underlying its Kron reduction $\bar{Y}$, depending on the context.

Since there are no internal measured nodes the admittance matrix Y reduces to the following form

$$Y =: \left[\begin{array}{c|c} Y_{11} & Y_{12} \\ \hline Y_{21} & Y_{22} \end{array}\right] =: \left[\begin{array}{c|cc} Y_{11,22} & Y_{12,21} & 0 \\ \hline & Y_{22,11} & Y_{22,12} \\ & & Y_{22,22} \end{array}\right] \quad (3a)$$

The given Kron reduction is $\bar{Y} =: Y/Y_{22}$. Each boundary measured node i has nonzero admittance submatrix $y_{ih(i)} \in \mathbb{C}^{3\times 3}$ for exactly one hidden node $h(i)$ in the tree. Otherwise if i is adjacent to two hidden nodes, there is a loop in G. Since every boundary measured node is adjacent to a hidden node, no boundary measured nodes can be adjacent to each other in G; otherwise there is a loop in G. Therefore $Y_{11,22}$ and $Y_{12,21}$ are of the form

$$Y_{11,22} = \operatorname{diag}\begin{bmatrix} y_{1h(1)} \\ \vdots \\ y_{Mh(M)} \end{bmatrix}, \quad Y_{12,21} = \begin{bmatrix} -e_{h(1)}^{\mathsf{T}} \otimes y_{1h(1)} \\ \vdots \\ -e_{h(M)}^{\mathsf{T}} \otimes y_{Mh(M)} \end{bmatrix} \quad (3b)$$

where $e_i \in \{0,1\}^{H_b}$ is the unit vector with a single 1 in the ith entry and 0 elsewhere, and $y_{ij} \in \mathbb{C}^{3\times 3}$ is the three-phase series admittance of line (i,j). Here $H_b \leq H$ is the number of boundary hidden nodes.

In this section we design an algorithm that uses the structure in (3) to recover Y for a single maximal clique by iteratively reversing Kron reduction, starting from $\bar{Y}$. We start by decomposing the forward Kron reduction into a sequence of iterations that maintain an invariant structure.

2) Iterative Kron reduction: It is more convenient to describe iterative Kron reduction in terms of an arbitrary $3n \times 3n$ complex matrix A^0 on a graph $G^0 := (N^0, E^0)$ where its 3×3 (i,j)th blocks $A^0[i,j]$ are given by:

$$A^0[i,j] \;=\; \begin{cases} -y_{jk} & (i,j) \in E^0 \\ \sum_{k:(i,k)\in E^0} y_{ik} & i = j \\ 0 & \text{otherwise} \end{cases}$$

We refer to A^0 as the admittance matrix of the graph G^0, or equivalently $G^0 = G(A^0)$. Suppose the graph and its admittance matrix (G^0, A^0) satisfy Assumption 1.

Let $A^0 =: \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^{\mathsf{T}} & A_{22} \end{bmatrix}$ with a $3k \times 3k$ nonsingular submatrix A_{22}, $1 \leq k < n$ (in our case, $k = H$). To simplify exposition, we will refer to nodes in A_{22} to be Kron reduced as "hidden nodes" and nodes in A_{11} as "measured nodes". We can compute the Schur complement A^0/A_{22} of A_{22} of the admittance matrix A^0 by eliminating hidden nodes on the graph G^0 one by one through Kron reduction. Following [20], we define

$$A^1 := A^0/A^0[n,n], \qquad \cdots, \quad (4a)$$

$$A^k := A^{k-1}/A^{k-1}[n-k+1, n-k+1] \;=\; A^0/A_{22} \quad (4b)$$

i.e., A^{l+1} is the admittance matrix for the graph after the last node in A^l has been Kron reduced. Conversely a sequence of matrices $A^0, A^1, \ldots, A^k$ computed according to (4) defines a sequence of graphs $G^0, G^1, \ldots, G^k$ with $G^l = (N^l, E^l) := G(A^l)$ defined by ($N^0 := \{1, \ldots, n\}$): for $l = 0, 1, \ldots, k$,

$$\begin{aligned} N^l \;&:=\; N^0 \setminus \{n, n-1, \ldots, n-l+1\} \\ E^l \;&:=\; \big\{(i,j) : A^l[i,j] \neq 0\big\} \end{aligned}$$

We refer to $G^l := G(A^l)$ as the graph underlying A^l. Explicitly, for $l = 0, \ldots, k-1$, $i, j = 1, \ldots, n-l-1$,

$$A^{l+1}[i,j] = A^l[i,j] \;-\; A^l[i, n-l]\left(\alpha^l\right)^{-1} A^l[j, n-l] \quad (5)$$

where $\alpha^l := A^l[n-l, n-l]$. Starting from A^0, (5) iteratively computes the Kron-reduced admittance matrix $A/^0 A_{22} = A^k$. The iterative computation is useful for proving properties that are preserved under Kron reduction.

3) Maximal clique C^l of re-labeled matrix $\hat{A}^l$: Given a tree A^0, the forward iterative Kron reduction grows an initial maximal clique $A^0[n,n] \in \mathbb{C}^{3\times 3}$ consisting of a single node n into a single maximal clique $A^k \in \mathbb{C}^{3(n-k)\times 3(n-k)}$ consisting of $n-k$ nodes, while eliminating k hidden nodes from A^0 in the process, one hidden node in each step.

The basic idea of our identification method is to derive an invariant structure that is preserved under one-step Kron reduction and that is reversible. To this end, it is more convenient to focus, not on the sequence $A^l := A^{l-1}/A^l[n-l-1, n-l-1]$, but a permuted sequence $\hat{A}^l$ over which the invariant structure can be propagated and from which the sequence A^l in (4) can be extracted (see Section III-B). We will use C^l to denote the maximal clique in the permuted matrix $\hat{A}^l$ in iteration l. We will abuse notation to use the term "maximal clique" and the symbol C^l to refer to either the principal submatrix of $\hat{A}^l$, or the subgraph of $G(\hat{A}^l)$, or the nodes in the subgraph corresponding to the maximal clique; the meaning should be clear from the context.

The permuted sequence $\hat{A}^l$ results from re-labeling nodes in each step of the iterative Kron reduction. Specifically, given the permuted matrix $\hat{A}^l$ in each iteration l, after taking the Schur complement $\hat{A}^l/\hat{A}^l[n-l, n-l]$ to reduce node $n-l$, we will re-label nodes so that the next permuted matrix $\hat{A}^{l+1}$ has all (hidden and measured) nodes in the maximal clique indexed consecutively with the largest indices (as well as another convenient structure). This corresponds to multiplying the matrix $\hat{A}^l/\hat{A}^l[n-l, n-l]$ on the left and on the right by appropriate permutation matrices and its transpose respectively to obtain $\hat{A}^{l+1}$ (see [19] for examples). We explain in Section III-B the invariant structure of the permuted sequence $(\hat{A}^l, C^l)$ and how to compute A^l from $\hat{A}^l$.

The Kron reduction A^0/A_{22} is generally not equal to the matrix $\hat{A}^k$ at the end of the forward iterations on the permuted sequence $(\hat{A}^0, \ldots, \hat{A}^k)$. The Kron reduction A^0/A_{22} can, however, be recovered from $\hat{A}^k$ since the re-labeling in each iteration l does not re-label node $n-l$ that will be Kron reduced in that iteration. Consider an arbitrary square matrix $A \in \mathbb{C}^{3(m_1+m_2)\times 3(m_1+m_2)}$ partitioned as $A =: \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^\mathsf{T} & A_{22} \end{bmatrix}$ where A_{11} is $m_1 \times m_1$ and A_{22} is $m_2 \times m_2$. The permutation matrix that re-labels nodes in A_{11} takes the form

$$P = \begin{bmatrix} P_{m_1} & 0 \\ 0 & \mathbf{I}_{m_2} \end{bmatrix} \tag{6a}$$

and the permuted matrix is

$$\hat{A} := PAP^\mathsf{T} = \begin{bmatrix} P_{m_1} A_{11} P_{m_1}^\mathsf{T} & P_{m_1} A_{12} \\ A_{21} P_{m_1}^\mathsf{T} & A_{22} \end{bmatrix}$$

The Kron reduction of the permuted matrix is

$$\hat{A}/A_{22} = P_{m_1} \left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right) P_{m_1}^\mathsf{T}$$

Since the square of any permutation matrix is an identity matrix, the Kron reduction A/A_{22} of the original matrix can be recovered as

$$A/A_{22} = P_{m_1} \left(\hat{A}/A_{22}\right) P_{m_1}^\mathsf{T} \tag{6b}$$

where the permutation matrix P_{m_1} is given in (6a).

B. Forward Kron reduction: growing C^l

We now design an alternative iterative Kron reduction that is equivalent to the computation in (4). The alternative procedure grows the maximal clique from $C^0 := A[n,n]$ corresponding to the single node n to $C^k := A/A_{22}$ corresponding to the Kron reduced network after removing k hidden nodes. It has the advantage that each step is easy to reverse, as we will explain in Section III-C.

Consider the admittance matrix A^0 of a single maximal clique of the form in (3). Then initially the maximal clique $C^0 := A^0[n,n] \in \mathbb{C}^{3\times 3}$ is in the lower-right corner of A^0. As we take Schur complements, the components of C^l may be spread across A^l. To facilitate reversing each iteration of Kron reduction we will work with a sequence of permuted matrices $\hat{A}^l$, as summarized in Figure 1 and given next.

Initialization. Let the set of neighbors of the hidden node n in the graph $G(A^0)$ be

$$N_n := \left\{j \neq n : A^0[j,n] \neq 0 \in \mathbb{C}^{3\times 3}\right\}$$

This set may contain both measured and hidden nodes (e.g., if n is a boundary hidden node). Let $n_0 := |N_n|$ be the number of these neighbors. Order them in a way that maintains their relative order in A^0:

$$\underbrace{i_1 < \cdots < i_{m_0}}_{\text{measured nodes}} < \underbrace{i_{m_0+1} < \cdots < i_{n_0}}_{\text{hidden nodes}}$$

Let $y_0 \in \mathbb{C}^{3n_l \times 3}$ denote the admittance submatrices of lines connecting node n, to be Kron reduced in iteration $l=0$, to these neighbors:

$$y_0[j,1] := A^0[i_j, n], \qquad j = 1, \ldots, n_0$$

Re-label nodes in A^0 except node n so that y_0 immediately precedes C^0 in the lower-right corner of the permuted matrix $\hat{A}^0$. Let the permutation matrix be denoted by P^0. Then $\hat{A}^0$ and its maximal clique C^0 take the form (the structure of $\hat{A}^0$ will be proved in Theorem 1 below):

$$\hat{A}^0 := P^0 A^0 \left(\hat{P}^0\right)^\mathsf{T} =: \left[\begin{array}{cc|c} A_{11}^1 & A_{12,1}^1 & 0 \\ & \operatorname{diag}(\hat{y}_0) & -y_0 \\ \hline & & \alpha^0 \end{array}\right] \tag{7a}$$

$$C^0 := \alpha^0 := A^0[n,n] \tag{7b}$$

$$\hat{C}^0 := \left[\begin{array}{c|c} \operatorname{diag}(\hat{y}_0) & -y_0 \\ \hline -y_0^\mathsf{T} & \alpha^0 \end{array}\right] \tag{7c}$$

$$P^0 = \begin{bmatrix} P_-^0 & 0 \\ 0 & \mathbf{I}_3 \end{bmatrix} \tag{7d}$$

where $P_-^0 \in \{0,1\}^{3(n-1)\times 3(n-1)}$ and (since $\hat{A}^0$ has zero row-block sums)

$$\hat{y}_0 := y_0 - \left(A_{12,1}^1\right)^\mathsf{T} (\mathbf{1} \otimes \mathbf{I}_3) \tag{7e}$$

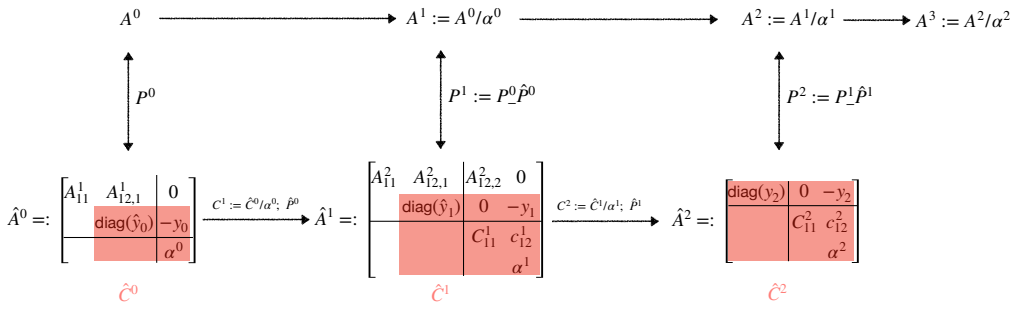

Fig. 1. Reversible iterative forward Kron reduction with three hidden nodes. The focus of iterative Kron reduction and its reverse process will be on the sequence of permuted matrices and their maximal clqiues $(\hat{A}^0, C^l), \ldots, (\hat{A}^k, C^k)$.

From (7a) we have $A^0 = P^0\hat{A}^0(P^0)^\mathsf{T}$ since the square of a permutation matrix is an identity matrix.

Given A^0 the initialization step thus produces $(\hat{A}^0, C^0, P^0)$ where C^0 is the maximal clique of $\hat{A}^0$ and A^0 can be obtained from $\hat{A}^0$ through permutation matrix P^0.

Iteration. For $l = 0, \ldots, k-1$, given $(\hat{A}^l, C^l, P^l)$ that takes the form:

$$\hat{A}^l =: \left[\begin{array}{c|c} A^l_{11} & A^l_{12} \\ \hline (A^l_{12})^\mathsf{T} & C^l \end{array}\right] \tag{8a}$$

$$= \left[\begin{array}{c|cc|c} A^{l+1}_{11} & A^{l+1}_{12,1} & A^{l+1}_{12,2} & 0 \\ \hline & \mathrm{diag}(\hat{y}_l) & 0 & -y_l \\ & & C^l_{11} & c^l_{12} \\ & & & \alpha^l \end{array}\right] \tag{8b}$$

$$C^l =: \begin{bmatrix} C^l_{11} & c^l_{12} \\ (c^l_{12})^\mathsf{T} & \alpha^l \end{bmatrix}, \tag{8c}$$

$$\hat{C}^l := \begin{bmatrix} \mathrm{diag}(\hat{y}_l) & 0 & -y_l \\ & C^l_{11} & c^l_{12} \\ & & \alpha^l \end{bmatrix} \tag{8d}$$

$$P^l = \begin{bmatrix} P^l_- & 0 \\ 0 & \mathbf{I}_3 \end{bmatrix} \tag{8e}$$

where (since $\hat{A}^l$ has zero row-block sums)

$$\hat{y}_l := y_l - \left(A^{l+1}_{12,1}\right)^\mathsf{T}(\mathbf{1} \otimes \mathbf{I}_3) \tag{8f}$$

and $\mathbf{1}$ is the vector of 1s of size n_l, such that $A^l = P^l\hat{A}^l(P^l)^\mathsf{T}$, we compute $(\hat{A}^{l+1}, C^{l+1}, P^{l+1})$ so that:

- The permuted matrix $\hat{A}^{l+1}$ has the same structure as that of $\hat{A}^l$.
- C^{l+1} is the maximal clique of $\hat{A}^{l+1}$.
- The Schur complement $A^{l+1} = A^l/\alpha^l = P^{l+1}\hat{A}^{l+1}(P^{l+1})^\mathsf{T}$ (re-labeling nodes in $\hat{A}^{l+1}$).

As we will see, the one-step Kron reduction, and its reversal, boils down to the propagation of the maximal clique:

$$C^{l+1} := \hat{C}^l/\alpha^l = \begin{bmatrix} \mathrm{diag}(\hat{y}_l) & 0 \\ 0 & C^l_{11} \end{bmatrix} - \begin{bmatrix} -y_l \\ c^l_{12} \end{bmatrix}(\alpha^l)^{-1}\begin{bmatrix} -y_l^\mathsf{T} & (c^l_{12})^\mathsf{T} \end{bmatrix} \tag{9}$$

Specifically we compute $(\hat{A}^{l+1}, C^{l+1}, P^{l+1})$ and A^{l+1} from $(\hat{A}^l, C^l, P^l)$ in (8) in four steps.

1) From (8b), the (unpermuted) Kron reduction $\hat{A}^l/\alpha^l$ is equal to:

$$\hat{A}^l/\alpha^l = \left[\begin{array}{c|c} \hat{A}^{l+1}_{11} & \hat{A}^{l+1}_{12} \\ \hline & \hat{C}^l/\alpha^l \end{array}\right] =: \left[\begin{array}{c|c} \hat{A}^{l+1}_{11} & \hat{A}^{l+1}_{12} \\ \hline & C^{l+1} \end{array}\right]$$

where $\hat{A}^{l+1}_{12} := \begin{bmatrix} A^{l+1}_{12,1} & A^{l+1}_{12,2} \end{bmatrix}$ and C^{l+1} is from (9).

2) It can be checked that the last node $n-(l+1)$ remains the node to be Kron reduced in the next iteration because the way the nodes in y_l have been ordered. Let the set of neighbors of node $n-(l+1)$ in the graph $G(\hat{A}^l/\alpha^l)$ that are not already in the maximal clique C^{l+1} be

$$N_{n-(l+1)} := \left\{ j \notin C^{l+1} : (\hat{A}^l/\alpha^l)[j, n-(l+1)] \neq 0 \right\}$$

This set may contain both measured and hidden nodes. Let $n_{l+1} := \left|N_{n-(l+1)}\right|$ be the number of these neighbors. Order them in a way that maintains their relative order in A^0:

$$\underbrace{i_1 < \cdots < i_{m_{l+1}}}_{\text{measured nodes}} < \underbrace{i_{m_{l+1}+1} < \cdots < i_{n_{l+1}}}_{\text{hidden nodes}}$$

Let $y_{l+1} \in \mathbb{C}^{3n_{l+1}\times 3}$ denote the admittance submatrices of lines connecting node $n-(l+1)$, to be Kron reduced in iteration $l+1$, to these neighbors:

$$y_{l+1}[j,1] := (\hat{A}^l/\alpha^l)[i_j, n-(l+1)]$$

Partition the maximal clique C^{l+1} of $\hat{A}^l/\alpha^l$ into the last row and column block and other submatrices:

$$C^{l+1} =: \begin{bmatrix} C^{l+1}_{11} & c^{l+1}_{12} \\ (c^{l+1}_{12})^\mathsf{T} & \alpha^{l+1} \end{bmatrix} \tag{10a}$$

where $\alpha^{l+1} := (A^l/\alpha^l)[n-(l+1), n-(l+1)] \in \mathbb{C}^{3\times 3}$.

3) Re-label nodes in $\hat{A}^l/\alpha^l$, other than node $n-(l+1)$, so that the permuted matrix $\hat{A}^{l+1}$ has the same structure as that in (8). Let the permutation matrix for re-labeling

be $\hat{P}^l$ so that

$$\hat{A}^{l+1} := \hat{P}^l \left(\hat{A}^l/\alpha^l\right) \left(\hat{P}^l\right)^{\mathsf{T}} =: \left[\begin{array}{c|c} A_{11}^{l+1} & A_{12}^{l+1} \\ \hline \left(A_{12}^{l+1}\right)^{\mathsf{T}} & C^{l+1} \end{array}\right] \tag{10b}$$

$$=: \left[\begin{array}{cc|cc} A_{11}^{l+2} & A_{12,1}^{l+2} & A_{12,2}^{l+2} & 0 \\ & \mathrm{diag}(\hat{y}_{l+1}) & 0 & -y_{l+1} \\ \hline & & C_{11}^{l+1} & c_{12}^{l+1} \\ & & & \alpha^{l+1} \end{array}\right] \tag{10c}$$

where (to maintain zero row-block sums of $\hat{A}^{l+1}$)

$$\hat{y}_{l+1} \;:=\; y_{l+1} \;-\; \left(A_{12,1}^{l+2}\right)^{\mathsf{T}} (\mathbf{1} \otimes \mathbf{I}_3) \tag{10d}$$

and $\mathbf{1}$ is the vector of 1s of size n_{l+1}.

4) Define the permutation matrix

$$P^{l+1} \;:=\; P_-^l \, \hat{P}^l \tag{10e}$$

where P_-^l is defined in (8e) and $\hat{P}^l$ is defined in (10c). Then from (6) and the fact that the square of a permutation matrix is the identity matrix, the one-step Kron reduction of $A^{l+1} := A^l/\alpha^l$ can be recovered as

$$A^{l+1} \;=\; P^{l+1}\hat{A}^{l+1}(P^{l+1})^{\mathsf{T}} \tag{10f}$$

Return. A clique $\hat{A}^k = C^k$ of size $n-k$ and the Kron reduction $A^0/A_{22} = A^k = P^k\hat{A}^k(P^k)^{\mathsf{T}}$.

The important feature of this procedure is that the structure of the permuted matrix $\hat{A}^l$ in (8) is preserved under one-step Kron reduction where the maximal clique C^l is a contiguous block in the lower-right corner of $\hat{A}$. This structure together with the way its maximal clique C^l propagates through (9) make it possible to reverse the Kron reduction as we explain in Section III-C. We now justify (8) and prove the correctness of the forward iterative Kron reduction procedure above. Consider $A^0 =: \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^{\mathsf{T}} & A_{22} \end{bmatrix} \in \mathbb{C}^{3n\times 3n}$ with a $3k \times 3k$ nonsingular submatrix A_{22}, $1 \le k < n$.

Theorem 1 (Invariant structure of $\hat{A}^l$ [19])**.** *Suppose A^0 is the admittance matrix of a single maximal clique of the form in* (3) *and it satisfies Assumption 1. The procedure above computes the Kron reduction A^0/A_{22}, i.e., for $l = 0, \ldots, k-1$,*

1) *The permuted matrix $\hat{A}^l$ has the form in* (8)*. In particular, the entries of $A_{11}^{l+1}, A_{12,1}^{l+1}, A_{12,2}^{l+1}$ in $\hat{A}^l$ are equal to the corresponding entries in A^0.*
2) *The matrix A^{l+1} computed in* (10f) *is equal to A^l/α^l.*
3) *At $l = k-1$, $A_{11}^k = 0$, $A_{12}^k = 0$, $\hat{A}^k = C^k$, and $A^k = A^0/A_{22}$.*

Moreover

4. *In* (8f)*, if node i in y_l is a measured node (i.e., in A_{11} with an appropriate label in light of all the permutations by iteration l), then $\hat{y}_l[i,1] = y_l[i,1]$.*
5. *If Assumption 2 holds then for each $l = 0, \ldots, k-1$, the number n_l of neighbors of node $n-l$ not in C^l is at least 2.*

We now use Theorem 1 (proved in [19]) to devise a method to compute Y from its Kron reduction $\bar{Y}$ by reversing each step in the forward iterative Kron reduction.

C. Reverse Kron reduction: shrinking C^l

Given A^0/A_{22}, it is generally not possible to reverse the Kron reduction to recover the original admittance matrix A^0. This turns out to be possible, using the invariance structure of A^l in Theorem 1, when the network is a tree.

The basic idea is as follows. Given the Kron reduction $A^k := A^0/A_{22}$, we will reverse each step in the process illustrated in Figure 1, focusing on computing the permuted sequence $\hat{A}^l$ in the reverse direction. Each reverse iteration l will involve three steps:

- From $\hat{A}^{l+1}$ with the structure in (8), identify a set of "sibling" nodes in C^{l+1} that are adjacent to a unique "parent" hidden node (this step requires Assumption 3 below). These sibling nodes define the set N_{n-l} of nodes in y_l, to be determined, and their parent node will be added in iteration l to the set of identified nodes.
- Reverse the Kron reduction of the maximal clique, i.e., compute $\hat{C}^l$ from C^{l+1}. This allows us to construct $\hat{A}^l$ which will have the structure in (8).
- Permute $\hat{A}^l$ to obtain A^l (this step can also be done only on $\hat{A}^0$ at the end).

The following assumption is important for constructing the permuted matrix $\hat{A}^l$ from $\hat{A}^{l+1}$.

Assumption 3 (Parent node)**.** *Given any $\hat{A}^{l+1}$ and its maximal clique C^{l+1}, it is possible to determine the identity of a set of all "sibling" nodes in C^{l+1} that are adjacent to a common "parent" node in the original graph $G(A^0)$ but not in the graph $G(\hat{A}^{l+1})$.* □

The parent node not in $G(\hat{A}^{l+1})$ will be labeled by $n-l$ and added to $G(\hat{A}^l)$ in iteration l. In the forward direction, y_l is constructed from the graph $G(\hat{A}^{l-1}/\alpha^{l-1})$ (or $G(\hat{A}^l)$). In the reverse direction, Assumption 3 allows us to determine the identity of the nodes in (but not the value of) y_l from the graph $G(\hat{A}^{l+1})$ and construct (9). We provide a sufficient condition in [19] for Assumption 3 to hold and an algorithm to identify these sibling nodes.

The procedure to identify A^0 from its Kron reduction A^0/A_{22} is as follows.

Algorithm: reverse iterative Kron reduction

Given: $A^k := A^0/A_{22} \in \mathbb{C}^{3(n-k)\times 3(n-k)}$.
Initialize: Let

$$\hat{A}^k \;:=\; A^k, \qquad C^k \;:=\; \hat{A}^k$$
$$\tilde{P}^k \;:=\; \mathbf{I}_{3(n-k)\times 3(n-k)}$$

Iterate for $l = k-1, k-2, \ldots,$ until the maximal clique $C^l \in \mathbb{C}^{3\times 3}$:

Given $(\hat{A}^{l+1}, C^{l+1}, \tilde{P}^{l+1})$ with

$$\hat{A}^{l+1} =: \left[\begin{array}{c|c} A_{11}^{l+1} & A_{12}^{l+1} \\ \hline (A_{12}^{l+1})^{\mathsf{T}} & C^{l+1} \end{array}\right] =: \left[\begin{array}{cc|cc} A_{11}^{l+2} & A_{12,1}^{l+2} & A_{12,2}^{l+2} & 0 \\ & \mathrm{diag}(\hat{y}_{l+1}) & 0 & -y_{l+1} \\ \hline & & C_{11}^{l+1} & c_{12}^{l+1} \\ & & & \alpha^{l+1} \end{array}\right] \tag{11}$$

we compute $(\hat{A}^l, C^l, \tilde{P}^l)$ as follows.

1. Identify all the "sibling" nodes in C^{l+1} that are adjacent to a common "parent" hidden node in $G(\hat{A}^0)$ but not in $G(\hat{A}^{l+1})$ guaranteed by Assumption 3. These nodes in C^{l+1} define the set N_{n-l} of nodes in y_l. The parent node is labeled $(n-k)+\tilde{l} = n-l$ in $\hat{A}^l$ (to be determined).
2. Solve (9) for $(y_l, \hat{y}_l)$ and $(C_{11}^l, c_{12}^l, \alpha^l)$. Due to page limit, we leave the details to [19].
3. Substitute $(y_l, \hat{y}_l)$ and $(C_{11}^l, c_{12}^l, \alpha^l)$ as well as $(A_{11}^{l+1}, A_{12}^{l+1})$ from (11) into (8) to obtain $(\hat{A}^l, C^l)$. Note that $\hat{A}^l$ has the same structure as $\hat{A}^{l+1}$ in (11) by construction, i.e., the invariance structure is preserved in both the forward and the reverse directions.
4. The matrix $y_l \in \mathbb{C}^{3n_l \times 3n_l}$ may include both measured and hidden nodes and therefore the order in which the measured nodes appear in $\hat{A}^l$ may not agree with that in the given $A^k = \tilde{A}^0$. Re-label the nodes in $\hat{A}^l$ so that they agree and let the permutation matrix be $\tilde{P}^l$. Then set

$$A^l \;:=\; \tilde{P}^l(\hat{A}^l)(\tilde{P}^l)^{\mathsf{T}}$$

Return: $A^{\tilde{k}} = A^0$.

IV. Concluding remarks

In this paper we study the problem of identifying a three-phase admittance matrix from partial measurements of voltage and current phasors. It boils down to computing the inverse Y of any given Kron reduced admittance matrix $\bar{Y}$. We have presented the solution for the special case of a single maximal clique, to which the general identification can be reduced (see [19]). The key idea is to derive an invariant structure that is preserved in each step of an iterative Kron reduction and use it to reverse Kron reduction step-by-step.

The main limitations of the paper are specific to unbalanced three-phase system. Assumption 1 requires $\mathrm{Re}(y_{jk}) \succ 0$ which may not be necessary. Assumption 2 assumes that all hidden nodes have zero injections which may not hold in practice. This assumption is required for both single-phase or three-phase network. Assumption 3, which identifies the common parent hidden node of a set of sibling nodes that have been identified, is crucial for the reverse Kron reduction. For an unbalanced three-phase network, a sufficient condition in [19] that ensures Assumption 3 requires all lines to be of the same type and differ from each other only in their lengths. This is too restrictive and likely not necessary.

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