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Distributed-Parameter Port-Hamiltonian Systems in Discrete-Time

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***Abstract*— This paper presents a design framework of discrete-time regulators for linear, port-Hamiltonian, boundary control systems. The contribution is twofold. At first, a discrete-time approximation of the plant dynamics originally described by a linear PDE with boundary actuation is introduced. The discretisation is performed in time only. Thus, the "distributed nature" of the state is maintained. Such a system inherits the passivity of the original one and is well-posed, namely the "next" state always exists. The second result is the characterisation of discrete-time, linear controllers in the port-Hamiltonian form that render the closed-loop dynamics asymptotically stable. A numerical example illustrates the effectiveness of the proposed framework.**

I. Introduction

Port-Hamiltonian systems have been introduced to model lumped-parameter continuous-time physical systems in a unified manner, [1]. Later, the paradigm has been extended to deal with distributed-parameter systems, mathematically described by partial differential equations (PDEs), [2]. Recently, the notion of discrete-time port-Hamiltonian system has been introduced, and has become a popular framework for the geometric integration of ordinary differential equations (ODEs), and the synthesis of digital control systems, [3]–[5]. The goal is to stabilise finite-dimensional continuous-time systems via digital regulators whose design is based on a discrete-time approximation of the plant. In this paper, such an approach is extended to infinite-dimensional port-Hamiltonian system with boundary actuation and sensing.

At first, a discrete-time approximation for the linear, port-Hamiltonian boundary control systems with one-dimensional domain introduced in [6] is proposed. The idea is to maintain the "distributed nature" of the state, and discretise in time. Thus, a difference equation where the "sampled" state is a function of the spatial coordinate is obtained. The result is a discrete-time infinite-dimensional system, see e.g. [7]–[9] for more details. Because the starting point is a PDE model with boundary actuation, the discrete-time dynamics is associated to a boundary-value problem to be solved at each sampling interval. It is shown that this problem always admits a unique solution, and so the sampled dynamics is well-posed.

The approximation of the PDE model has been inspired by the energy-preserving integrators of ODEs [10], and with the (lumped-parameters) discrete-time systems in [5], [11], [12] shares the property that the dynamics depends on the discrete gradient of the Hamiltonian (density) function. As in finite dimensions, the advantage is that an energy-balance relation is easily obtained for the discrete-time system, thus making the control design and stability analysis based on Lyapunov arguments simpler and similar to the continuous-time counterpart. This property has been exploited to characterise the linear, discrete-time boundary controllers capable to asymptotically stabilise the closed-loop discrete-time system.

For the discrete-time approximation of the PDE models studied here, general conditions to achieve asymptotic stability with a linear, discrete-time port-Hamiltonian regulator are obtained. The control system shares the same features of the plant, namely the property that the dynamics depends on the discrete gradient of the Hamiltonian function. The stability result is general, and it can be interpreted as the discrete-time re-formulation of [13], [14]. Similarly to the continuous-time case, asymptotic stability is achieved if the control system meets two requirements. The first one is that the state variable is fully damped, and the second one is that a non-zero feedthrough term is present and imposes full dissipation on at least one extremity of the spatial domain. The effectiveness of the design framework is illustrated with the help of an example in which the stabilisation problem of the longitudinal vibrations in an elastic beam is tackled.

II. Background on distributed-parameter port-Hamiltonian systems

We refer to the class of linear port-Hamiltonian systems on real Hilbert spaces described by the PDE, [6], [15]:

$$\frac{\partial x}{\partial t}(t,z) = P_1 \frac{\partial}{\partial z}\big(\mathcal{L}x(t,z)\big) + P_0 \mathcal{L}x(t,z) \tag{1}$$

with $x \in X = L^2(a,b;\mathbb{R}^n)$, and $\mathcal{L} \in \mathbb{R}^{n\times n}$ is constant matrix such that $\mathcal{L} = \mathcal{L}^{\mathrm{T}} > 0$. X is endowed with the inner product $\langle x_1 \mid x_2\rangle_{\mathcal{L}} = \langle x_1 \mid \mathcal{L}x_2\rangle$ and norm $\|x_1\|^2_{\mathcal{L}} = \langle x_1 \mid x_1\rangle_{\mathcal{L}}$, where $\langle\cdot\mid\cdot\rangle$ denotes the natural L^2-inner product. X is the space of energy variables,

$$H(x(t)) = \frac{1}{2}\|x(t)\|^2_{\mathcal{L}} = \int_a^b \underbrace{\frac{1}{2}x^{\mathrm{T}}(t,z)\mathcal{L}x(t,z)}_{=h(x(t,z))}\,\mathrm{d}z \tag{2}$$

is the Hamiltonian function, with $h(x)$ the "energy" density, and $\mathcal{L}x(t,z)$ is the co-energy variable. Moreover, $P_1, P_0 \in \mathbb{R}^{n\times n}$, with $P_1 = P_1^{\mathrm{T}}$ and invertible, and $P_0 + P_0^{\mathrm{T}} \leq 0$. For (1), we define the boundary variables $f_\partial, e_\partial \in \mathbb{R}^n$ as

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}}_{=:R}\begin{pmatrix} \mathcal{L}x(b) \\ \mathcal{L}x(a) \end{pmatrix}.$$

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The characterisation of the (boundary) inputs and outputs for (1) in terms of f_∂ and e_∂ to have a boundary control system has been addressed in the next proposition, a particular case of the framework introduced in [6, Theorems 4.4 and 5.3].

Proposition 2.1: Denote by W a full rank $n \times 2n$ matrix, and define the input $u(t) \in \mathbb{R}^n$ as

$$u(t) = W \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}. \tag{3}$$

Given $\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$, if W satisfies $W\Sigma W^{\mathrm{T}} = 0$, then (1) with input (3) so that $u \in C^2(0,\infty;\mathbb{R}^n)$ and initial condition $x(0) \in H^1(a,b;\mathbb{R}^n)$ is a boundary control system on X in the sense of the semigroup theory, [9, Definition 3.3.2]. Moreover, let $\tilde{W}$ be a full rank $n \times 2n$ matrix such that $\begin{pmatrix} W^{\mathrm{T}} & \tilde{W}^{\mathrm{T}} \end{pmatrix}$ is invertible, $\tilde{W}\Sigma\tilde{W}^{\mathrm{T}} = 0$, and $W\Sigma\tilde{W}^{\mathrm{T}} = I$, and define the output as

$$y(t) = \tilde{W} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \in \mathbb{R}^n. \tag{4}$$

Then, along system's trajectories we have that

$$\begin{aligned} \dot{H}(x(t)) &= \int_a^b \big(\mathcal{L}x(t,z)\big)^{\mathrm{T}} P_0 \mathcal{L}x(t,z)\,\mathrm{d}z + y^{\mathrm{T}}(t)u(t) \\ &\le y^{\mathrm{T}}(t)u(t) \end{aligned} \tag{5}$$

where $H(x)$ has been defined in (2).

The design of a linear control system able to exponentially stabilise the PDE models of Proposition 2.1 is discussed in [14]. Let us consider the linear port-Hamiltonian system

$$\begin{cases} \dot{w}(t) = FQw(t) + Gu_c(t) \\ y_c(t) = G^{\mathrm{T}}Qw(t) + Su_c(t) \end{cases} \tag{6}$$

in which $w \in \mathbb{R}^{n_c}$, $u_c, y_c \in \mathbb{R}^n$, $F, Q \in \mathbb{R}^{n_c\times n_c}$ with $F + F^{\mathrm{T}} \le 0$ and $Q = Q^{\mathrm{T}} > 0$, $G \in \mathbb{R}^{n_c\times n}$, and $S \in \mathbb{R}^{n\times n}$ and such that $S = S^{\mathrm{T}}$. System (6) is a dynamic controller that is interconnected to the boundary port (u, y) of (1) defined thanks to (3) and (4) in power-conserving way

$$\begin{pmatrix} u(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} u_c(t) \\ y_c(t) \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} r(t), \tag{7}$$

being $r \in \mathbb{R}^n$ an additional input. Exponential stability is studied in the next proposition, see [14, Theorem IV.2].

Proposition 2.2: Let us consider the closed-loop system that consists of the interconnection of (1) and (6) via (7), with $r(t) = 0$. If FQ is Hurwitz, G is full-rank and $S > 0$, then the closed-loop system is exponentially stable.

III. From continuous- to discrete-time

The aim of this section is to obtain a discrete-time formulation of the boundary control system introduced in Proposition 2.1. The idea is to "preserve" the continuous dependance on the spatial coordinate z of the energy variable x, but to have a discrete-time evolution of the dynamics. In a similar way as in [5], we start by approximating the time-derivative of the state with the finite difference as

$$\frac{\partial x}{\partial t}(t_k, z) \simeq \tfrac{1}{\tau}\left[x_{k+1}(z) - x_k(z)\right] \tag{8}$$

being $t_k = k\tau$, with $\tau > 0$ and $k \in \mathbb{N}$, the time samples, and $x_k(z) = x(t_k, z) \in L^2(a,b;\mathbb{R}^n)$. Besides, from (1), it is clear that the dynamics depends on the co-energy variable $\mathcal{L}x$ or, equivalently, on the gradient of the energy density $h(x)$ defined in (2). This property is valid also in the nonlinear case, see e.g. [2]. So, an option that mimics what has been done in [5] for the lumped-parameter case, is to rely on a discrete approximation of the gradient operator, [10, Definition 3.1].

Definition 3.1: Let $\Phi : \mathbb{R}^p \to \mathbb{R}$ be a continuously differentiable function. A *discrete gradient* $\bar{\nabla}\Phi : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ is a continuous map such that for all $\varphi, \varphi_+ \in \mathbb{R}^p$ we have

$$\begin{gathered} (\varphi_+ - \varphi)^{\mathrm{T}}\bar{\nabla}\Phi(\varphi,\varphi_+) = \Phi(\varphi_+) - \Phi(\varphi), \\ \lim_{\varphi_+\to\varphi} \bar{\nabla}\Phi(\varphi,\varphi_+) = \nabla\Phi(\varphi). \end{gathered} \tag{9}$$

Typical examples are the mean value [16, Theorem 2.1], or the Gonzalez discrete gradients, [10, Proposition 3.1]. If we have a quadratic function $\Phi(\varphi) = \varphi^{\mathrm{T}}Q\varphi$ with $Q = Q^{\mathrm{T}}$, it is easy to check that $\bar{\nabla}\Phi(\varphi,\varphi_+) = Q(\varphi + \varphi_+)$. Then, from (8) and Definition 3.1, the discrete-time re-formulation of the PDE (1) becomes

$$\begin{aligned} \tfrac{1}{\tau}\left[x_{k+1}(z) - x_k(z)\right] &= \tfrac{1}{2}P_1\mathcal{L}\frac{\mathrm{d}}{\mathrm{d}z}\left[x_{k+1}(z) + x_k(z)\right] \\ &\quad + \tfrac{1}{2}P_0\mathcal{L}\left[x_{k+1}(z) + x_k(z)\right] \end{aligned} \tag{10}$$

since the energy density $h(x) = \frac{1}{2}x^{\mathrm{T}}\mathcal{L}x$ introduced in (2) is a quadratic function. It is immediate to re-formulate (10) as

$$P_1\mathcal{L}\frac{\mathrm{d}x_{k+1}}{\mathrm{d}z}(z) = \left(\tfrac{2}{\tau}I - P_0\mathcal{L}\right)x_{k+1}(z) - \Psi(x_k(z)), \tag{11}$$

where

$$\Psi(x_k(z)) = \left(\tfrac{2}{\tau}I + P_0\mathcal{L}\right)x_k(z) + P_1\mathcal{L}\frac{\mathrm{d}x_k}{\mathrm{d}z}(z). \tag{12}$$

Note that the discrete-time system (10) is in implicit form, as in the finite-dimensional case treated in [5].

Remark 3.1: This time discretisation corresponds to the Cayley transform [17], but its application to boundary control systems is new, even if boundary control and state space forms are essentially equivalent. Such a transformation has been employed e.g. in optimal control in [18], and in model predictive control of PDEs in [19]. However, the approach proposed here is less abstract than the mentioned ones.

Under the hypothesis that the trajectories of (10) exist (this point is clarified later), we can compute the variation of the total energy (2) between two consecutive samples. From the first relation in (9), we get

$$H(x_{k+1}) - H(x_k) = \int_a^b \tfrac{1}{2}(x_{k+1} + x_k)^{\mathrm{T}}\mathcal{L}(x_{k+1} - x_k)\,\mathrm{d}z,$$

and then

$$\begin{aligned} &H(x_{k+1}) - H(x_k) = \\ &\quad = \tau\int_a^b \tfrac{1}{2}(x_{k+1} + x_k)^{\mathrm{T}}\mathcal{L}P_1\mathcal{L}\frac{\mathrm{d}}{\mathrm{d}z}\left[\tfrac{1}{2}(x_{k+1} + x_k)\right]\mathrm{d}z \\ &\quad\quad + \tau\int_a^b \tfrac{1}{2}(x_{k+1} + x_k)^{\mathrm{T}}\mathcal{L}P_0\mathcal{L}\left[\tfrac{1}{2}(x_{k+1} + x_k)\right]\mathrm{d}z \end{aligned} \tag{13}$$

where (10) has been taken into account. Since $P_0 + P_0^{\mathrm{T}} \leq 0$, the second term is always lower than 0 and represents the dissipated energy in the interval $[k\tau, (k+1)\tau)$ due to the internal resistive structure of the PDE model (1). The first contribution, instead, is related to the energy supplied to the system through the boundary port (u_k, y_k) that is now defined by adapting the machinery proposed in [6] to (10).

More precisely, if

$$\begin{pmatrix} f_{\partial k} \\ e_{\partial k} \end{pmatrix} = R \begin{pmatrix} \frac{1}{2}\mathcal{L}\left[x_{k+1}(b) + x_k(b)\right] \\ \frac{1}{2}\mathcal{L}\left[x_{k+1}(a) + x_k(a)\right] \end{pmatrix}, \tag{14}$$

with simple passages we get that

$$\int_a^b \tfrac{1}{2}(x_{k+1} + x_k)^{\mathrm{T}} \mathcal{L} P_1 \mathcal{L} \frac{\mathrm{d}}{\mathrm{d}z}\left[\tfrac{1}{2}(x_{k+1} + x_k)\right] \mathrm{d}z = \\ = \frac{1}{2} \begin{pmatrix} f_{\partial k} \\ e_{\partial k} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} f_{\partial k} \\ e_{\partial k} \end{pmatrix},$$

where the same steps illustrated in [6] have been followed. If the boundary input and output are defined as

$$u_k = W \begin{pmatrix} f_{\partial k} \\ e_{\partial k} \end{pmatrix} \qquad y_k = \tilde{W} \begin{pmatrix} f_{\partial k} \\ e_{\partial k} \end{pmatrix}, \tag{15}$$

with the W and $\tilde{W}$ matrices chosen as in Proposition 2.1, then (13) becomes

$$\begin{aligned} & H(x_{k+1}) - H(x_k) = \\ & \quad = \tau \int_a^b \tfrac{1}{2}(x_{k+1} + x_k)^{\mathrm{T}} \mathcal{L} P_0 \mathcal{L}\left[\tfrac{1}{2}(x_{k+1} + x_k)\right] \mathrm{d}z \\ & \qquad + \tau\, y_k^{\mathrm{T}} u_k \leq \tau\, y_k^{\mathrm{T}} u_k \end{aligned} \tag{16}$$

which is the discrete-time counterpart of (5).

The definition of the input u_k in (15) assures the well-posedness of the discrete-time dynamics (10) if and only if at each sample k for a given actual state x_k and input u_k there exists only one x_{k+1} such that (10) or, equivalently, (11)-(12) holds. From the latter formulation of the discrete-time system, such an ODE is required to have a unique solution. Note that for a given u_k, from (14) and (15) we get that the following "boundary" conditions on x_{k+1} are imposed:

$$WR \begin{pmatrix} \mathcal{L}x_{k+1}(b) \\ \mathcal{L}x_{k+1}(a) \end{pmatrix} = 2u_k - WR \begin{pmatrix} \mathcal{L}x_k(b) \\ \mathcal{L}x_k(a) \end{pmatrix}. \tag{17}$$

Thus, it is necessary to verify that the linear boundary problem defined by (10) and (17) has a unique solution for all x_k and u_k. This problem is tackled in the next proposition.

Proposition 3.1: Let us consider the discrete-time approximation of the boundary control system in port-Hamiltonian form of Proposition 2.1 given by the difference equation (10) equipped with the boundary input defined as in (15). Then, for all $x_k \in L^2(a, b; \mathbb{R}^n)$ and $u_k \in \mathbb{R}^n$ there is only one $x_{k+1} \in L^2(a, b; \mathbb{R}^n)$ that satisfies (10) and (15).

Proof: The idea is to show that the linear boundary value problem defined by (10) and (17) has a unique solution for all x_k and u_k. Following e.g. [20, Theorem 1.1], this is true if and only if the homogeneous problem has only the zero solution. From (10) and (17), such a problem is

$$\begin{aligned} P_1 \mathcal{L} \frac{\mathrm{d}}{\mathrm{d}z}\xi(z) &= \left(\frac{2}{\tau} I - P_0 \mathcal{L}\right)\xi(z) \\ WR \begin{pmatrix} \mathcal{L}\xi(b) \\ \mathcal{L}\xi(a) \end{pmatrix} &= 0. \end{aligned} \tag{18}$$

Let us introduce the operator J defined as $J\xi = P_1 \mathcal{L} \frac{\mathrm{d}}{\mathrm{d}z}\xi + P_0 \mathcal{L}\xi$, with domain $D(J) = \{\xi \in H^1(a, b; \mathbb{R}^n)\}$. The PDE (1) can be written in abstract form as $\dot{\xi} = J\xi$. Besides, from the proof of Proposition 2.1, see [6, Theorems 4.4 and 5.3], the operator $J'\xi = P_1 \mathcal{L} \frac{\mathrm{d}}{\mathrm{d}z}\xi + P_0 \mathcal{L}\xi$, with domain

$$D(J') = \left\{\xi \in H^1(a, b; \mathbb{R}^n) \mid 0 = WR \begin{pmatrix} \mathcal{L}\xi(b) \\ \mathcal{L}\xi(a) \end{pmatrix}\right\}$$

generates a contraction semigroup. This means that for all $\alpha > 0$ and $\xi \in D(J')$, $\|(\alpha I - J')\xi\|_{\mathcal{L}} \geq \alpha \|\xi\|_{\mathcal{L}}$, see [9, Theorem 2.3.2]. Now, (18) can be re-written as $(\alpha I - J')\xi = 0$, with $\alpha = \frac{2}{\tau}$ and $\xi \in D(J')$. Consequently, we get that $0 = \left\|\left(\frac{2}{\tau} I - J'\right)\xi\right\|_{\mathcal{L}} \geq \frac{2}{\tau}\|\xi\|_{\mathcal{L}}$, and so the only solution of the homogeneous boundary value problem (18) is $\xi = 0$. ■

The problem is now to determine under which conditions the discrete-time version of the boundary controller (6) is able to asymptotically stabilise the discrete-time dynamics (10) and (17). By following the same steps that lead to (10), but see also [5], the digital controller takes the form

$$\begin{cases} w_{k+1} = w_k + \tau\left[\frac{1}{2} FQ(w_{k+1} + w_k) + G u_{ck}\right] \\ \quad y_{ck} = \frac{1}{2} G^{\mathrm{T}} Q(w_{k+1} + w_k) + S u_{ck} \end{cases}. \tag{19}$$

Note that (19) is in implicit form. However, because of the linearity and since $I - \frac{\tau}{2} FQ$ is non-singular, the explicit formulation always exists.

IV. Stabilisation via static or dynamic controllers

The extension of Proposition 2.2 to the discrete-time scenario in which the distributed parameter system is described by (10) and (17), and the controller by (19) requires to study at first the conditions that the static feedback law

$$u_k = -K y_k, \quad K = K^{\mathrm{T}} \geq 0 \tag{20}$$

has to meet so that x_k asymptotically converges to the origin. It is easy to check that in the continuous-time case, such a control action combined with the requirement $P_0 + P_0^{\mathrm{T}} < 0$ makes the origin of (1) an exponentially stable equilibrium. In discrete-time, instead, from (16), for all $k \in \mathbb{N}$ we get

$$H(x_{k+1}) - H(x_k) \leq -\kappa \|x_{k+1} + x_k\|_{\mathcal{L}}^2 \tag{21}$$

for some $\kappa > 0$, so energy is decreasing until $x_{k+1} + x_k \neq 0$. In steady state we have that $x_{k+1} = -x_k$, which combined with (10) implies that $x_k = 0$, thus proving the convergence to the origin. In the next proposition, sufficient conditions on K in (20) so that balance relation (21) holds for the general case $P_0 + P_0^{\mathrm{T}} \leq 0$ are presented.

Proposition 4.1: Under the conditions of Proposition 3.1, let us consider the discrete-time dynamics (10) equipped with the boundary input and output (17). Then, the static control action (20) makes the trajectories to converge to the origin if $K \geq 0$ is such that

$$\begin{aligned} & y_k^{\mathrm{T}} u_k \leq -\kappa_b \left\| x_{k+1}(b) + x_k(b) \right\|^2 \\ \text{or} \quad & y_k^{\mathrm{T}} u_k \leq -\kappa_a \left\| x_{k+1}(a) + x_k(a) \right\|^2 \end{aligned} \tag{22}$$

for some $\kappa_a, \kappa_b > 0$ and all $k \in \mathbb{N}$.

Proof: The proof relies on an extension of the energy-multiplier method [21] for the discrete-time dynamics (10). Such a technique has been already applied in [22] to study the exponential stability of boundary control systems in port-Hamiltonian belonging to the general class characterised in Proposition 2.1. Here, a similar notation has been adopted, and analogously we consider the function

$$W(x) = \frac{1}{2} \int_a^b m(z) x^{\mathrm{T}}(z) P_1^{-1} x(z) \, \mathrm{d}z,$$

with $m \in C^1(a, b; \mathbb{R})$ to be specified later. It is easy to show that there exists $\varepsilon_0 > 0$ so that $\varepsilon_0 |W(x)| \leq H(x)$ for all $x \in X$. Along the trajectories of (10), we have that

$$\begin{aligned} W(x_{k+1}) - W(x_k) = \\ = \frac{\tau}{4} \int_a^b m(z) (x_{k+1} + x_k)^{\mathrm{T}} \mathcal{L} \frac{\mathrm{d}}{\mathrm{d}z} (x_{k+1} + x_k) \, \mathrm{d}z \\ + \frac{\tau}{4} \int_a^b m(z) (x_{k+1} + x_k)^{\mathrm{T}} P_1^{-1} P_0 \mathcal{L} (x_{k+1} + x_k) \, \mathrm{d}z. \end{aligned} \tag{23}$$

With simple passages, the first integral in (23) becomes

$$\begin{aligned} & \frac{1}{2} m(b) \left\| \sqrt{\mathcal{L}} [x_{k+1}(b) + x_k(b)] \right\|^2 \\ & \quad - \frac{1}{2} m(a) \left\| \sqrt{\mathcal{L}} [x_{k+1}(a) + x_k(a)] \right\|^2 - \\ & \quad - \frac{1}{2} \int_a^b m'(z) (x_{k+1} + x_k)^{\mathrm{T}} \mathcal{L} (x_{k+1} + x_k) \, \mathrm{d}z. \end{aligned}$$

Now, let us assume that the first relation in (22) holds; the second one is treated in a similar manner. If we require that $m(a) = 0$, (23) can be equivalently re-written as

$$\begin{aligned} W(x_{k+1}) - W(x_k) \leq \tau \kappa_1 m(b) \left\| x_{k+1}(b) + x_k(b) \right\|^2 \\ + \frac{\tau}{2} \int_a^b (x_{k+1} + x_k)^{\mathrm{T}} M(z) (x_{k+1} + x_k) \, \mathrm{d}z \end{aligned} \tag{24}$$

for some $\kappa_1 > 0$, and where

$$M(z) = \tfrac{1}{2} m(z) P_1^{-1} P_0 \mathcal{L} - m'(z) \mathcal{L}. \tag{25}$$

The function $V(x) = H(x) + \varepsilon W(x)$ is a Lyapunov function for (10) with input defined in (20) if $0 < \varepsilon < \varepsilon_0$, since

$$\left(1 - \tfrac{\varepsilon}{\varepsilon_0}\right) H(x) \leq V(x) \leq \left(1 + \tfrac{\varepsilon}{\varepsilon_0}\right) H(x)$$

for all $x \in X$. Besides, from (16), (24) and (25), we have

$$\begin{aligned} & V(x_{k+1}) - V(x_k) \leq \\ & \quad \leq -\tau \left[\kappa_b - \varepsilon \kappa_1 m(b)\right] \left\| x_{k+1}(b) + x_k(b) \right\|^2 \\ & \quad + \frac{\tau \varepsilon}{2} \int_a^b (x_{k+1} + x_k)^{\mathrm{T}} M(z) (x_{k+1} + x_k) \, \mathrm{d}z. \end{aligned} \tag{26}$$

Let us assume that in (25) it is possible to select $m(z)$ so that $M(z) < 0$ for all $z \in [a, b]$. Then, from (26) and if ε is sufficiently small, $V(x_{k+1}) - V(x_k) \leq -\bar{\kappa} \left\| x_{k+1} + x_k \right\|_{\mathcal{L}}^2$, with $\bar{\kappa} > 0$. This implies that $V(x)$ decreases as long as $x_{k+1} + x_k \neq 0$. As discussed after (21), this shows the convergence to the origin for the trajectories of (10) with static control action (20). The last point to check if it is possible to select $m(z)$ so that $m(a) = 0$ and $M(z) < 0$. In fact, an admissible choice is $m(z) = \mathrm{e}^{\mu(z-a)} - 1$ (see e.g. [23]), with $\mu > 0$, which implies that for all $z \in [a, b]$

$$M(z) = \mathrm{e}^{\mu(z-a)} \left[\tfrac{1}{2}(1 - \mathrm{e}^{-\mu(z-a)}) P_1^{-1} P_o \mathcal{L} - \mu \mathcal{L}\right] < 0$$

if μ is sufficiently large, since $\mathcal{L}$ is positive definite. ■

Based on the previous result, an extension of Proposition 2.2 to the discrete-time scenario is now discussed.

Proposition 4.2: Under the conditions of Proposition 3.1, let us consider the discrete-time dynamics (10) interconnected to the boundary controller (19) in power-conserving way, i.e. for all $k \in \mathbb{N}$ we have

$$\begin{pmatrix} u_k \\ y_k \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} u_{ck} \\ y_{ck} \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} r_k \tag{27}$$

being $r_k \in \mathbb{R}^n$ an additional input. If F is such that $F + F^{\mathrm{T}} < 0$, Q, $S > 0$ and the input-output pair for (10) has been selected so that

$$\begin{aligned} & \|u_k\|^2 + \|y_k\|^2 \geq \kappa_b' \left\| x_{k+1}(b) + x_k(b) \right\|^2 \\ \text{or} \quad & \|u_k\|^2 + \|y_k\|^2 \geq \kappa_a' \left\| x_{k+1}(a) + x_k(a) \right\|^2 \end{aligned} \tag{28}$$

for some $\kappa_a', \kappa_b' > 0$ and all $k \in \mathbb{N}$, then the trajectories of the closed-loop system converge to zero as long as $r_k = 0$.

Proof: Let $H_c(w) = \frac{1}{2} w^{\mathrm{T}} Q w$ be the Hamiltonian function of (19). We get $H_c(w_{k+1}) - H_c(w_k) = \tau y_{ck}^{\mathrm{T}} u_{ck} - \tau u_{ck}^{\mathrm{T}} S u_{ck} + \frac{\tau}{2} (w_{k+1} + w_k)^{\mathrm{T}} Q F Q (w_{k+1} + w_k)$. With the proof of Proposition 4.1 in mind, denote by

$$V'(x, w) = V(x) + H_c(w) \tag{29}$$

a Lyapunov function for the closed-loop system. From (27), we obtain that $y_k^{\mathrm{T}} u_k + y_{ck}^{\mathrm{T}} u_{ck} = 0$ for all $k \in \mathbb{N}$. Thus, for (29) we get that

$$\begin{aligned} V'(x_{k+1}, w_{k+1}) - V'(x_k, w_k) \leq -\tau u_{ck}^{\mathrm{T}} S u_{ck} \\ + \tfrac{\tau}{2} (w_{k+1} + w_k)^{\mathrm{T}} Q F Q (w_{k+1} + w_k) \\ + \tau \varepsilon \kappa_1' m(b) \left\| x_{k+1}(b) + x_k(b) \right\|^2 \\ - \tau \varepsilon \kappa_2' m(a) \left\| x_{k+1}(b) + x_k(b) \right\|^2 \\ + \frac{\tau \varepsilon}{2} \int_a^b (x_{k+1} + x_k)^{\mathrm{T}} M(z) (x_{k+1} + x_k) \, \mathrm{d}z. \end{aligned}$$

where κ_1' and κ_2' are two positive constants. Again from (27), we see that $u_{ck} = y_k$. Now, as in the proof of Proposition 4.1, assume that the first condition in (28) holds. Consequently, if we select the function $m(z)$ so that $m(a) = 0$ and $M(z) < 0$, from the previous relation we obtain that

$$\begin{aligned} V'(x_{k+1}, w_{k+1}) - V'(x_k, w_k) \leq -\tau \kappa_w \left\| w_{k+1} + w_k \right\|^2 \\ - \tau y_k^{\mathrm{T}} S y_k + \tau \varepsilon \kappa_1' m(b) \left\| x_{k+1}(b) + x_k(b) \right\|^2 \\ - \tau \bar{\kappa}' \left\| x_{k+1} + x_k \right\|_{\mathcal{L}}^2, \end{aligned}$$

where $\kappa_w > 0$ is a positive constant that exists since the symmetric part of F is negative definite, and $\bar{\kappa}' > 0$ plays the same role of $\bar{\kappa}$ in the proof of Proposition 4.1. Since $S > 0$, we have that $y_k^{\mathrm{T}} S y_k \geq \kappa_S \|y_k\|^2 + \varepsilon_S \|u_k\|^2 - \varepsilon_S \|u_k\|^2$, where κ_S and ε_S are positive constants. Now, from (19) and (27), there must exist $\delta_1, \delta_2 > 0$ such that $\|u_k\|^2 = \|y_{ck}\|^2 \leq \delta_1 \|w_{k+1} + w_k\|^2 + \delta_2 \|y_k\|^2$, which implies that

$$
\begin{aligned}
V'(x_{k+1}, w_{k+1}) - V'(x_k, w_k) &\leq -\tau\varepsilon_S \|u_k\|^2 \\
&- \tau(\kappa_S - \varepsilon_S\delta_2) \|y_k\|^2 - \tau\bar{\kappa}' \|x_{k+1} + x_k\|_{\mathcal{L}}^2 \\
&- \tau(\kappa_w - \varepsilon_S\delta_1) \|w_{k+1} + w_k\|^2 \\
&+ \tau\varepsilon\kappa_1' m(b) \|x_{k+1}(b) + x_k(b)\|^2 .
\end{aligned}
$$

If ε_S is sufficiently small so that $\kappa_S > \varepsilon_S\delta_2$ and $\kappa_w > \varepsilon_S\delta_1$, and if $\delta = \min\{\varepsilon_S, \kappa_S - \varepsilon_S\delta_2\}$, the result is proved by selecting ε sufficiently small so that $\delta\kappa_b' - \varepsilon\kappa_1' m(b) > 0$, where κ_b' appears in the first relation in (28). In fact, we get $\delta(\|u_k\|^2 + \|y_k\|^2) \geq \delta\kappa_b' \|x_{k+1}(b) + x_k(b)\|^2$, and so $V'(x_{k+1}, w_{k+1}) - V'(x_k, w_k) \leq -\tau\kappa_w \|w_{k+1} + w_k\|^2 - \tau\bar{\kappa}' \|x_{k+1} + x_k\|_{\mathcal{L}}^2$. Convergence towards the origin then follows in the same way as in the proof of Proposition 4.1. ■

Remark 4.1: These results represent an extension of [13], [14] to the discrete-time systems introduced here. As far as the stability conditions in Proposition 4.2, it is possible to match the ones of the continuous-time case of Proposition 2.2. Note that (28) is equivalent to [14, Assumption 2].

V. Example: the Longitudinal Vibration of a Beam

In this section, it is shown how to design the controller (19) to damp the longitudinal vibration in a beam. This is the same system studied e.g. in [24] that admits the boundary control system representation formalised in Proposition 2.1. There, $x = (\varepsilon, p)$ is state variable, being ε and p the beam deformation and linear momentum density, respectively, while

$$
P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad P_0 = \begin{pmatrix} 0 & 0 \\ 0 & -D \end{pmatrix} \quad \mathcal{L} = \begin{pmatrix} Es & 0 \\ 0 & \frac{1}{\rho S_b} \end{pmatrix}.
$$

Besides, S_b is the bar section, supposed constant, E the Young elasticity modulus, ρ the mass density and $D \geq 0$ the internal friction coefficient. Finally, for the spatial coordinate we have that $z \in [0, L]$, being L the length of the beam. The boundary input and output are $u = (v(0), \sigma(L))$ and $y = (-\sigma(0), v(L))$, where $v = \frac{p}{\rho S_b}$ and $\sigma = ES_b\varepsilon$ are the co-energy variables, i.e. the velocity of the cross-section and the axial stress, respectively.

As discussed in [24, Section VI.D], the beam is clamped at $z = 0$ and controlled at the other side, i.e. $u(t) = (0, u'(t))$. Thus, the controller is interconnected at the boundary port $(u'(t), y'(t))$, with $y'(t) = v(t, L)$. The discrete-time approximation (10) is endowed with the input-output pair

$$
\begin{aligned}
u_k' &= \tfrac{1}{2}(\sigma_{k+1}(L) + \sigma_k(L)) = \tfrac{ES_b}{2}(\varepsilon_{k+1}(L) + \varepsilon_k(L)) \\
y_k' &= \tfrac{1}{2}(v_{k+1}(L) + v_k(L)) = \tfrac{1}{2\rho S_b}(p_{k+1}(L) + p_k(L)).
\end{aligned} \tag{30}
$$

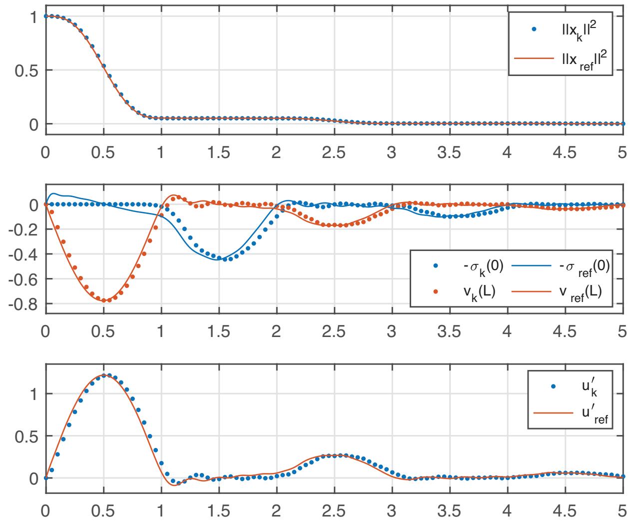

Fig. 1. Response of the discrete-time closed-loop system compared to the continuous-time one.

As far as the control system (19) is concerned, we select $n_c = 1$, and so the regulator is characterised by a PD-like structure. A preliminary step requires to check if the first condition in (28) is met. This property can be easily verified from (30), so Proposition 4.2 provides the class of discrete-time controllers that asymptotically stabilise the system.

The effectiveness of the synthesis methodology has been verified with a numerical example. As far as the bar parameters are concerned, we let $D = 0$, while ρ, S_b and E are so that $\mathcal{L}$ is the identity. On the other hand, the parameters of the controllers (6) and (19) are $F = -10$, $G = 1$, $Q = 0.5$ and $S = 2$. Besides, the sampling interval of the discrete-time regulator is $\tau = 0.05$. The closed-loop behaviour in the design scenario, i.e. when control system and plant are evolving in discrete-time, is reported in Fig. 1. As shown in the top graph, the norm of the state converges to zero, thus the closed-loop system is asymptotically stable. In the other two plots, instead, the evolution of the output and of the control input are reported and compared to the ones in the "ideal" case, i.e. when controller and plant are continuous-time systems. Note that the convergence to zero of the norm of the state is almost identical, while there are differences in the output and in the control action. However, it is possible to verify that such a discrepancy decreases quickly as long as the sampling rate increases.

Even if this problem has not been tackled from a theoretical point of view, in a second simulation the behaviour of the closed-loop system in which the controller is the same as before, and the plant is modelled as a boundary control system is investigated. The interconnection is realised by means of a zero-order-hold. The results are in Fig. 2. It is interesting to note that the behaviour is almost identical to the one in the scenario where also the controller is contioustime. Differently from before, also the output evolution is closer to the one in which both the controller and the plant are continuous-time systems. Even if the interconnection

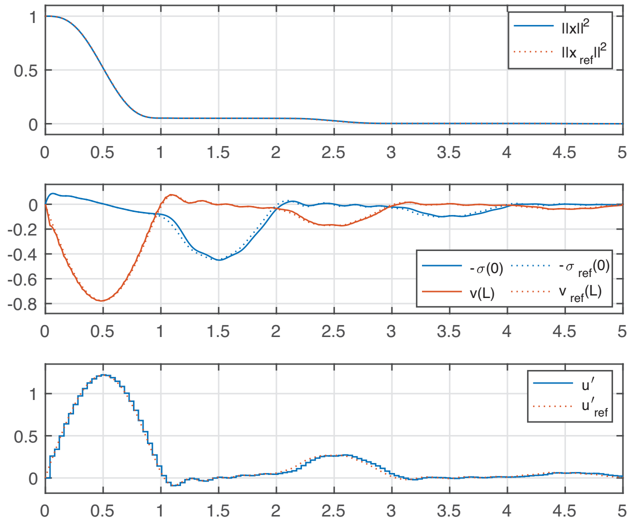

Fig. 2. Response of the closed-loop system in which the discrete-time controller is coupled with the continuous-time plant, and comparison with the scenario in which also the controller is contious-time.

between controller and plant is not power-conserving because of the presence of a zero-order-hold, for the specify choice of τ, namely 0.05, stability is still guaranteed. It is possible to verify, however, that such a result is no longer valid for example if $\tau = 0.1$. To overcome such a limitation, the interconnection should be modified by including additional dissipative effects tuned to preserve the passivity of the interconnection [25], or to get the at least the same energy decrease as in the design scenario (i.e., in discrete-time), [26]. These approaches, however, are applicable only when the plant is a lumped-parameter system.

VI. Conclusions

In this paper, a framework for the design of digital regulators for linear, port-Hamiltonian boundary control systems is presented. To avoid the difficulties related to plant dynamics described by a PDE, a discrete-time approximation is proposed. The discretisation is performed in time only, thus the state maintains its "infinite-dimensional" nature. Such an approximation is not meant for simulating the plant dynamics, since its accuracy has not been quantified yet. However, it is shown that the discrete dynamics is well-posed, i.e. given the current state and the input, the "future" state always exists and is unique. This model is meant for control design, and in the second part of the manuscript, a characterisation of linear, discrete-time controllers capable of asymptotically stabilising the closed-loop system is provided.

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