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Non-overshooting tracking controllers based on combinatorial polynomials

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Abstract— **This paper presents a technique for designing two-parameter compensators that stabilize a plant and provide offset-free tracking of set-points without overshooting or undershooting. We first represent the impulse response of linear systems using combinatorial polynomials, based on which a new set of conditions is derived for the system to be externally positive. This result is then used in control synthesis to achieve monotonic tracking. In contrast to the methods available in the literature, the proposed technique always gives a solution whenever the problem is feasible, can yield as small a settling time as desired, and provides the freedom to choose the closed-loop poles arbitrarily inside the unit circle, all obtained by low-degree controllers.**

I. INTRODUCTION

Offset-free tracking of a reference signal is a central problem to classical control theory. However, the mere stabilization and asymptotic set-point tracking is deemed inadequate in applications that are sensitive to overshoots and oscillations. Such applications are found in diverse areas such as transportation systems [14], robotics [8], and electronics [9]. Their needs have initiated a long quest for controller synthesis techniques that can eliminate both overshoots and undershoots in the closed-loop system response.

A popular way to ensure that no overshoots or undershoots are present in the system is to remove all oscillations by imposing monotonicity on the closed-loop system response. For linear systems, this is equivalent to enforcing that the closed-loop system has a non-negative impulse response. Such systems are known as Externally Positive (EP). There have been many interesting results on the analysis and control of these systems in the literature [2, 4, 5, 6]. Nevertheless, finding conditions that ensure that a system is EP and designing controllers that ensure such property in closed-loop are still two open problems in control theory [14, 2].

The exact conditions under which a given transfer function is EP are only known for system of orders up to three in continuous-time and up to order two in discrete time [10, 15, 16]. For higher-order systems, there are conditions that are only sufficient or only necessary but not both [5, 14, 17]. The conditions characterizing EP systems are ultimately used to synthesize controllers that eliminate overshoots and undershoots from the system response by enforcing external positivity on the closed-loop. The problem of designing such controllers for general linear systems is not completely solved yet [14]. Nevertheless, several design procedures have been proposed in the literature that achieve monotonic tracking using different control structures [4, 6, 13, 16, 17]. Each of these methods targets a different type of systems. For example, modal synthesis was used in [13] to design state feedback controllers which can be applied to non-minimum phase systems. However, the design in [13] is restricted to systems with relative degree one. The control synthesis methods proposed in [15, 16] can be applied to systems with any relative degree. However, the approach in [15] is restricted to minimum-phase systems and the one in [16] is restricted to not have any zeros beyond the vertical line $\{z \in \mathbb{C} | \operatorname{Re}(z) \geq 1\}$. Both the restrictions on relative degrees and the location of zeros are lifted in the control designs offered in [4, 17]. However, both these methods still have some restrictions on where the closed-loop poles can be located: In [4], all the closed-loop poles must be real and non-negative, with some of them forced to be at the same location, while [17] only permits a single closed-loop pole to be real and positive.

In this paper, we develop a new set of conditions that ensure a given system of arbitrary order is externally positive. Our conditions are derived from a combinatorial representation of the impulse response and are used to obtain controllers that achieve offset-free monotonic tracking in closed-loop. The proposed control design is applicable to *all* linear systems (regardless of their relative degree and zero locations) and the closed-loop poles can be placed anywhere (on the negative real axis, on the positive real axis, or as complex-conjugate pairs) inside the unit circle.

A. Notation

$\mathbb{R}_+$ ($\mathbb{R}_{++}$) refers to the set of non-negative (positive) reals, $z^\downarrow \in \mathbb{R}^n$ is the vector $z \in \mathbb{R}^n$ with components sorted in descending order, $|z| \in \mathbb{R}^n_+$ is the vector of the absolute values of the components of $z \in \mathbb{C}^n$. The notation $[A]_{ij}$ refers to the element in row i and column j of the matrix A.

II. PRELIMINARIES

A. Problem statement

Consider the linear discrete-time systems with the transfer functions of the form

$$H(z) = \frac{B(z)}{A(z)} = K\frac{\prod_{i=1}^{m}(z - z_i)}{\prod_{i=1}^{n}(z - p_i)} = \frac{b_0 z^n + b_1 z^{n-1} + \ldots + b_{n-1}z + b_n}{z^n + a_1 z^{n-1} + \ldots + a_{n-1}z + a_n} \quad (1)$$

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where $K \neq 0$ to rule out the degenerate case $H(z) \equiv 0$. Assume that the transfer function (1) has n_+ real non-negative poles and $n_\mathrm{o} = n - n_+$ other poles in the region $\mathbb{C}\backslash\mathbb{R}_+$. We denote by $p^+ \in \mathbb{R}_+^n$ the vector of all the non-negative poles of $H(z)$ with additional zeros added as $p_i^+ = 0$ $(n_+ < i \leq n)$ if $n_+ < n$. Likewise, we let $p^\mathrm{o} \in \mathbb{C}^n$ denote the vector of all the other poles of $H(z)$ with additional zeros appended when $n_\mathrm{o} < n$, *i.e.*, $p_i^\mathrm{o} = 0$ for $i = n_\mathrm{o}+1, n_\mathrm{o}+2, \ldots, n$.

We are interested in controlling the plant (1) such that the closed-loop system is stable and tracks a set-point reference with no overshoot, no undershoot, and no steady-state error.

Definition 1: The system (1) is called Externally Positive (EP) if $h_t = \mathbb{Z}^{-1}\{H(z)\} \geq 0$ for all $t \in \mathbb{N}_0$.

Definition (1) indicates that a linear EP system has a monotonic step response. Therefore, one way to eliminate both overshoots and undershoots in a control system is to design a controller that renders the closed-loop system EP. For this purpose, we first study the family of EP systems.

B. Externally positive systems: the impact of pole locations

The conditions that make a discrete-time transfer function EP can be neatly presented when the order of the transfer function is lower than three [16]. Unfortunately, external positivity conditions rapidly grow complicated for higher-order systems and even for third-order discrete-time systems, there is no set of conditions that exposes all the EP systems known yet.

Whether an arbitrary-order system (1) is EP or not depends on the exact location of its poles and zeros. Intuitively, when the system has large negative or complex-conjugate poles, it may not have a non-negative impulse response because its modes p_i^t corresponding to these poles keep changing signs with time t. The following proposition makes this point accurate, by asserting that a transfer function cannot be EP unless its "dominant pole" is positive.

Proposition 1 ([4]): Let $\max_i |p_i| > 0$. If system (1) is EP, then the pole with the maximum modulus is positive. If there are other poles with the same maximum modulus, the multiplicity of this real positive pole is greater than or equal to the multiplicity of such other poles.

Proposition 1 only provides a necessary condition for (1) to be EP: there are transfer functions whose dominant poles are positive, but who are not EP. In the special case when all the poles are positive, one can guarantee the system is EP under certain assumptions, as shown in the next proposition.

Proposition 2 ([18]): The system (1) is externally positive if $p_i \geq 0$ for all $i = 1, 2, \cdots, n$ and $b_k \geq 0$ for all $k = 0, 1, \cdots, n$.

Now consider a system whose dominant pole is positive, but in addition to its positive pole(s), it also has complex-conjugate and negative poles. Such a system passes the necessity test of Proposition 1 but does not satisfy the requirements of Proposition 2. Therefore, it may or may not be EP. These kinds of systems are in fact very common in practice, as negative poles already appear in second-order systems and complex-conjugate poles already appear in third-order systems and have a critical say in the non-negativeness of the impulse response by creating oscillations. One can expect that these systems are EP, if their positive poles prevail over all their negative and complex-conjugate poles in some sense. This dominance should be stronger than the necessary condition of Proposition 1 but more relaxed than the sufficient condition of Proposition 2. In Section IV we show that this dominance can be elegantly described by a majorization inequality over the set of poles. We use the properties of combinatorial polynomials to derive this result. The next section is therefore devoted to a brief review of the combinatorial polynomials used in this paper.

III. COMBINATORIAL POLYNOMIALS

In this section, we focus on four different families of combinatorial polynomials that are useful in deriving the main results in this paper. The *t-th elementary symmetric polynomial* $e_t : \mathbb{C}^n \mapsto \mathbb{R}$ $(0 \leq t \leq n)$ is the sum of all monomials with total degree t and distinct variables,

$$e_t(x) = \sum_{j \in \mathcal{D}_n^t} x_{j_1} x_{j_2} \ldots x_{j_t} \tag{2}$$

where $\mathcal{D}_n^t = \{j \in \mathbb{N}^t \mid 1 \leq j_1 < j_2 < \ldots < j_t \leq n\}$. These polynomials relate the roots of a polynomial, say $B(z)$ in (1), to its coefficients via Vieta's formula [3, p.192]:

$$e_k(-z) = b_{n-m+k}/b_{n-m}, \quad 0 \leq k \leq m \tag{3}$$

Allowing for repeated variables in the monomials of (2) leads to the *t-th complete homogeneous symmetric polynomial* $\eta_t : \mathbb{C}^n \mapsto \mathbb{R}$, the sum of all monomials of total degree t

$$\eta_t(x) = \sum_{j \in \mathcal{S}_t^n} x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n} \tag{4}$$

where $\mathcal{S}_t^n = \{j \in \mathbb{N}_0^n \mid j_1 + j_2 + \ldots + j_n = t\}$. The first few samples of these polynomials are given by

$$\eta_0(x) = 1$$
$$\eta_1(x) = \textstyle\sum_{i=1}^n x_i$$
$$\eta_2(x) = \textstyle\sum_{1 \leq i_1 \leq i_2 \leq n} x_{i_1} x_{i_2}.$$

By convention, $\eta_k(x) \equiv 0$ for $k < 0$. A closely related family of polynomials are the *power sums* $\Pi_t : \mathbb{C}^n \mapsto \mathbb{R}$ where

$$\Pi_t(x) = x_1^t + x_2^t + \ldots + x_n^t. \tag{5}$$

The last family of combinatorial polynomials that we will use are the t^{th} *complete exponential Bell polynomials*, $B_t(x) : \mathbb{C}^t \mapsto \mathbb{R}$, defined as

$$B_t(x) = \sum_{j \in \mathcal{J}_t} \frac{n!}{j_1! j_2! \ldots j_t!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_t}{t!}\right)^{j_t} \tag{6}$$

where $\mathcal{J}_t = \{j \in \mathbb{N}_0^t \mid j_1 + 2j_2 + \cdots + tj_t = t\}$. The first few samples of these polynomials are

$$B_0(x) = 1$$
$$B_1(x) = x_1$$
$$B_2(x) = x_1^2 + x_2.$$

It follows from (6) that Bell polynomials are non-negative on the non-negative orthant, *i.e.*

$$x_1, x_2, \cdots, x_t \geq 0 \Rightarrow B_t(x) \geq 0. \tag{7}$$

One relationship between the three families of polynomials, complete homogeneous symmetric polynomials (4), power sum polynomials (5) and Bell polynomials (6) is given by the identity [3, p.433]

$$\eta_t(p) = \frac{1}{t!} B_t\left(\Pi_1, 1!\,\Pi_2, \cdots, (t-1)!\,\Pi_t\right). \tag{8}$$

The importance of combinatorial polynomials in linear control theory stems from their ability to provide alternative representations for the inverse $\mathcal{Z}$ transform. For example, consider the following lemma.

Lemma 1 ([18]): The impulse response of (1) is given by

$$h_t = (b * \eta(p))_t = \sum_{k=0}^{n} b_k \eta_{t-k}(p). \tag{9}$$

Unlike the conventional partial fractions expansions, the representation (9) allows for multiple poles as well as zero-pole cancellations in the same compact expression.

IV. EXTERNAL POSITIVITY CONDITIONS

In this section, a sufficient condition is derived which ensures that a transfer function (1) of arbitrary order is EP. This condition is obtained and described using the concept of majorization, introduced next.

Definition 2: We say that $x \in \mathbb{R}^n$ (weakly) majorizes $y \in \mathbb{R}^n$ and write $x \succ_w y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \geq \sum_{i=1}^{k} y_i^{\downarrow}$$

for all $k = 1, 2 \cdots, n$.

The majorization inequality in Definition 2 can be interpreted as the components of x are generally greater than or more spread out than those of y. Majorization has an important consequence described by Karamata's inequality below.

Lemma 2 ([20]): Let $\mathcal{I} \subseteq \mathbb{R}$, $x, y \in \mathcal{I}^n$ and $g : \mathcal{I} \to \mathbb{R}$ be convex and increasing on $\mathcal{I}$. If $x \succ_w y$, then

$$\sum_{i=1}^{n} g(x_i) \geq \sum_{i=1}^{n} g(y_i).$$

We are now ready to present our main results.

Theorem 1: All linear systems on the form (1) with non-negative numerator coefficients and poles satisfying

$$p^+ \succ_w |p^{\circ}| \tag{10}$$

are externally positive.

Proof: Since $b_k \geq 0$ for all $0 \leq k \leq n$, the impulse response (9) is non-negative if $\eta_t(p) \geq 0$ holds for all $t \in \mathbb{N}_0$. Due to (8), this condition is equivalent to

$$B_t\left(\Pi_1, 1!\Pi_2, \cdots, (t-1)!\Pi_t\right) \geq 0, \quad t \in \mathbb{N}_0. \tag{11}$$

By (7), this inequality holds if

$$\Pi_t(p) \geq 0, \quad t \in \mathbb{N}_0. \tag{12}$$

It hence remains to show that (12) holds true to prove that (1) is EP. To this aim, we observe that

$$\begin{aligned}
\Pi_t(p) &= \sum_{i=1}^{n} p_i^t \\
&= \sum_{i=1}^{n_+} \left(p_i^+\right)^t + \sum_{i=1}^{n_\circ} \left(p_i^{\circ}\right)^t \\
&= \sum_{i=1}^{n_+} \left(p_i^+\right)^t + \sum_{i=1}^{n_\circ} |p_i^{\circ}|^t \cos(\mathrm{Arg}(p_i^{\circ})t) \\
&\geq \sum_{i=1}^{n} \left(p_i^+\right)^t - \sum_{i=1}^{n} |p_i^{\circ}|^t.
\end{aligned} \tag{13}$$

Now, from the majorization inequality (10) and the fact that the scalar function $g(x) := x^t$ is both increasing and convex in $\mathcal{I} = \mathbb{R}_+$, Lemma 2 can be used to deduce

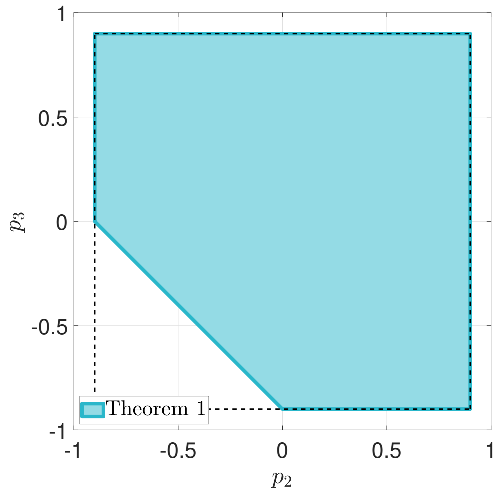
$$\sum_{i=1}^{n} \left(p_i^+\right)^t \geq \sum_{i=1}^{n} |p_i^{\circ}|^t.$$

Therefore the right-hand side of (13) is non-negative and (12) holds true. This proves that the system (1) is EP. ■

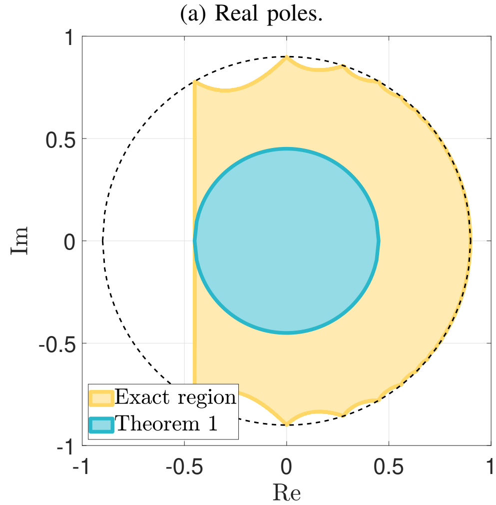
An attractive feature of Theorem 1 is that the external positivity conditions involving the zeros and poles are decoupled from each other. As long as the numerator coefficients are positive, external positivity is guaranteed by a single condition (10) on the system poles. As we will see in Section V, this simplifies the control synthesis considerably. Although the requirement that all numerator coefficients are non-negative limits the use of Theorem 1 for analysis, it does not exclude any systems when it comes to control design. The reason is that a controller can always meet this requirement by multiplying $B(z)$ with a suitable polynomial (see Lemma 3 in Section V).

When the system poles are real, the condition (10) is both necessary and sufficient for systems of up to order four, as shown by Proposition 4 in the appendix. Proposition 5 in appendix demonstrates that the condition (10) in Theorem 1 is optimal in an interesting sense: it provides the largest circle around the origin where the poles can be placed arbitrarily while ensuring all the systems with non-negative numerator coefficients are EP. The following example illustrates these points for a third-order system.

Example 1: Consider a third-order strictly proper system (1) with arbitrary but non-negative numerator coefficients. We fix a positive pole at $p_1 = 0.9$ and let it be dominant, *i.e.*, $p_1 > |p_2|, |p_3|$. This system satisfies the necessary condition offered in Proposition 1 for being EP. The regions in which placing the two other poles p_2, p_3 proves the system to be EP using Theorem 1 are shown in Figure 1. As it can be seen, Theorem 1 exposes all the EP systems in case of real poles and yields the largest disk centered at the origin that contains the complex-conjugate poles that result in an EP system. This result holds regardless of the numerator polynomial, as long as its coefficients are non-negative. For comparison, Proposition 2 only detects the set of poles in the square $\{(p_2, p_3) \,|\, 0 \leq p_2, p_3 < 0.9\}$ in the upper-right corner of Figure 1a as EP.



(a) Real poles.



(b) Complex-conjugate poles.

Fig. 1: The region of poles that results in an EP system in Example 1. The region specified by Theorem 1 is always a subset of the exact region and the two coincide in case of real poles in this example.

V. NON-OVERSHOOTING TRACKING CONTROLLER DESIGN

In this section, we use the results of Section IV to design an output-feedback controller that renders the closed-loop system stable and externally positive with a unit steady-state gain. Such a controller tracks reference set-points monotonically with no overshoot or undershoot. Consider a two-parameter compensator as [19, p.101]

$$U(z) = C_1(z)R(z) - C_2(z)Y(z) \tag{14}$$

where $R(z) = \mathcal{Z}^{-1}\{r_t\}$ and $Y(z) = \mathcal{Z}^{-1}\{y_t\}$ are the reference input and plant output respectively, and

$$C_1(z) = K_c \frac{N(z)}{D(z)}, \quad C_2(z) = \frac{F(z)}{G(z)} \tag{15}$$

are the controller transfer functions. The control signal $U(z)$ obtained from (14) is fed as input to the plant (1). The control law (14) constitutes the most general linear time-invariant output-feedback control structure and results in the closed-loop transfer function [19, §5.6]

$$H^{\text{cl}}(z) = \frac{K_c N(z)G(z)B(z)}{D(z)B(z)F(z) + D(z)A(z)G(z)}. \tag{16}$$

In order to render the closed-loop externally positive, we need to assume that the plant does not have any zeros in $z \in [1, +\infty)$, according to the following proposition.

Proposition 3 ([4]): An output-feedback controller exists that results in a stable externally positive closed-loop system, if and only if the plant (1) does not have any real positive non-minimum-phase zeros, *i.e.* in the range $z \in [1, +\infty)$.

For simplicity, we first assume that $B(z)$ does not have any non-negative zeros and then extend the result to also handle the plant zeros in the range $z \in [0, 1)$. In the remainder of this Section, we will show that it is possible to find controller polynomials $N(z)$, $D(z)$, $F(z)$, $G(z)$ and a static gain K_c for the control law (14) which results in a closed-loop system (16) that is stable, externally positive, and has a unit steady-state gain.

A. Stability

For the closed-loop system (16) to be internally stable, it is required that all the roots of $D(z)$ on or outside the unit circle are contained in the roots of $G(z)$ [19, p.103]. Therefore, we let

$$D(z) = G(z)\hat{D}(z) \tag{17}$$

where $\hat{D}(z)$ is a Schur stable polynomial. By this choice, the closed-loop system is internally stable if and only if

$$\hat{A}(z) = B(z)F(z) + A(z)G(z) \tag{18}$$

has all its roots inside the unit circle. This is always possible to achieve with $\deg(G) = n - 1$ if $B(z)$ and $A(z)$ are relatively prime [7, p.180]. By (17)-(18), the closed-loop system (16) takes the form

$$H^{\text{cl}}(z) = K_c \frac{N(z)B(z)}{A^{\text{cl}}(z)} \tag{19}$$

where

$$A^{\text{cl}}(z) = \hat{D}(z)\hat{A}(z) = \prod_{i=1}^{n_{\text{cl}}}(z - p_i^{\text{cl}}) \tag{20}$$

denotes the closed-loop denominator polynomial.

B. External positivity

We use the results of Theorem 1 to ensure the closed-loop system (16) is EP. As Theorem 1 requires the numerator coefficients to be non-negative, we multiply $B(z)$ by an appropriate polynomial $N(z)$ to make all the coefficients of the closed-loop numerator $N(z)B(z)$ non-negative. This is always possible, as shown in the following lemma.

Lemma 3 ([12]): There is a polynomial $N(z)$ of degree at most $k \leq \bar{k}$ such that $N(z)B(z)$ has non-negative coefficients, where

$$\bar{k} = \sum_{i \in \mathcal{I}_+} \left(\lceil \pi / \operatorname{Arg}(z_i) \rceil - 2 \right) \tag{21}$$

and $\mathcal{I}_+ = \{i \in [1,m] \,|\, \mathrm{Im}(z_i) > 0\}$. If $B(z)$ is a second-order polynomial, then the minimum order of such multiplier is equal to (21).

Such a multiplier polynomial can be expressed as $N(z) = \sum_{i=0}^{k} \nu_i z^{k-i}$, where the coefficients $\nu = [\nu_0, \ldots, \nu_k]^T \in \mathbb{R}^k$ are found from the inequality

$$\mathcal{T}_b \nu \geq 0 \tag{22}$$

where $\mathcal{T}_b \in \mathbb{R}^{(k+n+1)\times(k+1)}$ is a Toeplitz matrix that satisfies $[\mathcal{T}_b]_{ij} = b_{i-j}$ if $i - j \in [0, n]$ and $[\mathcal{T}_b]_{ij} = 0$ otherwise. Inequality (22) can be easily verified using a linear program that is feasible with $k = \bar{k}$ given by (21). In addition, one may use a simple bisection search over k to find the least-order multiplier that satisfies (22). Once $\deg(N) = k$ is found, one needs to ensure that

$$\deg(D) = \deg(G) + \deg(\hat{D}) = n - 1 + \deg(\hat{D}) \geq k$$

to make $C_1(z)$ in (15) proper. Thereby, the order of closed-loop system (19) can be chosen as

$$\begin{aligned} n_{\mathrm{cl}} &= \deg(A^{\mathrm{cl}}) = \deg(\hat{D}) + \deg(\hat{A}) \\ &\geq \max\{k - n + 1, 0\} + 2n - 1 \\ &= \max\{k + n,\, 2n - 1\}. \end{aligned} \tag{23}$$

The n_{cl} closed-loop poles are assigned in the unit circle such that the real non-negative ones (weakly) majorize the rest, *i.e.*, $p^{\mathrm{cl}+} \succ_w \left|p^{\mathrm{clo}}\right|$. This makes the closed-loop system (19) EP according to Theorem 1. There is still a degree of freedom left in how to divide the set of chosen closed-loop poles between $\hat{D}(z)$ and $\hat{A}(z)$ in (20). Among the n_{cl} poles, $2n-1$ are allocated to $\hat{A}(z)$ as

$$\hat{A}(z) = \sum_{i=0}^{2n-1} \hat{a}_i z^{2n-1-i} = \prod_{i=1}^{2n-1} (z - p_i^{\mathrm{cl}}) \tag{24}$$

and the remaining $n_{\mathrm{cl}} - (2n-1)$ are allocated to $\hat{D}(z)$. Once $\hat{A}(z)$ in (18) is determined by (24), the coefficients f and g of the polynomials $F(z)$ and $G(z)$ are determined in a simple pole-placement process from the unique solution of the algebraic equation [16]

$$M \begin{bmatrix} f \\ g \end{bmatrix} = \hat{a} \tag{25}$$

where $\hat{a}$ is given by (24) and $M = \begin{bmatrix} \mathcal{T}_b & \mathcal{T}_a \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$.

C. Unit steady-state gain

The last step to complete the controller design is to adjust the closed-loop steady-state gain to guarantee the reference tracking error $r_t - y_t$ tends to zero asymptotically. This is done by choosing the controller static gain $K_c > 0$ such that $H^{\mathrm{cl}}(1) = 1$, *i.e.*

$$K_c = \frac{A^{\mathrm{cl}}(1)}{N(1)B(1)} \tag{26}$$

D. Control synthesis procedure in summary

In summary, the proposed control synthesis procedure comprises the following steps.

1) Use linear programming (possibly combined with bisection) to find $N(z)$ of order $k \leq \bar{k}$ that satisfies (22).
2) Choose $n_{\mathrm{cl}} \geq \max\{k + n,\, 2n - 1\}$ closed-loop poles inside the unit circle, such that the non-negative ones weakly majorize the other ones.
3) Use $2n - 1$ of the chosen closed-loop poles to form $\hat{A}(z)$ as in (24). Obtain $F(z)$ and $G(z)$ from (25).
4) Use the remaining $n_{\mathrm{cl}} - (2n - 1)$ closed-loop poles to form $\hat{D}(z)$. Obtain $D(z)$ from (17).
5) Determine the controller static gain K_c via (26).

VI. Discussion

To provide additional insight into the proposed control synthesis procedure, this section discusses some key aspects of our approach supported by numerical examples.

A. Plants with positive zeros

The control synthesis procedure proposed in Section V can be easily extended to handle the plant zeros in the range $z \in [0, 1)$. Assume that the first $\hat{m}$ plant zeros are in $z \in [0, 1)$ and the rest are located in $z \in \mathbb{C}\backslash[0, 1)$. The plant's numerator can then be factorized as $B(z) = K \prod_{i=1}^{\hat{m}} (z - z_i)\hat{B}(z)$ where $\hat{B}(z)$ does not have any real non-negative roots. The multiplier $N(z)$ is then designed to make all the coefficients of $N(z)\hat{B}(z)$ non-negative. The first $\hat{m}$ closed-loop poles are chosen as $p_i^{\mathrm{cl}} \in (z_i, 1)$ to make $H_1^{\mathrm{cl}}(z) = K \prod_{i=1}^{\hat{m}} (z - z_i)/(z - p_i^{\mathrm{cl}})$ a series connection of first-order systems that are all EP [16]. The remaining $n_{\mathrm{cl}} - \hat{m}$ poles are chosen as before based on Theorem 1, to make $H_2^{\mathrm{cl}}(z) = N(z)\hat{B}(z)/\prod_{i=\hat{m}+1}^{n_{\mathrm{cl}}} (z - p_i^{\mathrm{cl}})$ EP. Now, since the closed-loop system can be written as the product of the two EP transfer functions $H_1^{\mathrm{cl}}(z)$ and $H_2^{\mathrm{cl}}(z)$ (later multiplied by some $K_c > 0$), it is itself EP. Everything else in the controller synthesis remains the same.

B. Controller order

The order of the two-parameter compensator (14) is

$$\begin{aligned} O &:= \deg(D) + \deg(G) \\ &= \deg(\hat{D}) + 2\deg(G) \\ &= n_{\mathrm{cl}} - \deg(\hat{A}) + 2\deg(G) \\ &= n_{\mathrm{cl}} - (2n - 1) + 2(n - 1) \\ &= n_{\mathrm{cl}} - 1 \\ &\geq \max\{k + n - 1, 2n - 2\}. \end{aligned} \tag{27}$$

The lowest controller order $O = 2n - 2$ is achievable when $n \geq 1$ and all the plant zeros are located on the left half-plane. For these systems, all the coefficients of $B(z)$ are non-negative and hence no multiplier is needed, *i.e.* $N(z) = 1$ and $k = 0$. In contrast, much higher controller orders may be needed for plants that have complex-conjugate zeros with positive real parts, as demonstrated in the following example.

Example 2: Consider a plant with the second-order numerator $B(z)$ with complex-conjugate zeros $z_{1,2} = u \pm iv$

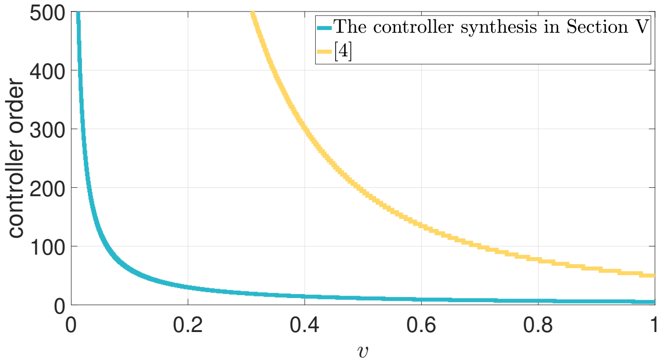


Fig. 2: The order of controllers designed for monotonic tracking of the system described in Example 2 using two different synthesis techniques.

where $u > 0$. If the zeros approach the real axis *i.e.* $\text{Arg}(z_i) \to 0$, one has $\bar{k} \to +\infty$ in (21). Therefore according to Lemma 3, the minimum multiplier order k tends to $+\infty$ and thereby, the controller order O in (27) also grows arbitrarily large when the plant zeros approach the real axis. The controller orders provided in Section V and in [4] are compared for a fixed $u = \text{Re}(z_{1,2}) = 2$ and different values of $v = \text{Im}(z_{1,2})$ in Figure 2. As it can be seen, the synthesis procedure in this paper provides much smaller controller orders than that of [4]. In addition, the control design in Section V allows for a flexible assignment of the closed-loop poles inside the unit circle based on Theorem 1. In contrast, the closed-loop poles in [4] are all located at $p_i^{\text{cl}} \in \{0, \rho\}$, where $\rho = 0.5$ was chosen for this experiment.

It seems natural to require an increasingly high controller order for plants with non-minimum phase complex-conjugate zeros very close to the real axis, since Proposition 3 establishes that monotonic tracking is impossible when the non-minimum phase zeros are located on the real axis.

C. Settling time

In discrete-time linear systems, the spectral radius of poles

$$\rho(H^{\text{cl}}) = \max_i |p_i^{\text{cl}}| \tag{28}$$

is a good measure of the decay rate of the closed-loop system response and roughly determines its settling time [18]. Similarly to [13], the control synthesis proposed in Section V-D is always able to obtain any desired decay rate $\rho(H^{\text{cl}}) = \rho \in [0, 1)$. This may require setting $p_i^{\text{cl}} = z_i$ for $1 \le i \le \hat{m}$ when the plant has positive zeros. Note that such pole-zero cancellations still result in an internally stable controller.

D. Sensitivity

In pole-placement-based controller designs, robustness to model uncertainties and disturbances is usually handled by a suitable assignment of the closed-loop poles [1]. For example, to have a reasonable peak on the sensitivity function

$$S(z) = \frac{B(z)G(z)}{A(z)G(z) + B(z)F(z)} = \frac{B(z)G(z)}{\hat{A}(z)}$$

one typically matches the high-frequency process poles by corresponding closed-loop poles [1]. Such design rules remain fully functional in our controller synthesis approach as well, because of the freedom in choosing the closed-loop pole locations. One just needs to be careful in Step 3 of the procedure to include the poles assigned for treating sensitivity and robustness in $\hat{A}(z)$ rather than in $\hat{D}(z)$.

E. MIMO systems

The proposed design method has a direct extension to MIMO systems where $u_t \in \mathbb{R}^{l_u}$, $r_t \in \mathbb{R}^{l_r}$ and $y_t \in \mathbb{R}^{l_y}$. The difference is that Lemma 3 is replaced by the following lemma to make the numerator coefficients of all channels non-negative using the same multiplier polynomial $N(z)$.

Lemma 4 ([11]): The polynomial $N(z)B(z)$ has non-negative coefficients, where $K > 0$, $N(z) = (z+1)^k$ and

$$k = \left\lceil \frac{\binom{m}{2} \max_i \{|b_{n-i}|/\binom{m}{i}\}}{\min_{z\in[0,1]}\{(1-z)^m B(z/(1-z))\}} \right\rceil - m. \tag{29}$$

Let $H = N_r D_r^{-1}$ and $H = D_l^{-1} N_l$ be any right and left co-prime factorizations of the plant $H(z) \in \mathbb{C}^{l_y \times l_u}$. Choosing the controller in (14) as

$$\begin{bmatrix} C_1 & C_2 \end{bmatrix} = (Y - Q_2 N_l)^{-1} \begin{bmatrix} Q_1 & X + Q_2 D_l \end{bmatrix} \tag{30}$$

makes the closed-loop system $H^{\text{cl}} = N_r Q_1$ internally stable, where Q_1 and Q_2 are arbitrary stable transfer matrices, $|Y - Q_2 N_l| \neq 0$, and X and Y are stable solutions to the Diophantine equation $XN_r + YD_r = I$ [19, §5.6]. With suitable conditions and a high enough order of $N(z)$ in Lemma 4, one can make all channels in the closed-loop system EP by choosing $Q_1 \in \mathbb{C}^{l_u \times l_r}$ *e.g.*, as $[Q_1]_{ij} = N(z)\Gamma(z)/A^{\text{cl}}(z)$ where $\Gamma(z)$ is the product of all the denominators in N_r and $A^{\text{cl}}(z)$ is the desired closed-loop denominator with poles assigned based on Theorem 1.

VII. Design Example

We conclude this section with a numerical example that demonstrates how the control synthesis procedure developed in Section V is applied in a simple case.

Example 3: We would like to control the unstable plant $H(z) = \left(z^2 - z + 1.25\right)/(z - 0.1)(z - 1.5)$ to obtain a stable closed-loop system with monotonic reference tracking and zero steady-state error. For this purpose, we follow the control design procedure in Section V-D. First, the maximum multiplier order (21) is calculated as $\bar{k} = \lceil \pi/\tan^{-1}(2) \rceil - 2 = 1$. Considering $N(z) = \nu_0 z + \nu_1$ of order $k = \bar{k} = 1$, the linear condition (22) given by $\nu_1 \ge \nu_0 \ge (4/5)\nu_1 \ge 0$ is satisfied with $\nu_0 = \nu_1 = 1$. Hence, $N(z) = z + 1$. We choose $n_{\text{cl}} = 3$ closed-loop system poles

$$p_i^{\text{cl}} \in \{0, 0.1, 0.2\} \tag{31}$$

to obtain a small settling time according to (28) and a low sensitivity by matching the high-frequency pole $p_1 = 0.1$ in $H(z)$. With $\deg(\hat{A}) = 2n - 1 = 3$ and we find $F(z) = 0.97z - 0.098$ and $G(z) = 0.025z + 0.81$. Since there are $n_{\text{cl}} - (2n-1) = 0$ poles remaining, $\hat{D} = 1$ and $D(z) = G(z)$. Finally, $K_c = 0.288$ is obtained from (26).

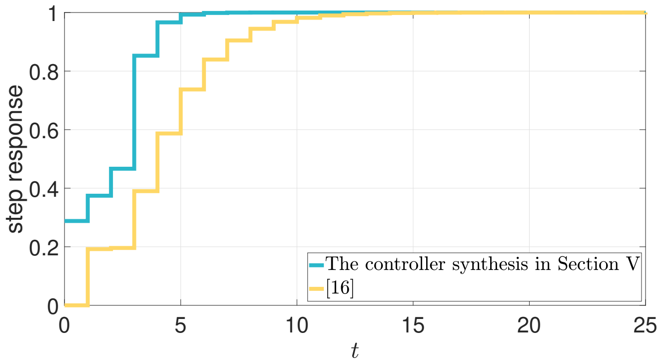


Fig. 3: The closed-loop step response of the system considered in Example 3 using two different controllers.

Figure 3 compares the step response of the designed controller with that of a controller designed via the method in [16]. Since the controller in [16] can only place the poles in the region $p_i^{\text{cl}} \in (0.5, 1)$ our approach obtains a faster step-response. In addition, our procedure allows us to match the high-frequency pole of the open-loop system and obtain a maximum sensitivity of $M_s = 1.9389$, roughly half that of the design from [16] which has $M_s' = 4.1309$.

VIII. Conclusion

We have proposed a syntheis procedure for output-feedback controllers that ensures a stable and externally positive closed-loop system. The framework is based on theoretical developments that combine results from combinatorics, majorization, and the theory of exactifying multipliers. It is applicable to all systems that admit monotonic tracking, and the resulting controllers can follow a set-point reference with no overshoot, no undershoot, and no steady-state error. The design technique can also account for important performance objectives in the time and frequency domain. Numerical examples demonstrate its advantages over the previous state-of-the-art.

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APPENDIX

Lemma 5: If system (1) is externally positive, then

$$\sum_{i=1}^{n} p_i \geq \sum_{i=1}^{m} z_i. \tag{32}$$

Proof: For each time instant $t \in \mathbb{N}_0$, the inequality $h_t \geq 0$ constitutes a necessary condition for $H(z)$ to be EP. For $t = n - m + 1$, Lemma 1 implies that this condition reads

$$\begin{aligned} h_{n-m+1} &= \sum_{k=0}^{n} b_k \eta_{n-m+1-k}(p) \\ &= b_{n-m}\eta_1(p) + b_{n-m+1}\eta_0(p) \\ &= b_{n-m}e_0(-z)\eta_1(p) + b_{n-m}e_1(-z)\eta_0(p) \\ &= K\sum_{i=1}^{n} p_i - K\sum_{i=1}^{m} z_i \geq 0. \end{aligned} \tag{33}$$

where we have used (3) and the fact that $b_k = 0$ for $k < n - m$ in (1). Dividing both sides of (33) by $K > 0$ results in (32). ■

Proposition 4: In (1) let $n \leq 4$ and $p_i \in \mathbb{R}$ for $1 \leq i \leq n$. Then all the systems on the form (1) with non-negative numerator coefficients are externally positive if and only if (10) holds.

Proof: We only prove this result for $n = 4$, as the other cases ($n = 1, 2, 3$) can be proved in a similar way. Since sufficiency is already established by Theorem 1, the proof follows by showing that condition (10) is necessary for $H(z)$ to be EP under the stated assumptions. To this end, we expand the inequality (10) as

$$\sum_{i=1}^{k} p_i^{+\downarrow} \geq \sum_{i=1}^{k} |p_i^{\mathrm{o}\downarrow}|, \quad k = 1, 2, \ldots, n. \tag{34}$$

The proof is divided into 5 different cases, depending on the number of non-negative poles $n_+ \in [0, n]$.

Case $n_+ = 1$: In this case, the inequality (34) is satisfied for all $k = 1, 2, \ldots, n$ if and only if it is satisfied for $k = n$. Therefore, inequality (10) is reduced to $p_1^+ \geq |p_1^{\mathrm{o}}| + |p_2^{\mathrm{o}}| + |p_3^{\mathrm{o}}|$, which is equivalent to

$$\sum_{i=1}^{n} p_i \geq 0 \tag{35}$$

as $p_i^{\mathrm{o}} \leq 0$ for $1 \leq i \leq n$. Now if (35) is violated, the necessary condition (32) is violated for the systems satisfying $b_k \geq 0$ and $b_{n-m+1} = 0$, because $-b_{n-m+1}/b_{n-m} = \sum_{i=1}^{m} z_i = 0$ holds by (3). Hence, such systems would not be EP, even though they satisfy $b_k \geq 0$ for all $0 \leq k \leq n$. This proves that (10) is necessary in this case.

Case $n_+ = 2$: In this case, inequality (34) can be written as

$$\max_i p_i^+ \geq \max_i |p_i^{\mathrm{o}}| \tag{36}$$

for $k = 1$ and as (35) for $k = 2, 3, 4$. Condition (36) is necessary for $H(z)$ to be EP, according to Proposition 1 and condition (35) was shown to be necessary above.

Case $n_+ = 3$: In this case, inequality (34) can be written as (36) for $k = 1$ and as

$$\sum_{i=1}^{k} p_i^{+\downarrow} \geq |p_1^{\mathrm{o}\downarrow}| \tag{37}$$

for $k = 2, 3, 4$. Since $\sum_{i=1}^{k} p_i^{+\downarrow} \geq \max_i p_i^+$, the inequality (37) follows from (36) which is known to be necessary from Proposition 1.

Case $n_+ = 4$: All the systems in this case satisfy (10) and are EP. ■

Proposition 5: In (1) fix the first n_x poles in $\mathbb{R}_{++}$. All the systems (1) with the remaining $n - n_\mathrm{x} \geq n_\mathrm{x}$ poles satisfying

$$|p_i| \leq \frac{p_1 + p_2 + \cdots + p_\mathrm{x}}{n - n_\mathrm{x}} := r, \quad n_\mathrm{x} < i \leq n \tag{38}$$

that have non-negative numerator coefficients are EP. Furthermore, the constant r in (38) is the maximum achievable.

Proof: It is first shown that such systems satisfy (10) and are therefore EP due to Theorem 1. It is enough to show this for the extreme case when $p_i = -r < 0$ for all $n_\mathrm{x} < i \leq n$, which maximizes the right-hand side of (34) and minimizes its left-hand side by not adding more positive zeros in p^+, *i.e.* $n_+ = n_\mathrm{x}$. For all $1 \leq k \leq n_\mathrm{x}$, one can write

$$\begin{aligned} \sum_{i=1}^{n_\mathrm{x}} p_i &\leq \sum_{i=1}^{k} p_i^{+\downarrow} + (n_\mathrm{x} - k) \max_{k+1 \leq i \leq n_\mathrm{x}} \{p_i^{+\downarrow}\} \\ &\leq \sum_{i=1}^{k} p_i^{+\downarrow} + \frac{n_\mathrm{x} - k}{k} \sum_{i=1}^{k} p_i^{+\downarrow} = \frac{n_\mathrm{x}}{k} \sum_{i=1}^{k} p_i^{+\downarrow}. \end{aligned} \tag{39}$$

Using (39) one can bound the the right-hand side of (34) as

$$\sum_{i=1}^{k} |p^{\mathrm{o}\downarrow}| = kr = \sum_{i=1}^{n_\mathrm{x}} \frac{k p_i}{n - n_\mathrm{x}} \leq \sum_{i=1}^{k} \frac{n_\mathrm{x} p_i^{+\downarrow}}{n - n_\mathrm{x}} \leq \sum_{i=1}^{k} p_i^{+\downarrow}$$

which verifies (34). It remains to show (34) for $n_\mathrm{x} < k \leq n$. However since $p_i^{+\downarrow} = 0$ for $n_\mathrm{x} < i \leq n$ and $p_i^{\mathrm{o}\downarrow} = 0$ for $n - n_\mathrm{x} < i \leq n$, it is enough to show that (34) holds for $k = n - n_\mathrm{x}$. This is simply done by noting $\sum_{i=1}^{n-n_\mathrm{x}} |p^{\mathrm{o}\downarrow}| = (n - n_\mathrm{x})r = \sum_{i=1}^{n_\mathrm{x}} p_i = \sum_{i=1}^{n-n_\mathrm{x}} p_i^{+\downarrow}$.

To show that r is the largest possible constant in (38), we use contradiction: If $|p_i| > r$ were allowed, one could choose $p_i < -r$ for $n_\mathrm{x} < i \leq n$ and deduce that

$$\sum_{i=1}^{n} p_i < \sum_{i=1}^{n_\mathrm{x}} p_i - (n - n_\mathrm{x})r = 0. \tag{40}$$

Equality (40) indicates that system (1) would no longer be EP with numerators satisfying $b_k \geq 0$ and $b_{n-m+1} = 0$. Because according to (3), one has $-b_{n-m+1}/b_{n-m} = \sum_{i=1}^{m} z_i = 0$ which violates the necessary condition (32). ■