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Robust, Nonparametric Backstepping Control over Reproducing Kernel Hilbert Spaces

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***Abstract*— This paper derives robust, nonparametric backstepping controllers for nonlinear systems that include functional uncertainties known to lie in a reproducing kernel Hilbert space (RKHS) of vector-valued functions. In contrast to the classical backstepping control architecture, including its robust formulations, the proposed controllers are defined in terms of the operator kernel underlying the RKHS. Hence, the proposed controllers do not require a regressor vector or other finite-dimensional parameterizations to capture uncertainties, and guarantee robustness to substantially larger classes of uncertainties. Furthermore, the proposed approach relieves the user from seeking bases that provide a parametric representation of the functional uncertainty. The qualitative behavior and validation of performance guarantees for the proposed controllers are shown through numerical examples.**

I. Introduction

A. Motivation

Contemporary treatments of both robust adaptive control [1]–[3], and, more generally, robust control for linear and nonlinear systems [4]–[6], focus for a large part on cases where the system uncertainty is characterized by a finite number of real parameters. Recently, several researchers studying robust adaptive control theory have noted the advantages of framing such problems in terms of a nonparametric control theory in which the system uncertainty is described by some unknown function that resides in a suitable function space. Early proponents of such a philosophy include [7]–[9]. More recent efforts along these lines include methods based on Gaussian processes like [10]–[14] or methods based on approximating a limiting distributed parameter system (DPS) in [15]–[19] that is represented by a partial differential equation (PDE), among many others. In all cases, one of the central themes of these nonparametric methods is to frame the adaptive control problem in a way that techniques of the theory of RKHS, and particularly associated methods of approximation and learning theory, can be brought to bear. This paper continues this trend.

In this paper, we study how analysis tools from RKHS theory can be used to develop general methods of *nonparametric robust control theory*. The key novelty of the proposed results lies in that the uncertainty class is lifted from a subset of an Euclidean space to a subset of an RKHS. Hence, the uncertainty class is now nonparametric and considerably broader than state-of-the-art.

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B. Summary of New Results

The key theoretical results of this paper generalize to an RKHS setting two sets of classical results, namely robust control laws for affine-in-the-control dynamical systems and backstepping control laws. Thus, the proposed results lift some iconic results in the control literature, namely Lemmas 2.26 and 2.28 of [5], to a nonparametric setting assuming that the uncertainty classes are subsets of an RKHS.

In the first part of this paper, we address the problem of designing robust control laws for affine-in-the-control dynamical systems. To this goal, we consider uncertainty classes of the form

$$\mathcal{C}_R \triangleq \{f \in \boldsymbol{\mathcal{H}} \mid \|f\|_{\boldsymbol{\mathcal{H}}} \leq R\} \tag{1}$$

where $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}(\mathbb{X}, \mathbb{U})$ is an RKHS of vector-valued functions from the state space $\mathbb{X} \triangleq \mathbb{R}^n$ to the space of control values $\mathbb{U} \triangleq \mathbb{R}^m$, and $R \geq 0$. Thus, in Theorem III.1, we provide a feedback control law for affine-in-the-control plants such that if no matched uncertainty were present, then the closed-loop plant trajectory $x(\cdot)$ is steered to zero, and if the plant is affected by matched uncertainties in $\mathcal{C}_R$, then

$$\limsup_{t\to\infty} \|x(t)\|_{\mathbb{X}} \approx O(\|f\|_{\boldsymbol{\mathcal{H}}}) \approx O(R). \tag{2}$$

As explained in Section III, Theorem III.1 holds for general vector-valued control inputs and uncertainties, while the result in Lemma 2.26 of [5] only holds for $m = 1$ and scalar-valued matched uncertainties. More importantly, Lemma 2.26 of [5] makes the *a priori* assumption that we can write the matched uncertainty as the product of user-defined regressor vector $\Phi(\cdot)$ by an unknown vector Θ, that is, $f(x) = \Phi(x)^{\mathrm{T}}\Theta$ for all $x \in \mathbb{X}$, where $\Phi(x) \triangleq [\phi_1(x), \ldots, \phi_N(x)]^{\mathrm{T}} \in \mathbb{R}^N$ is a *known collection of basis functions*. Furthermore, Lemma 2.26 of [5] guarantees the performance bound

$$\limsup_{t\to\infty} \|x(t)\|_{\mathbb{X}} \approx O(\|\Theta\|_{\mathbb{R}^N}) \approx O(R), \tag{3}$$

Although (2) and (3) closely resemble one another, a substantial difference lies in the fact that (3) yields for functional uncertainties f in the uncertainty class

$$\mathcal{C}_{\Phi_N,R} \triangleq \left\{g \in \boldsymbol{\mathcal{H}} \mid g = \Phi^{\mathrm{T}}\Theta, \|\Theta\|_{\mathbb{R}^N} \leq R\right\}, \tag{4}$$

which is a subset of $\mathcal{C}_R$. Consequently, the proposed results are more advantageous for the following reasons. Firstly, if $\boldsymbol{\mathcal{H}}$ is infinite-dimensional, then $\mathcal{C}_{\Phi_N,R}$ is always a proper subset of $\mathcal{C}_R$. Thus, the proposed control laws guarantee satisfactory performance despite uncertainties that are impossible to counter by means of existing results, such as those in [5] and variations thereof. Secondly, the definition of the uncertainty class $\mathcal{C}_R$ is basis-free and not determined by a finite set of real parameters. For this reason, we say that the proposed control systems are nonparametric. Finally, the proposed feedback control systems are also basis-free. In other words, while (3) only holds for uncertainties having a very specific form in terms of a known basis, (2) yields the same form of performance guarantee *for any function in* $\boldsymbol{\mathcal{H}}$ with $\|f\|_{\boldsymbol{\mathcal{H}}} \leq R$.

In the second part of this paper, the robust control laws for affine-in-the-control dynamical systems are leveraged to construct backstepping control laws for dynamical systems in cascaded forms. Similarly to existing results on robust backstepping control, both parts of these plants in the cascaded form are affected by uncertainties. In contrast to existing approaches, which assume a fixed basis, the functional uncertainties are assumed to lie in possibly infinite-dimensional RKHSs. Specifically, part of the system's dynamics is affected by uncertainties in the class $\mathcal{C}_{R_1}(\boldsymbol{\mathcal{H}}_1)$ and the cascaded part of the system's dynamics is affected by uncertainties in the class $\mathcal{C}_{R_2}(\boldsymbol{\mathcal{H}}_2)$. In these cases, as shown by Theorem IV.1, which generalizes Lemma 2.28 of [5], an ultimate performance guarantee for the backstepping problem is given by

$$\limsup_{t\to\infty} \left(\|x(t)\|_{\mathbb{X}} + \|\xi(t)\|_{\mathbb{U}}\right) \approx O(\|F\|_{\boldsymbol{\mathcal{H}}_1} + \|G\|_{\boldsymbol{\mathcal{H}}_2}) \quad \text{(5a)}$$

$$\approx O(R_1 + R_2) \quad \text{(5b)}$$

for all functional uncertainties $F \in \mathcal{C}_{R_1}(\boldsymbol{\mathcal{H}}_1)$ and $G \in \mathcal{C}_{R_2}(\boldsymbol{\mathcal{H}}_2)$. This guarantee is basis-free, as are the definitions of the feedback laws and the uncertainty classes. In contrast, the corresponding expression in Lemma 2.28 of [5] holds only for fixed choices of bases, the feedback controllers are expressed in terms of these selections of bases, and they hold for typically smaller uncertainty classes defined in terms of the basis selection. Theorem IV.2 provides a special case of Theorem IV.1 under tighter assumptions on the functional uncertainties affecting the plant. Indeed, whereas Theorem IV.1 assumes that the uncertainties in the plant model are functions of time, the state, and the control input, Theorem IV.2 assumes that uncertainties are functions of time and the plant state only.

For brevity, proofs are omitted. Complete proofs and extended discussions on results can be found in Chapter 5 of [20].

II. Notation and Preliminaries

In this paper, we denote by $\mathcal{L}(W_1, W_2)$ the collection of all bounded linear operators between the normed vector spaces W_1, W_2. We use the typeface H for any generic Hilbert space, $\boldsymbol{\mathcal{H}}$ for an RKHS of real, vector-valued functions, and $\mathcal{H}$ for an RKHS of scalar-valued functions. If $g : \mathbb{R}^n \to \mathbb{R}$ is a real-valued function, then we define ∇g as the Jacobian of g with respect to its argument x.

This paper uses analysis tools from the theory of RKHS. Good general accounts of this theory can be found in [21]–[23]. Let $\boldsymbol{\mathcal{H}} \triangleq \boldsymbol{\mathcal{H}}(\Omega, \mathbb{U})$ denote a Hilbert space of functions defined over $\Omega \subseteq \mathbb{X} \triangleq \mathbb{R}^n$ that take values in $\mathbb{U} \triangleq \mathbb{R}^m$. Initially, we let $\Omega \equiv \mathbb{X}$. We subsequently restrict Ω to proper subsets of $\mathbb{X}$ in some cases. By definition, a Hilbert space $\boldsymbol{\mathcal{H}}$ is an RKHS if each evaluation operator $E_x : f \in \boldsymbol{\mathcal{H}} \mapsto f(x) \in \mathbb{U}$ is a bounded linear operator $E_x \in \mathcal{L}(\boldsymbol{\mathcal{H}}, \mathbb{U})$ for each $x \in \mathbb{X}$. The *reproducing property* of the RKHS states that, for each $x \in \mathbb{X}$, there is a bounded linear operator $\mathcal{K}_x : \mathbb{U} \to \boldsymbol{\mathcal{H}}$ such that, for all $u \in \mathbb{U}$ and $f \in \boldsymbol{\mathcal{H}}$,

$$(\mathcal{K}_x u, f)_{\boldsymbol{\mathcal{H}}} = (u, f(x))_{\mathbb{U}} = (u, E_x f)_{\mathbb{U}}, \quad (6)$$

where $(\cdot,\cdot)_S$ denotes an inner product over some inner product space S. This identity implies that the adjoint $\mathcal{K}_x^* \triangleq (\mathcal{K}_x)^*$ is given by $\mathcal{K}_x^* = E_x$. The reproducing kernel $\mathcal{K}(x, y) \in \mathcal{L}(\mathbb{U})$ is defined to be

$$\mathcal{K}(x, y) = E_x \mathcal{K}_y = E_x E_y^* = \mathcal{K}_x \mathcal{K}_y, \quad (7)$$

where $\mathcal{K}_y(\cdot) \triangleq \mathcal{K}(\cdot, y)$.

There are many ways to construct such operator kernels $\mathcal{K}(x, y) \in \mathcal{L}(\mathbb{U})$. Diagonal kernels are among the simplest ones and take the form

$$\mathcal{K}(x, y) \triangleq \operatorname{diag}\left(\mathfrak{K}_1(x, y), \ldots, \mathfrak{K}_m(x, y)\right), \quad (8)$$

where each kernel $\mathfrak{K}_i : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ is scalar-valued and defines the RKHS (also known as native space) $\mathcal{H}_i$ of scalar-valued functions over $\mathbb{X}$. Such an operator-valued kernel defines the native space $\boldsymbol{\mathcal{H}} \triangleq \mathcal{H}_1 \times \ldots \times \mathcal{H}_m$ that is the Cartesian product of the native spaces $\mathcal{H}_1, \ldots, \mathcal{H}_m$. It is natural to build such spaces in terms of well-known native spaces of scalar-valued kernels, including exponential, Sobolev-Matern, Wendland, and inverse multiquadric, to name a few, as described in [24]. However, we emphasize that this paper applies to non-diagonal operator kernels as well. See the operator kernel studied recently in [19] for an example of a generic nondiagonal, operator-valued kernel.

The results presented in this paper use types of multiplication operators between RKHSs of vector-valued functions. Suppose that both the RKHS $\boldsymbol{\mathcal{H}}_1$ and $\boldsymbol{\mathcal{H}}_2$ consist of functions that map from $\mathbb{X}$ into $\mathbb{U} \triangleq \mathbb{R}^m$, and $M(x) \in \mathbb{R}^{m\times m}$ is an $m \times m$ matrix function. We say that the operator $\mathcal{M}$ given by

$$(\mathcal{M}f)(x) = M(x)f(x) \quad \text{for all } x \in \mathbb{X}.$$

is a *multiplication operator* from $\boldsymbol{\mathcal{H}}_1$ to $\boldsymbol{\mathcal{H}}_2$ if it is a bounded linear operator, that is, $\mathcal{M} \in \mathcal{L}(\boldsymbol{\mathcal{H}}_1, \boldsymbol{\mathcal{H}}_2)$. A typical bound on the operator norm $\|\mathcal{M}\|$ is denoted by $\bar{\mathcal{M}}_\infty \in [0, \infty)$, for which we have $\|\mathcal{M}\| \leq \bar{\mathcal{M}}_\infty < \infty$.

III. Robust Control Lyapunov Functions in Reproducing Kernel Hilbert Spaces

Consider the *plant model*

$$\dot{x}(t) = a(x(t)) + b(x(t))\left(u(t) + E_{x(t)} f\right), \quad x(0) = x_0, \quad (9)$$

for all $t \geq 0$, where $x(t) \in \mathbb{R}^n \triangleq \mathbb{X}$ denotes the *state vector*, $u(t) \in \mathbb{R}^m \triangleq \mathbb{U}$ denotes the *control input*, $a : \mathbb{X} \to \mathbb{X}$ denotes the *drift function*, $b : \mathbb{X} \to \mathbb{R}^{n\times m}$ denotes the *control influence operator*, and $f \in \boldsymbol{\mathcal{H}} \triangleq \boldsymbol{\mathcal{H}}(\mathbb{X}, \mathbb{U})$ denotes the functional uncertainty. The RKHS $\boldsymbol{\mathcal{H}}$ is induced by the operator-valued kernel $\mathcal{K}(x,y) \in \mathcal{L}(\mathbb{U})$ for all $x, y \in \mathbb{X}$, and it contains functions that map $\mathbb{X}$ into $\mathbb{U}$. In this section, we seek a feedback control law $\mu : \mathbb{X} \to \mathbb{U}$ such that $u(t) = \mu(x(t))$, $t \geq 0$, is bounded and drives the state $x(t) \to 0$ as $t \to \infty$, or, at least, assures uniform ultimate boundedness of the controlled trajectory.

To construct regulators for (9), we recall the notion of *control Lyapunov function* (CLF).

Definition III.1 (Control Lyapunov Functions)**.** *The continuously differentiable function $V : \mathbb{X} \to \mathbb{R}$ is a* control Lyapunov function *for* (9) *with $f \equiv 0$ if there exist class $\mathcal{K}_\infty$ functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ such that*

$$\gamma_1(\|x\|_{\mathbb{X}}) \leq V(x) \leq \gamma_2(\|x\|_{\mathbb{X}}) \quad \textit{for all } x \in \mathbb{X}, \tag{10}$$

and there exists a class $\mathcal{K}_\infty$ function $\gamma_3(\cdot)$ such that

$$\frac{\partial V(x)}{\partial x}\left[a(x) + b(x)\alpha(x)\right] \leq -\gamma_3(\|x\|_{\mathbb{X}}) \tag{11}$$

for all $x \in \mathbb{X}$, where $\alpha : \mathbb{X} \to \mathbb{U}$ and $\|\cdot\|_{\mathbb{X}}$ denotes a norm on $\mathbb{X}$.

Definition III.1 requires that $f \equiv 0$ in (9). It follows from Definition III.1 that if a CLF exists, then $x(t) \equiv 0$ is an asymptotically stable equilibrium point of (9) with $f \equiv 0$ and $u(t) = \alpha(x(t))$, $t \geq t_0$. In the presence of a functional uncertainty f, one must resort to the notion of *robust CLF* (RCLF). For the statement of this definition, note that the nonlinearities in (9) have the form $E_{x(t)} f \equiv f(x(t))$, $t \geq 0$. Furthermore, let $\mathring{\mathcal{D}}$ denote the interior of a set $\mathcal{D} \subset \mathbb{X}$.

Definition III.2 (Classical Robust Control Lyapunov Functions)**.** *The continuously differentiable function $V : \mathbb{X} \to \mathbb{R}$ is a* robust control Lyapunov function *for* (9) *with $E_{x(t)} f \equiv f(x(t))$, $t \geq 0$, if* (10) *is verified and there exists a class $\mathcal{K}_\infty$ function $\gamma_3(\cdot)$ such that*

$$\frac{\partial V(x)}{\partial x}\left[a(x) + b(x)\left(\alpha(x) + f(x)\right)\right] \leq -\gamma_3(\|x\|) \tag{12}$$

for all $x \in \mathbb{X} \setminus \mathcal{D}$, where $\mathcal{D}$ is closed, bounded, and such that $0 \in \mathring{\mathcal{D}}$, and $\alpha : \mathbb{X} \to \mathbb{U}$.

It follows from Definition III.2 that if a RCLF exists, then (9) with $E_{x(t)} f \equiv f(x(t))$, $t \geq 0$, is uniformly ultimately bounded. There are many examples of Lyapunov functions, CLFs, and RCLFs, such as (9) with $f \equiv 0$ or (9) with $E_{x(t)} f \equiv f(x(t))$, $t \geq 0$. Such examples can be found in classical treatises on adaptive control including, such as [5], [6], [25], [26] to name a few. In many instances, as described in [5], RCLF is constructed based on the assumption that f lies in a known finite-dimensional space. This section addresses for the first time the problem of constructing an RCLF $V(\cdot)$ in closed form for a system as general as (9), whose uncertainty f is not restricted to a fixed finite-dimensional subspace whose basis is known.

The next theorem captures the first original contribution of this paper. This result is a generalization to vector-valued uncertainty in a native space of a well-known approach to robust control exemplified by Lemma 2.26 in [5]. It is used subsequently in the same text for the development of robust backstepping methods. Lemma 2.26 treats the case when $m = 1$ and the scalar functional uncertainty is defined in terms of a fixed, finite number of regressors.

Theorem III.1. *Consider the nonlinear dynamical model in* (9)*. Let the RKHS $\boldsymbol{\mathcal{H}}(\mathbb{X}, \mathbb{U})$ be induced by the matrix-valued kernel $\mathcal{K} : \mathbb{X} \times \mathbb{X} \to \mathcal{L}(\mathbb{U})$. Assume that $f \in \boldsymbol{\mathcal{H}}(\mathbb{X}, \mathbb{U})$ and let $V(\cdot)$ denote a CLF for* (9) *with $f \equiv 0$, and let $\alpha : \mathbb{X} \to \mathbb{U}$ be such that* (11) *is verified. If $u(t) = \mu(x(t))$, $t \geq t_0$, where*

$$\mu(x) \triangleq \alpha(x) - \beta\mathcal{K}(x,x)b^{\mathrm{T}}(x)\left[\frac{\partial V(x)}{\partial x}\right]^{\mathrm{T}} \quad \textit{for all } x \in \mathbb{X}, \tag{13}$$

where $\beta > 0$, guarantees global uniform boundedness of (9) *with $u(t) = \mu(x(t))$, $t \geq t_0$, and*

$$\limsup_{t \to \infty} \|x(t)\| \leq \gamma_1^{-1} \circ \gamma_2 \circ \gamma_3^{-1}\left(\frac{\|f\|_{\boldsymbol{\mathcal{H}}}}{4\beta}\right), \tag{14}$$

where the class $\mathcal{K}_\infty$ functions $\gamma_1(\cdot)$, $\gamma_2(\cdot)$, and $\gamma_3(\cdot)$ verify (10) *and* (11).

Theorem III.1 assumes the existence of a CLF for (9) with $f \equiv 0$ and the associated asymptotically stabilizing feedback control law. As discussed in [6], the assumption on the existence of a CLF is a standard assumption to deduce RCLFs. Once a CLF is identified, the feedback-stabilizing control law can be produced, for instance, by applying Theorem 6.7 in [27].

Next, we review Lemma 2.26 [5] and make a few observations to better understand the implications of casting this problem in terms of native space embedding.

Example III.1 (Comparison to [5, Lemma 2.26])**.** *The approach in [5] assumes that the functional uncertainty is scalar with $m = 1$ and that the functional uncertainty is expressed in terms of a fixed, finite collection of known basis functions, so that*

$$f(x) \triangleq \Theta_N^{\mathrm{T}}\Phi_N(x) = \sum_{k=1}^{N} \theta_k\phi_k(x) \quad \textit{for all } x \in \mathbb{X}, \tag{15}$$

where $\Theta_N \triangleq [\theta_1, \ldots, \theta_N]^{\mathrm{T}}$ and $\Phi_N(x) \triangleq [\phi_1(x), \ldots, \phi_N(x)]^{\mathrm{T}}$. This means that the uncertainty is parameterized by a fixed, finite number $N > 0$ of real parameters Θ_N, and the uncertain function f is contained in $H_N \triangleq \operatorname{span}\{\phi_1(x), \ldots, \phi_N(x)\} = \operatorname{span}(\Phi_N(x))$.

With the parameterization of uncertainty in (15)*, the approach in [5] proposes the feedback control law*

$$\mu(x) \triangleq \alpha(x) - \beta\Phi_N^{\mathrm{T}}(x)\Phi_N(x)b^{\mathrm{T}}(x)\left[\frac{\partial V(x)}{\partial x}\right]^{\mathrm{T}}, \tag{16}$$

for all $x \in \mathbb{X}$, which is qualitatively similar to (13)*. Note that the inner product $\Phi_N^{\mathrm{T}}(x)\Phi_N(x)$ in* (16) *is replaced with the operator kernel $\mathcal{K}(x,x) \in \mathcal{L}(\mathbb{U})$ in* (13), which does not depend on the basis nor on the dimension N.

Lemma 2.26 of [5] shows that, employing (16) *with* $m = 1$,

$$\limsup_{t\to\infty} \|x(t)\| \leq \gamma_1^{-1} \circ \gamma_2 \circ \gamma_3^{-1} \left(\frac{\|\Theta_N\|_\infty}{4\beta} \right). \tag{17}$$

If we assume that $\|\Theta_N\|_\infty \leq R$*, where* $R \geq 0$ *is known, then* (17) *reduces to*

$$\limsup_{t\to\infty} \|x(t)\| \leq \gamma_1^{-1} \circ \gamma_2 \circ \gamma_3^{-1} \left(\frac{R}{4\beta} \right) \tag{18}$$

for all affine-in-the-control uncertain systems with $m = 1$ *and functional uncertainty* f *residing in the parametric uncertainty class*

$$\mathcal{C}_{\Phi_N,R} \triangleq \{f = \Theta_N^{\mathrm{T}} \Phi_N(\cdot) \mid \Theta_N \in \mathbb{R}^N, \|\Theta_N\|_\infty \leq R\}. \tag{19}$$

We emphasize that the uncertainty class (19) *depends on the dimension* N *and on the choice of basis in* $\Phi_N(\cdot)$*. Theorem III.1, in contrast, allows for a broader uncertainty class since it holds for all systems whose uncertainty lies in the uncertainty class*

$$\mathcal{C}_R \triangleq \left\{ h : \mathbb{R}^n \to \mathbb{R}^m \;\middle|\; \|h\|_{\boldsymbol{\mathcal{H}}} \leq R \right\} \subset \boldsymbol{\mathcal{H}}. \tag{20}$$

The uncertainty class $\mathcal{C}_{\Phi_N,R}$ *is smaller than the uncertainty class* $\mathcal{C}_R$ *since* $\mathcal{C}_{\Phi_N,R}$ *is tied to the dimension* N *and the choice of basis* Φ_N*. To be more specific, suppose, without loss of generality, that the basis of* N *functions in* Φ_N *is contained in a finite-dimensional scalar-valued native space* $\mathcal{H}$ *such that* $\|\phi_i\|_{\mathcal{H}} = 1$ *for* $1 \leq i \leq N$*. Then, it holds that*

$$C_{R',\Phi_N} \subset C_R \tag{21}$$

for a constant $R' \triangleq R/\sqrt{\|\mathbb{K}_N\|N}$*, where*

$$\mathbb{K}_N \triangleq \left[\langle \phi_i, \phi_j \rangle_{\mathcal{H}} \right] \in \mathbb{R}^{N\times N} \tag{22}$$

denotes the Grammian matrix*. The relationship* (21) *follows from the fact that* $f = \Phi_N^{\mathrm{T}} \Theta_N$ *for any* $f \in \mathcal{C}_{\Phi_N,R'}$ *and some* Θ_N *such that* $\|\Theta_N\|_\infty \leq R'$*. Thus,*

$$\begin{aligned} \|f\|_{\mathcal{H}}^2 &= \Theta_N^{\mathrm{T}} \mathbb{K}_N \Theta_N \leq \|\Theta_N\|_2^2 \|\mathbb{K}_N\|_2 \\ &\leq \|\Theta_N\|_\infty^2 \|\mathbb{K}_N\|_2 N \leq N(R')^2 \|\mathbb{K}_N\|_2 \leq R^2, \end{aligned}$$

which implies that $f \in \mathcal{C}_R$*. If* $\mathcal{H}$ *is infinite-dimensional, then no such converse result holds:* $\mathcal{C}_R \not\subset \mathcal{C}_{\Phi_N,R'}$ *for any* $N, R, R' > 0$*.* △

In light of Theorem III.1, we extend for the first time Definition III.2 to plant models affected by infinite-dimensional matched uncertainties.

Definition III.3 (Robust Control Lyapunov Functions)**.** *The continuously differentiable function* $V : \mathbb{X} \to \mathbb{R}$ *is a* robust control Lyapunov function *for* (9) *if* (10) *is verified and there exists a class* $\mathcal{K}_\infty$ *function* $\gamma_3(\cdot)$ *such that*

$$\frac{\partial V(x)}{\partial x} \left[a(x) + b(x) \left(\alpha(x) + f(x) \right) \right] \leq -\gamma_3(\|x\|) \tag{23}$$

for all $x \in \mathbb{X} \setminus \mathcal{D}$*, where* $\mathcal{D}$ *is closed, bounded, and such that* $0 \in \mathring{\mathcal{D}}$*, and* $\alpha : \mathbb{X} \to \mathbb{U}$*.*

IV. Robust Nonparametric Backstepping

In this section, we show how the results in Section III can be employed to extend classical robust parametric backstepping methods for nonlinear systems, whose uncertainties lie in RKHSs. Consider plant models in the form

$$\begin{aligned} \dot{x}(t) = a(x(t)) + b(x(t)) \left(\xi(t) + E_{1,x(t)} F(\cdot, \xi(t), u(t), t) \right), \\ x(t_0) = x_0, \qquad t \geq t_0, \end{aligned} \tag{24a}$$

$$\dot{\xi}(t) = u(t) + E_{2,x(t)} G(\cdot, \xi(t), u(t), t), \quad \xi(t_0) = \xi_0, \tag{24b}$$

where $x(t) \in \mathbb{R}^n \triangleq \mathbb{X}$, $u(t) \in \mathbb{R}^m \triangleq \mathbb{U}$, and $F : \mathbb{X} \times \mathbb{U} \times \mathbb{U} \times [t_0, \infty) \to \mathbb{U}$ and $G : \mathbb{X} \times \mathbb{U} \times \mathbb{U} \times [t_0, \infty) \to \mathbb{U}$ denote functional undertainties. The plant model (24a) and (24b) modifies (9) by introducing a cascaded architecture and generalizes the structure of the system studied in (2.292) of [5, p. 82] by considering uncertainties $F(\cdot, \xi, u, t)$ and $G(\cdot, \xi, u, t)$ in infinite-dimensional native spaces.

The functional uncertainties $F(\cdot, \xi, u, t)$ and $G(\cdot, \xi, u, t)$ in (24a) and (24b) are assumed to lie in different RKHSs, $\boldsymbol{\mathcal{H}}_1$ and $\boldsymbol{\mathcal{H}}_2$, respectively, for fixed ξ, u, and t. Thus, two different evaluation operators, namely $E_{1,x}$ and $E_{2,x}$ are employed. This way, for each $x \in \mathbb{X}$ and for each $f \in \boldsymbol{\mathcal{H}}_1$, we let $E_{1,x} f = f(x) \in \mathbb{U}$, and $E_{2,x} h = h(x) \in \mathbb{U}$ for each $h \in \boldsymbol{\mathcal{H}}_2$. Two alternative assumptions are considered for the definition of functional uncertainties.

Assumption 1. *It holds that* $F(\cdot, \xi, u, t) \in \boldsymbol{\mathcal{H}}_1$ *for each* $(\xi, u, t) \in \mathbb{U} \times \mathbb{U} \times [t_0, \infty)$*, where* $\boldsymbol{\mathcal{H}}_1 \triangleq \boldsymbol{\mathcal{H}}_1(\mathbb{X}, \mathbb{U})$ *is an RKHS of vector-valued functions induced by the operator kernel* $\mathcal{K}_1 : \mathbb{X} \times \mathbb{X} \to \mathcal{L}(\mathbb{U})$*. Similarly, we assume that* $G(\cdot, \xi, u, t) \in \boldsymbol{\mathcal{H}}_2 \triangleq \boldsymbol{\mathcal{H}}_2(\mathbb{X}, \mathbb{U})$ *for each* $(\xi, u, t) \in \mathbb{U} \times \mathbb{U} \times [t_0, \infty)$*, where* $\boldsymbol{\mathcal{H}}_2 \triangleq \boldsymbol{\mathcal{H}}_2(\mathbb{X}, \mathbb{U})$ *is an RKHS of vector-valued functions induced by the operator kernel* $\mathcal{K}_2 : \mathbb{X} \times \mathbb{X} \to \mathcal{L}(\mathbb{U})$*. We assume that the uncertainties are bounded in the sense that there exist* $\bar{F}_\infty, \bar{G}_\infty \in (0, \infty)$ *such that*

$$\|F(\cdot, \xi, u, t)\|_{\boldsymbol{\mathcal{H}}_1} \leq \sup_{(\xi,u,t)\in\mathbb{U}\times\mathbb{U}\times[t_0,\infty)} \|F(\cdot, \xi, u, t)\|_{\boldsymbol{\mathcal{H}}_1} \leq \bar{F}_\infty, \tag{25}$$

$$\|G(\cdot, \xi, u, t)\|_{\boldsymbol{\mathcal{H}}_2} \leq \sup_{(\xi,u,t)\in\mathbb{U}\times\mathbb{U}\times[t_0,\infty)} \|G(\cdot, \xi, u, t)\|_{\boldsymbol{\mathcal{H}}_2} \leq \bar{G}_\infty. \tag{26}$$

Assumption 2. *We assume that, in* (24a)*,* $F(x, \xi, u, t) \triangleq f(x, t)$ *where* $f(\cdot, t) \in \boldsymbol{\mathcal{H}}_1$ *for all* $t \in [t_0, \infty)$*, and in* (24b)*,* $G(x, \cdot, u, t) = g(\cdot, t) \in \boldsymbol{\mathcal{H}}_2(\mathbb{X}, \mathbb{U})$*. Finally, we assume that there exist* $\bar{f}_\infty, \bar{g}_\infty \in (0, \infty)$ *such that*

$$\|f(\cdot, t)\|_{\boldsymbol{\mathcal{H}}_2} \leq \|f\|_{L^\infty([t_0,\infty),\boldsymbol{\mathcal{H}}_1)} \leq \bar{f}_\infty, \tag{27}$$

$$\|g(\cdot, t)\|_{\boldsymbol{\mathcal{H}}_2} \leq \|g\|_{L^\infty([t_0,\infty),\boldsymbol{\mathcal{H}}_2)} \leq \bar{g}_\infty. \tag{28}$$

The systematic development of a nonlinear controller for the system given by (24a) and (24b) begins, as in the last section, by assuming that we have an RCLF for the subsystem in (24a). Thus, we assume that there exist $V : \mathbb{X} \to \mathbb{R}$ positive-definite and $\mu : \mathbb{X} \to \mathbb{U}$ such that

$$\begin{aligned} \frac{\partial V(x)}{\partial x} \Bigg(a(x) + b(x) \Big(\mu(x) + E_{1,x} F(\cdot, \xi, u, t) \Big) \Bigg) \\ \leq -W(x) + \lambda \end{aligned} \tag{29}$$

for all $x \in \mathbb{X}$, $u \in \mathbb{U}$, $\xi \in \mathbb{U}$, and $t \in [t_0, \infty)$, for some $\lambda \in [t_0, \infty)$, and for some function $W(\cdot)$ that is positive definite and radially unbounded.

The next result provides sufficient conditions for a feedback control law to assure uniform ultimate boundedness of (24a) and (24b) under Assumption 1.

Theorem IV.1 (Robust Nonparametric Backstepping)**.** *Consider the dynamical model given by* (24a) *and* (24b)*, and suppose that Assumption 1 is verified. Suppose that there exist $V : \mathbb{X} \to \mathbb{R}$ positive-definite and $\mu : \mathbb{X} \to \mathbb{U}$ such that* (29) *is verified. Assume that the matrix function $\frac{\partial \mu^{\mathrm{T}}(x)}{\partial x} b(x) \in \mathcal{L}(\mathbb{U} \times \mathbb{U})$ defines the bounded linear multiplication operator $\mathcal{M} : \boldsymbol{\mathcal{H}}_1 \to \boldsymbol{\mathcal{H}}_2$ with operator norm $\|\mathcal{M}\| \leq \bar{\mathcal{M}}_\infty$. Then,* (24a) *and* (24b) *with control input*

$$\begin{aligned} u(t) = &-cz(t) + \frac{\partial \mu(x(t))}{\partial x} \left(a(x) + b(x)\xi(t)\right) \\ &- b^{\mathrm{T}}(x(t)) \frac{\partial V(x(t))}{\partial x}^{\mathrm{T}} - \beta \mathcal{K}_2(x(t), x(t)) z(t), \quad t \geq t_0, \end{aligned} \tag{30}$$

where $z(t) \triangleq \xi(t) - \mu(x(t))$, is uniformly ultimately bounded.

The next result provides sufficient conditions for a feedback control law to assure uniform ultimate boundedness of (24a) and (24b) under Assumption 2. In this case, (24a) and (24b) reduce to

$$\dot{x}(t) = a(x(t)) + b(x(t)) \left(\xi(t) + E_{1,x(t)} F(\cdot, t)\right), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{31}$$

$$\dot{\xi}(t) = u(t) + E_{2,\xi(t)} G(\cdot, t), \quad \xi(t_0) = \xi_0, \tag{32}$$

respectively. Note that, despite (24a), (31) allows only for uncertainties in the state x in the matched uncertainty $f(x,t)$, with $f(\cdot, t) \in \boldsymbol{\mathcal{H}}_1$. Similarly, despite (24b), (32) admits only uncertainties in the state ξ in the matched uncertainty $g(\eta, t)$, with $g(\cdot, t) \in \boldsymbol{\mathcal{H}}_2$.

Theorem IV.2. *Consider the dynamical model given by* (31) *and* (32)*, and suppose that Assumption 2 is verified. Suppose that there exist $V : \mathbb{X} \to \mathbb{R}$ positive-definite and $\mu : \mathbb{X} \to \mathbb{U}$ such that* (29) *is verified. Assume that the matrix function $\frac{\partial \mu(x)}{\partial x} b(x) \in \mathcal{L}(\mathbb{U} \times \mathbb{U})$ defines the bounded linear multiplication operator $\mathcal{M} : \boldsymbol{\mathcal{H}}_1 \to \boldsymbol{\mathcal{H}}_2$ with operator norm $\|\mathcal{M}\| \leq \bar{\mathcal{M}}_\infty$. Then,* (31) *and* (32) *with control input*

$$\begin{aligned} u(t) = &-cz(t) + \frac{\partial \mu(x)}{\partial x} \left(a(x) + b(x)\xi(t)\right) \\ &- b^{\mathrm{T}}(x) \frac{\partial V(x)}{\partial x}^{\mathrm{T}} - \beta \mathcal{K}_2(x(t), x(t)) z(t), \quad t \geq t_0, \end{aligned} \tag{33}$$

where $z(t) \triangleq \xi(t) - \mu(x(t))$, is uniformly ultimately bounded.

V. Numerical Example

Consider the dynamical model

$$\dot{x}(t) = \xi(t) + f(x(t))\delta(t), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{34}$$

$$\dot{\xi}(t) = u(t), \quad \xi(t_0) = \xi_0. \tag{35}$$

The goal is to regulate the trajectories of (34) and (35).

The dynamical model (34) and (35) is in the same form as (24a) and (24b) with $n = 1$, $m = 1$, $\mathbb{X} = \mathbb{R}$, $\mathbb{U} = \mathbb{R}$, $a(x) = 0$, $b(x) = 1$, $F(x, \xi, u, t) = f(x)\delta(t)$, $f \in \mathcal{H}$, where $\mathcal{H}$ is an RKHS of scalar-valued functions over Ω that is induced by the scalar-valued kernel $\mathfrak{K}(x,y)$ for $x, y \in \mathbb{R}$, and $G(x, \xi, u, t) = 0$. We fix the bounds $\|f\|_{\mathcal{H}} \leq \bar{f}_\infty \leq R$ and $|\delta(t)| \leq \bar{\delta}_\infty \leq 1$. We finally choose $V(x) \triangleq \frac{1}{2}x^2$, $x \in \mathbb{X}$, and $\alpha(x) = -c_1 x$, where $c_1 > 0$, so that $\dot{V}(x) = -c_1|x|^2 = -W(x)$ and $W(x) \triangleq c_1 x^2$. It follows from Theorem III.1 that the feedback control law (13) guarantees uniform ultimate boundedness of the closed-loop system. It is easy to show that the ultimate bound on the trajectories of (34) and (35) with control law (13) are such that

$$c_1|x|^2 + c|z|^2 \leq \frac{(1 + \bar{\mathcal{M}}_\infty^2)\bar{f}_\infty^2}{4\beta} \tag{36}$$

for all $t \geq t_0$ large enough.

Since the functional uncertainty is unknown, the integer $N \in \mathbb{N}$ and a collection of centers $\Xi_N \subset \Omega$ are selected randomly. With this random choice of N and Ξ_N, we define the subspace

$$\mathcal{H}_N = \mathrm{span}\{\mathfrak{K}_{\xi_i}(\cdot) \mid \xi_i \in \Xi_N \subset \Omega\}. \tag{37}$$

Thus, the functional uncertainty $f \in \mathcal{H}_N$ has a coordinate representation given by

$$f(x) = \sum_{i=1}^{N} \theta_i \mathfrak{K}_{\xi_i}(x), \tag{38}$$

where $\Theta_N = [\theta_1, \ldots, \theta_N]^{\mathrm{T}} \in \mathbb{R}^N$ is a coefficient vector. Initially, the vector $\Theta_N \in \mathbb{R}^N$ is chosen randomly. Since $\|f\|_{\mathcal{H}} = \sqrt{\Theta_N^{\mathrm{T}} \mathbb{K}_N \Theta_N}$, where $\mathbb{K}_N \triangleq [\mathfrak{K}(\xi_i, \xi_j)]$ denotes the Grammian matrix of $\mathcal{H}_N$, and the norm of f is rescaled so that $\|f\|_{\mathcal{H}} = R$. Notice that from the assumption that $|\delta(t)| \leq 1$, we have $\|f\delta(t)\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}} |\delta(t)| \leq \|f\|_{\mathcal{H}} = R$. For this specific example, in order to make the disturbance destabilizing, we choose $\delta(t) = \cos(t)$. The remaining simulation parameters are $R = 1$, $c_1 = 5$, $c = 8$, and $\beta = 10$.

The effect of multiple kernels is investigatd. The first kernel used to construct the controller and subspace $\mathcal{H}_N$ is the *normalized Gaussian kernel*

$$\mathfrak{K}(x,y) = \exp\left(-\frac{\|x-y\|^2}{2l^2}\right), \quad (x,y) \in \mathbb{X} \times \mathbb{X}, \tag{39}$$

with $l = 0.5$. An alternative kernel investigated is the 3/2 Matern Sobolev kernel

$$\mathfrak{K}(x,y) = \left(1 + \frac{\sqrt{3}\|x-y\|}{l}\right) \exp\left(-\frac{\sqrt{3}\|x-y\|}{l}\right), \quad (x,y) \in \mathbb{X} \times \mathbb{X}. \tag{40}$$

A hundred simulations were performed with such randomized f for each kernel function. Figure 1 illustrates the maximum steady-state error in $x(\cdot)$ across different simulations. We emphasize how conventional approaches that use explicitly some known basis for the subspace containing

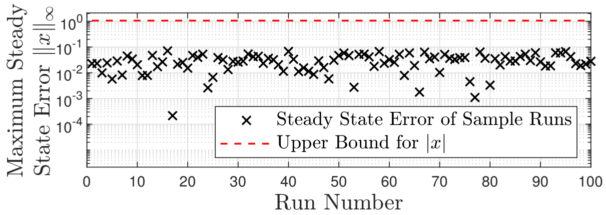

Fig. 1. Maximum steady-state error for the state $x(\cdot)$ employing the Gaussian kernel across 100 random simulations. Conventional parametric approaches derive a different basis-dependent controller for each case. The upper bound is given by $\sqrt{(1+\bar{\mathcal{M}}_\infty^2)f_\infty^2/4\beta c_1}$.

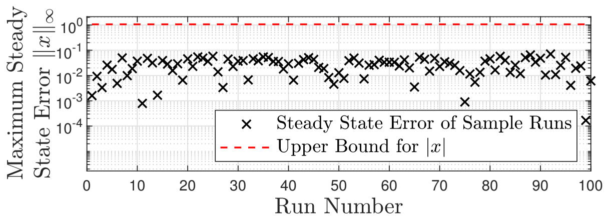

Fig. 2. Maximum steady-state error for the state $x(\cdot)$ employing the 3/2 Matern-Sobolev kernel across 100 random simulations. The difference between this case and the Gaussian Kernel is negligible.

uncertainty use a different controller, which depends on the basis, for each data point in the figure. This paper introduces one single controller that is employed for all uncertainty cases. Figure 2 depicts the analogous results when the 3/2 Matern-Sobolev kernel defines the RKHS.

VI. Conclusion

This paper provided a novel method to construct backstepping controllers. By viewing the unknown parts of the system as elements of a real, vector-valued native space determined by an operator kernel, the proposed controllers guarantee convergence across classes of functions that are substantially broader than the parametric classes typically used in classical backstepping control. The performance of the proposed control systems has been verified over 200 random simulations and different kernel functions.

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