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Exponential Extremum Seeking with Unbiased Convergence

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Abstract— **We present a multivariable extremum seeking (ES) algorithm for static and dynamic maps that achieves unbiased convergence to the optimum exponentially, referred to as exponential ES. The conventional ES approach, which uses constant amplitude sinusoids, results in steady-state oscillations around the optimum and is unable to guarantee unbiased convergence. In contrast, our ES approach employs exponential decay and growth functions to gradually decrease the amplitude of the perturbation signal and increase the amplitude of the demodulation signal, respectively. This eliminates the steady-state oscillation. To achieve unbiased convergence, we choose an adaptation gain that is sufficiently larger than the decay rate of the perturbation so that the learning process outpaces the perturbation's waning. The stability analysis is based on state transformation, averaging, and singular perturbation methods applied to the transformed system resulting in local stability of the transformed system as well as local exponential stability of the original system. For numerical simulation, we consider the problem of source seeking by a 2D velocity actuated point.**

I. Introduction

Extremum Seeking (ES), due to its model-free nature and convergence guarantees, has been a uniquely effective optimization technique for locating and tracking the optima of cost functions in static and dynamic systems. Since the proof of stability for ES is developed in [10], this technique has been extensively researched theoretically [4], [5], [14], [18] and applied in various practical settings [6], [12], [21]. The fundamental principle of ES is to introduce a small perturbation to the system through an excitation signal, observe the system's response, estimate the gradient by demodulating the output, and adjust the system's inputs towards the vicinity of the optima. However, due to the persistent excitation involved in the process, unbiased convergence to the extremum cannot be guaranteed and instead, steady-state oscillations around the extremum are observed.

In addressing the steady-state oscillation problem in classical ES, a scheme with a decaying perturbation amplitude is introduced in [19]. By choosing a sufficiently large initial value of the amplitude, convergence to an arbitrarily small neighborhood of the global extremum is guaranteed in the presence of local extrema. This is followed by [20], which claims exponential convergence to the extremum without steady-state oscillations by updating the amplitude based on the system output. However, this claim is later proven to be incorrect in [2], which shows that under certain conditions, the system converges to a point on a manifold, which is not necessarily extremum. Subsequently, a non-smooth ES design is presented in [13], which reduces perturbation as the system settles toward the extremum. Another non-smooth ES is introduced in [17] with exponential convergence exactly to the extremum. In [8], a formula for the design of input functions is provided, unifying and generalizing previous results [13], [17]. The asymptotic and exponential convergence to the extremum is guaranteed under some restrictive assumptions on the cost function. This result is later extended to dynamic systems in [7]. However, all results in [7], [8], [13], [17] are based on the assumption that the optimum point is unknown, but the value of the cost function at the optimum is known. This restrictive assumption is removed in [3] and [11], which achieve asymptotic convergence to a neighborhood of the extremum with vanishing oscillation by updating the amplitude based on the gradient estimate. An asymptotic convergence directly to optimum is achieved in [1] and [16] without requiring the knowledge of the extremum. However, none of the aforementioned papers, which are either based on classical averaging or Lie brackets, achieve exponential convergence directly to unknown extremum. Our aim is to achieve this for the framework of classical averaging-based ES. We call this approach exponential ES.

In this paper, we present an exponential ES with vanishing oscillation and unbiased convergence. The concept of exponential ES relies on an exponential decay function that reduces the effect of the perturbation signal and the use of its multiplicative inverse, which grows exponentially, to maximize the effect of the demodulation signal multiplied by the high-pass filtered output. The use of exponentially growing gain is mainly inspired by [15], which introduces an adaptive control that employs an unbounded gain to achieve exponential stabilization of unknown nonlinear systems in the absence of persistent excitation. Similar to the concept presented in [15], the convergence of our filtered output occurs at a faster rate than the divergence of the inverse function and the convergence of the perturbation, keeping the controller bounded. Although the same exponential decay perturbation is used, the amplitude of the demodulation signal is kept constant in [19]. For the stability analysis, we transform the system using the exponentially growing function and then apply classical averaging and singular perturbation methods to show the local stability of the transformed system, which in turn implies the local exponential stability of the original system as well as exponential convergence of the output to the extremum with proper choice of gains.

II. Exponential ES for Static Maps

We consider the following optimization problem

$$\max_{\theta\in\mathbb{R}^n} h(\theta), \tag{1}$$

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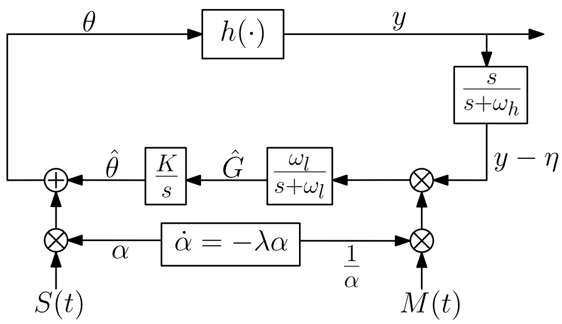

Fig. 1: Exponential ES scheme with unbiased convergence. The design uses an exponential decay function α to gradually reduce the effect of the perturbation signal $S(t)$ and its multiplicative inverse $\frac{1}{\alpha}$ to gradually increase the effect of the demodulation signal $M(t)$.

where $\theta \in \mathbb{R}^n$ is the input, $h \in \mathbb{R}^n \to \mathbb{R}$ is an unknown smooth function. We make the following assumption regarding the unknown static map $h(\cdot)$:

Assumption 1: The function h is $\mathcal{C}^4$, and there exists $\theta^* \in \mathbb{R}^n$ such that

$$\frac{\partial}{\partial\theta}h(\theta^*) = 0, \qquad \frac{\partial^2}{\partial\theta^2}h(\theta^*) = H < 0, \quad H = H^T. \quad (2)$$

Assumption 1 guarantees the existence of a maximum of the function $h(\theta)$ at $\theta = \theta^*$. It is important to note that Assumption 1 also requires that the function $h(\theta)$ be $\mathcal{C}^4$ to enable the application of the averaging theorem in [9], as will be discussed in the stability analysis. We measure the unknown function $h(\theta)$ in real time as follows

$$y(t) = h(\theta(t)), \qquad t \in [t_0, \infty), \quad (3)$$

in which $y \in \mathbb{R}$ is the output. Our aim is to design an ES algorithm using output feedback $y(t)$ in order to achieve exponential convergence of θ to θ^* while simultaneously maximizing the steady state value of y, without requiring prior knowledge of either θ^* or the function $h(\cdot)$. Our exponential ES design for static maps is schematically illustrated in Fig. 1, where K is an $n \times n$ positive diagonal matrix, the filter coefficients ω_h and ω_l are positive real numbers, the perturbation and demodulation signals are defined as

$$S(t) = \begin{bmatrix} a_1 \sin(\omega_1 t) & \cdots & a_n \sin(\omega_n t)\end{bmatrix}^T, \quad (4)$$

$$M(t) = \begin{bmatrix} \frac{2}{a_1} \sin(\omega_1 t) & \cdots & \frac{2}{a_n} \sin(\omega_n t)\end{bmatrix}^T, \quad (5)$$

respectively and the exponential decay function α is governed by

$$\dot{\alpha}(t) = -\lambda\alpha(t), \qquad \alpha(t_0) = \alpha_0. \quad (6)$$

The parameters α_0, λ are positive real numbers, the amplitudes a_i are real numbers, ω_i/ω_j are rational and the frequencies are chosen such that $\omega_i \neq \omega_j$ and $\omega_i + \omega_j \neq \omega_k$ for distinct i, j and k. We select the probing frequencies ω_i's as $\omega_i = \omega\omega_i', i \in \{1, 2, \dots, n\}$, where ω is a positive constant and and ω_i' is a rational number. In addition, the parameters should satisfy the following conditions:

$$\lambda < \frac{\omega_l}{2}, \frac{\omega_h}{2}, \quad (7)$$

$$K > (\omega_l - \lambda)\frac{\lambda}{\omega_l}\left(\frac{1}{-H}\right) > 0. \quad (8)$$

Note that if $K > \frac{\lambda}{-2H}$, stability is achieved for all admissible λ (not exceeding $\omega_l/2$). The algorithm can be used without the low-pass filter, in which case these conditions become, taking the limit $\omega_l \to \infty$, $\lambda < \frac{\omega_h}{2}, K > \frac{\lambda}{-H} > 0$. The interpretation of the conditions is that perturbation amplitude α and the demodulation amplitude $1/\alpha$ can decay and grow, respectively, but not too fast, while the estimate $\hat{\theta}$ needs to be updated fast enough, for the given rate of decay/growth of the amplitudes. In other words, learning needs to outpace the waning of the perturbation. We summarize closed-loop system depicted in Fig. 1 as follows

$$\frac{d}{dt}\begin{bmatrix}\tilde{\theta}\\ \hat{G}\\ \tilde{\eta}\\ \alpha\end{bmatrix} = \begin{bmatrix} K\hat{G}\\ -\omega_l\hat{G} + \omega_l(y - h(\theta^*) - \tilde{\eta})\frac{1}{\alpha}M(t)\\ -\omega_h\tilde{\eta} + \omega_h(y - h(\theta^*))\\ -\lambda\alpha\end{bmatrix}, \quad (9)$$

in view of the transformations

$$\tilde{\theta} = \hat{\theta} - \theta^*, \quad \tilde{\eta} = \eta - h(\theta^*), \quad (10)$$

where η is governed by

$$\dot{\eta} = -\omega_h\eta + \omega_h y. \quad (11)$$

Theorem 1: Consider the feedback system (9) with the parameters that satisfy (7), (8) under Assumption 1. There exists $\bar{\omega}$ and for any $\omega > \bar{\omega}$ there exists a neighborhood of the point $(\hat{\theta}, \hat{G}, \eta, \alpha) = (\theta^*, 0, h(\theta^*), 0)$ such that any solution of the system (9) from the neighborhood exponentially converges to that point. Furthermore, $y(t)$ exponentially converges to $h(\theta^*)$.

Proof: **Step 1: State transformation.** Consider the following transformations

$$\tilde{\theta}_f = \frac{1}{\alpha}\tilde{\theta}, \qquad \hat{G}_f = \frac{1}{\alpha}\hat{G}, \qquad \tilde{\eta}_f = \frac{1}{\alpha^2}\tilde{\eta}, \quad (12)$$

which transform (9) to the following system

$$\frac{d}{dt}\begin{bmatrix}\tilde{\theta}_f & \hat{G}_f & \tilde{\eta}_f & \alpha\end{bmatrix}^T = \begin{bmatrix} \lambda\tilde{\theta}_f + K\hat{G}_f\\ (\lambda - \omega_l)\hat{G}_f + \omega_l\left[\nu(\tilde{\theta}_f\alpha + S(t)\alpha) - \tilde{\eta}_f\alpha^2\right]\frac{M(t)}{\alpha^2}\\ (2\lambda - \omega_h)\tilde{\eta}_f + \omega_h\nu(\tilde{\theta}_f\alpha + S(t)\alpha)\frac{1}{\alpha^2}\\ -\lambda\alpha\end{bmatrix}, \quad (13)$$

where $\nu(z) = h(\theta^* + z) - h(\theta^*)$ with $z = \tilde{\theta}_f\alpha + S(t)\alpha$ in view of $\theta = \hat{\theta} + S(t)\alpha$ and (10). By Assumption 1, we get

$$\nu(0) = 0, \quad \frac{\partial}{\partial z}\nu(0) = 0, \quad \frac{\partial^2}{\partial z^2}\nu(0) = H < 0. \quad (14)$$

Step 2: Verification of the feasibility of (13) for averaging. We rewrite the system (13) in the time scale $\tau = \omega t$ as follows

$$\frac{d}{d\tau}\begin{bmatrix}\tilde{\theta}_f & \hat{G}_f & \tilde{\eta}_f & \alpha\end{bmatrix} = \frac{1}{\omega}\begin{bmatrix} \lambda\tilde{\theta}_f + K\hat{G}_f\\ (\lambda - \omega_l)\hat{G}_f + \omega_l\left[\nu(\tilde{\theta}_f\alpha + \bar{S}(\tau)\alpha) - \tilde{\eta}_f\alpha^2\right]\frac{\bar{M}(\tau)}{\alpha^2}\\ (2\lambda - \omega_h)\tilde{\eta}_f + \omega_h\nu(\tilde{\theta}_f\alpha + \bar{S}(\tau)\alpha)\frac{1}{\alpha^2}\\ -\lambda\alpha\end{bmatrix}, \quad (15)$$

where $\bar{S}(\tau) = S(\tau/\omega), \bar{M}(\tau) = M(\tau/\omega)$. Let us write the system (15) in compact form as

$$\frac{d\zeta_f}{d\tau} = (1/\omega)\mathcal{F}(\tau, \zeta_f), \tag{16}$$

where $\zeta_f = \begin{bmatrix} \tilde{\theta}_f & \hat{G}_f & \tilde{\eta}_f & \alpha \end{bmatrix}^T$. For the application of the averaging theorem in [9], we need to show that $\mathcal{F}(\tau, \zeta_f)$ and its partial derivatives with respect to ζ_f up to the second order on compact sets of ζ_f for all $\tau \geq \omega t_0$ are continuous and bounded. The proof is trivial for $\mathcal{F}(\tau, \zeta_f)$ excluding the term $\nu(\alpha\tilde{\theta}_f + \alpha\bar{S}(\tau))\frac{1}{\alpha^2}$. To complete the proof, we utilize Taylor's theorem to write

$$\nu(z) = \sum_{i=1}^{n}\sum_{j=1}^{n} z_i z_j \int_0^1 (1-s)\frac{\partial^2 \nu}{\partial z_i \partial z_j}(sz)ds \tag{17}$$

in view of (14). By substituting $z = \tilde{\theta}_f\alpha + \bar{S}(\tau)\alpha$ into (17) and multiplying both sides by $\frac{1}{\alpha^2}$, we obtain

$$\begin{aligned}\frac{1}{\alpha^2}\nu(\tilde{\theta}_f\alpha + \bar{S}(\tau)\alpha) = &\sum_{i=1}^{n}\sum_{j=1}^{n}(\tilde{\theta}_{f_i} + a_i\sin(\omega_i'\tau))(\tilde{\theta}_{f_j} + a_j \\ &\times \sin(\omega_j'\tau))\int_0^1 (1-s)\frac{\partial^2\nu}{\partial z_i\partial z_j}\left(s\tilde{\theta}_f\alpha + s\bar{S}(\tau)\alpha\right)ds,\end{aligned} \tag{18}$$

where $\tilde{\theta}_{f_i}$ is the ith element of $\tilde{\theta}_f$. Next, we apply the mean value theorem to obtain

$$\begin{aligned}\frac{1}{\alpha^2}\nu(\tilde{\theta}_f\alpha + \bar{S}(\tau)\alpha) = &\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\tilde{\theta}_{f_i} + a_i\sin(\omega_i'\tau))(\tilde{\theta}_{f_j} \\ &+ a_j\sin(\omega_j'\tau))\frac{\partial^2\nu}{\partial z_i\partial z_j}\left(\mathfrak{s}\tilde{\theta}_f\alpha + \mathfrak{s}\bar{S}(\tau)\alpha\right)\end{aligned} \tag{19}$$

for some $\mathfrak{s} \in [0,1]$. By Assumption 1, (19) is continuous and bounded on compact sets of $\tilde{\theta}_f$ and α. Considering the $\mathcal{C}^4$ property of ν and using the mean value theorem, we prove the continuity and boundedness of the partial derivatives of (18) with respect to $\tilde{\theta}_f$ and α up to the second order on compact sets of $\tilde{\theta}_f$ and α. Therefore, $\mathcal{F}(\tau, \zeta_f)$ satisfies the continuity and boundedness assumptions of the averaging theorem in [9].

Step 3: Averaging operation. Let us define the common period of the probing frequencies as follows

$$\Pi = 2\pi \times \mathrm{LCM}\left\{1/\omega_i\right\}, \qquad i \in \{1, 2 \ldots, n\}, \tag{20}$$

where LCM stands for the least common multiple. The average of the system (15) over the period Π is given by

$$\begin{aligned}&\frac{d}{d\tau}\begin{bmatrix}\tilde{\theta}_f^a & \hat{G}_f^a & \tilde{\eta}_f^a & \alpha^a\end{bmatrix}^T \\ &= \frac{1}{\omega}\begin{bmatrix}\lambda\tilde{\theta}_f^a + K\hat{G}_f^a \\ (\lambda - \omega_l)\hat{G}_f^a \\ (2\lambda - \omega_h)\tilde{\eta}_f^a \\ -\lambda\alpha^a\end{bmatrix} + \frac{1}{\omega}\begin{bmatrix}0 \\ \omega_l\frac{1}{\Pi}\int_0^\Pi \nu(\tilde{\theta}_f^a\alpha^a + \bar{S}(\sigma)\alpha^a)\frac{\bar{M}(\sigma)}{(\alpha^a)^2}d\sigma \\ \omega_h\frac{1}{\Pi}\int_0^\Pi \nu(\tilde{\theta}_f^a\alpha^a + \bar{S}(\sigma)\alpha^a)\frac{1}{(\alpha^a)^2}d\sigma \\ 0\end{bmatrix}\end{aligned} \tag{21}$$

It follows from (21) that the average equilibrium denoted as $\begin{bmatrix}\tilde{\theta}_f^{a,e} & \hat{G}_f^{a,e} & \tilde{\eta}_f^{a,e} & \alpha^{a,e}\end{bmatrix}^T$ satisfies that $\lambda\tilde{\theta}_f^{a,e} = -K\hat{G}_f^{a,e}, \alpha^{a,e} = 0$ and

$$\begin{aligned}&(\omega_l - \lambda)\hat{G}_f^{a,e} \\ &= \lim_{\alpha^{a,e}\to 0}\left[\frac{\omega_l}{\Pi}\int_0^\Pi \nu(\tilde{\theta}_f^{a,e}\alpha^{a,e} + \bar{S}(\sigma)\alpha^{a,e})\frac{\bar{M}(\sigma)}{(\alpha^{a,e})^2}d\sigma\right], \\ &(\omega_h - 2\lambda)\tilde{\eta}_f^{a,e} \\ &= \lim_{\alpha^{a,e}\to 0}\left[\frac{\omega_h}{\Pi}\int_0^\Pi \nu(\tilde{\theta}_f^{a,e}\alpha^{a,e} + \bar{S}(\sigma)\alpha^{a,e})\frac{1}{(\alpha^{a,e})^2}d\sigma\right].\end{aligned}$$

By performing a Taylor series approximation of ν in view of (14) as follows

$$\begin{aligned}\nu(z) = &\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\nu}{\partial z_i\partial z_j}(0)z_i z_j \\ &+ \frac{1}{3!}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^3\nu}{\partial z_i\partial z_j\partial z_k}(0)z_i z_j z_k + \mathcal{O}(|z|^4)\end{aligned}$$

with $z = \tilde{\theta}_f^{a,e}\alpha^{a,e} + \bar{S}(\sigma)\alpha^{a,e}$, we compute

$$\begin{aligned}&\lim_{\alpha^{a,e}\to 0}\left[\frac{1}{\Pi}\int_0^\Pi \nu(\tilde{\theta}_f^{a,e}\alpha^{a,e} + \bar{S}(\sigma)\alpha^{a,e})\frac{\bar{M}(\sigma)}{(\alpha^{a,e})^2}d\sigma\right] \\ &= \lim_{\alpha^{a,e}\to 0}\left[\frac{1}{\Pi}\int_0^\Pi \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\nu}{\partial z_i\partial z_j}(0)(\tilde{\theta}_{f_i}^{a,e} + a_i\sin(\omega_i'\sigma))\right. \\ &\left.\times(\tilde{\theta}_{f_j}^{a,e} + a_j\sin(\omega_j'\sigma))(\alpha^{a,e})^2\frac{\bar{M}(\sigma)}{(\alpha^{a,e})^2}d\sigma + \frac{(\alpha^{a,e})^3}{(\alpha^{a,e})^2}\mathcal{O}(|a|^2)\right], \\ &= H\tilde{\theta}_f^{a,e},\end{aligned} \tag{22}$$

and

$$\begin{aligned}&\lim_{\alpha^{a,e}\to 0}\left[\frac{1}{\Pi}\int_0^\Pi \nu(\tilde{\theta}_f^{a,e}\alpha^{a,e} + \bar{S}(\sigma)\alpha^{a,e})\frac{1}{(\alpha^{a,e})^2}d\sigma\right] \\ &\qquad = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}H_{i,j}\tilde{\theta}_{f_i}^{a,e}\tilde{\theta}_{f_j}^{a,e} + \frac{1}{4}\sum_{i=1}^{n}H_{i,i}a_i^2,\end{aligned} \tag{23}$$

by L'Hospital's rule, where $H_{i,j} = \frac{\partial^2\nu}{\partial z_i\partial z_j}(0)$ and $\tilde{\theta}_{f_i}^{a,e}$ is the ith element of $\tilde{\theta}_f^{a,e}$. Then, we obtain the equilibrium of the average system (21) as

$$\begin{aligned}&\begin{bmatrix}\tilde{\theta}_f^{a,e} & \hat{G}_f^{a,e} & \tilde{\eta}_f^{a,e} & \alpha^{a,e}\end{bmatrix}^T \\ &= \begin{bmatrix}0_{1\times n} & 0_{1\times n} & \frac{\omega_h}{4(\omega_h - 2\lambda)}\sum_{i=1}^n H_{i,i}a_i^2 & 0\end{bmatrix}^T,\end{aligned} \tag{24}$$

provided that $\omega_l \neq \lambda, \omega_h \neq 2\lambda$ and $K \neq \lambda(\lambda - \omega_l)\omega_l^{-1}H^{-1}$.

Step 4: Stability analysis. The Jacobian of the average system (21) at the equilibrium (24) is given by

$$J_f^a = \frac{1}{\omega}\begin{bmatrix}\lambda I_{n\times n} & K & 0_{n\times 1} & 0_{n\times 1} \\ \omega_l H & (\lambda - \omega_l)I_{n\times n} & 0_{n\times 1} & \frac{\omega_l}{\Pi}\int_0^\Pi \frac{\partial\left(\frac{\nu\bar{M}}{(\alpha^a)^2}\right)}{\partial\alpha^a}d\sigma \\ 0_{1\times n} & 0_{1\times n} & (2\lambda - \omega_h) & \frac{\omega_h}{\Pi}\int_0^\Pi \frac{\partial\left(\frac{\nu}{(\alpha^a)^2}\right)}{\partial\alpha^a}d\sigma \\ 0_{1\times n} & 0_{1\times n} & 0 & -\lambda\end{bmatrix}.$$

Note that J_f^a is block-upper-triangular and Hurwitz provided that (7) and (8) are satisfied. This proves the local exponential stability of the average system (21). Then, based on the averaging theorem [9], we show that there exists $\bar{\omega}$ and for any $\omega > \bar{\omega}$, the system (15) has a unique exponentially stable periodic solution $(\tilde{\theta}_f^\Pi(\tau), \hat{G}_f^\Pi(\tau), \tilde{\eta}_f^\Pi(\tau), \alpha^\Pi(\tau))$ of period Π and this solution satisfies

$$\begin{aligned}&\left|\begin{bmatrix}\tilde{\theta}_f^\Pi(\tau), & \hat{G}_f^\Pi(\tau), & \tilde{\eta}_f^\Pi(\tau) - \frac{\omega_h}{4(\omega_h - 2\lambda)}\sum_{i=1}^n H_{i,i}a_i^2, & \alpha^\Pi(\tau)\end{bmatrix}^T\right| \\ &\qquad \leq \mathcal{O}(1/\omega).\end{aligned} \tag{25}$$

In other words, all solutions $(\tilde{\theta}_f(\tau), \hat{G}_f(\tau), \tilde{\eta}_f(\tau), \alpha(\tau))$ exponentially converge to an $\mathcal{O}\left(1/\omega + |a|^2\right)$-neighborhood of the origin.

Step 5: Convergence to extremum. Considering the results in Step 4 and recalling from (12) and Fig. 1 that

$$\theta(t) = \alpha(t)\tilde{\theta}_f(t) + \theta^* + \alpha(t)S(t), \tag{26}$$

we conclude the exponential convergence of $\theta(t)$ to θ^*. This implies the convergence of the output $y(t)$ to $h(\theta^*)$ and completes the proof of Theorem 1. ■

III. Exponential ES for Dynamic Systems

In this section, we extend our results in Section II to dynamic systems. For this, we consider a general multi-input single-output nonlinear model

$$\dot{x} = f(x, u), \tag{27}$$
$$y = h(x), \tag{28}$$

where $x \in \mathbb{R}^m$ is the state, $u \in \mathbb{R}^n$ is the input, $y \in \mathbb{R}$ is the output and the unknown functions $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^m \to \mathbb{R}$ are smooth. Suppose there is a smooth control law $u = \phi(x, \theta)$ parametrized by a vector parameter $\theta \in \mathbb{R}^n$. The closed-loop system

$$\dot{x} = f(x, \phi(x, \theta)) \tag{29}$$

then has equilibria parameterized by θ. We make the following assumptions about the closed-loop system:

Assumption 2: There exists a smooth function $l : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(x, \phi(x, \theta)) = 0$ if and only if $x = l(\theta)$.

Assumption 3: For each $\theta \in \mathbb{R}^n$, the equilibrium $x = l(\theta)$ of (29) is locally exponentially stable uniformly in θ.

Assumption 4: The function $h \circ l$ is $\mathcal{C}^4$, and there exists $\theta^* \in \mathbb{R}^n$ such that

$$\frac{\partial}{\partial \theta}(h \circ l)(\theta^*) = 0,\ \frac{\partial^2}{\partial \theta^2}(h \circ l)(\theta^*) = H < 0,\ H = H^T.$$

We aim to design a controller u to drive the output y directly to its optimum $h \circ l(\theta^*)$ exponentially without any steady-state oscillation and without the need for knowledge of θ^*, h, or l. The perturbation and demodulation signals are defined by (4) and (5), respectively, and α is governed by (6). The probing frequencies ω_i's, the filter coefficients ω_h and ω_l, gain K and parameter λ are selected as $\omega_i = \omega\omega_i' = \mathcal{O}(\omega), i \in \{1, 2, \ldots, n\}, \omega_h = \omega\omega_H = \omega\delta\omega_H' = \mathcal{O}(\omega\delta), \omega_l = \omega\omega_L = \omega\delta\omega_L' = \mathcal{O}(\omega\delta), K = \omega K' = \omega\delta K'' = \mathcal{O}(\omega\delta), \lambda = \omega\lambda' = \omega\delta\lambda'' = \mathcal{O}(\omega\delta)$, where ω and δ are small positive constants, ω_i' is a rational number, ω_H', ω_L' and λ'' are $\mathcal{O}(1)$ positive constants, K'' is a $n \times n$ diagonal matrix with $\mathcal{O}(1)$ positive elements. In addition, the parameters should satisfy (7) and (8). We summarize the closed-loop system as follows

$$\frac{d}{dt}\begin{bmatrix} x \\ \tilde{\theta} \\ \hat{G} \\ \tilde{\eta} \\ \alpha \end{bmatrix} = \begin{bmatrix} f(x, \phi(x, \theta^* + \tilde{\theta} + S(t)\alpha)) \\ K\hat{G} \\ -\omega_l\hat{G} + \omega_l(y - h \circ l(\theta^*) - \tilde{\eta})\frac{M(t)}{\alpha} \\ -\omega_h\tilde{\eta} + \omega_h(y - h \circ l(\theta^*)) \\ -\lambda\alpha \end{bmatrix}. \tag{30}$$

Theorem 2: Consider the feedback system (30) with the parameters that satisfy (7), (8) under Assumptions 2–4. There exists $\bar{\omega} > 0$ and for any $\omega \in (0, \bar{\omega})$ there exists $\bar{\delta} > 0$ such that for the given ω and $\delta \in (0, \bar{\delta})$ there exists a neighborhood of the point $(x, \hat{\theta}, \hat{G}, \eta) = (l(\theta^*), \theta^*, 0, h \circ l(\theta^*))$ such that any solution of systems (30) from the neighborhood exponentially converges to that point. Furthermore, $y(t)$ exponentially converges to $h \circ l(\theta^*)$.

Proof: **Step 1: Time-scale separation.** We rewrite the system (30) in the time scale $\tau = \omega t$ as

$$\omega\frac{\mathrm{d}x}{\mathrm{d}\tau} = f(x, \phi(x, \theta^* + \tilde{\theta} + \bar{S}(\tau)\alpha)), \tag{31}$$

$$\frac{d}{d\tau}\begin{bmatrix} \tilde{\theta} \\ \hat{G} \\ \tilde{\eta} \\ \alpha \end{bmatrix} = \delta\begin{bmatrix} K''\hat{G} \\ -\omega_L'\hat{G} + \omega_L'(y - h \circ l(\theta^*) - \tilde{\eta})\frac{1}{\alpha}\bar{M}(\tau) \\ -\omega_H'\tilde{\eta} + \omega_H'(y - h \circ l(\theta^*)) \\ -\lambda''\alpha \end{bmatrix}, \tag{32}$$

where $\bar{S}(\tau) = S(\tau/\omega), \bar{M}(\tau) = M(\tau/\omega)$.

Step 2: State transformation. Consider the following transformations

$$\tilde{\theta}_f = \frac{1}{\alpha}\tilde{\theta}, \qquad \hat{G}_f = \frac{1}{\alpha}\hat{G}, \qquad \tilde{\eta}_f = \frac{1}{\alpha^2}\tilde{\eta}, \tag{33}$$

which transform (31), (32) to the following system

$$\omega\frac{\mathrm{d}x}{\mathrm{d}\tau} = f(x, \phi(x, \theta^* + \tilde{\theta}_f\alpha + \bar{S}(\tau)\alpha)), \tag{34}$$
$$\frac{d\zeta_f}{d\tau} = \delta E(\tau, x, \zeta_f), \tag{35}$$

where $\zeta_f = \begin{bmatrix} \tilde{\theta}_f & \hat{G}_f & \tilde{\eta}_f & \alpha \end{bmatrix}^T$ and

$$E(\tau, x, \zeta_f) = \begin{bmatrix} \lambda''\tilde{\theta}_f + K''\hat{G}_f \\ (\lambda'' - \omega_L')\hat{G}_f + \omega_L'(y - h \circ l(\theta^*) - \tilde{\eta}_f\alpha^2)\frac{\bar{M}(\tau)}{\alpha^2} \\ (2\lambda'' - \omega_H')\tilde{\eta}_f + \omega_H'(y - h \circ l(\theta^*))\frac{1}{\alpha^2} \\ -\lambda''\alpha \end{bmatrix}.$$

Step 3: Averaging analysis for reduced system. We first freeze x in (34) at its equilibrium value $x = L(\tau, \zeta_f) = l(\theta^* + \tilde{\theta}_f\alpha + \bar{S}(\tau)\alpha)$, substitute it into (35) and then get the reduced system

$$\frac{d\zeta_{f,r}}{d\tau} = \delta E(\tau, L(\tau, \zeta_{f,r}), \zeta_{f,r}), \tag{36}$$

where $\zeta_{f,r} = \begin{bmatrix} \tilde{\theta}_{f,r} & \hat{G}_{f,r} & \tilde{\eta}_{f,r} & \alpha \end{bmatrix}^T$,

$$E(\tau, L(\tau, \zeta_{f,r}), \zeta_{f,r}) = \begin{bmatrix} \lambda''\tilde{\theta}_{f,r} + K''\hat{G}_{f,r} \\ (\lambda'' - \omega_L')\hat{G}_{f,r} + \omega_L'(\nu(\tilde{\theta}_{f,r}\alpha + \bar{S}(\tau)\alpha) - \tilde{\eta}_{f,r}\alpha^2)\frac{\bar{M}(\tau)}{\alpha^2} \\ (2\lambda'' - \omega_H')\tilde{\eta}_{f,r} + \omega_H'\nu(\tilde{\theta}_{f,r}\alpha + \bar{S}(\tau)\alpha)\frac{1}{\alpha^2} \\ -\lambda''\alpha \end{bmatrix}$$

and $\nu(z) = h \circ l(z + \theta^*) - h \circ l(\theta^*)$ with $z = \tilde{\theta}_{f,r}\alpha + \bar{S}(\tau)\alpha$. From Assumption 4, we get $\nu(0) = 0$, $\frac{\partial}{\partial z}\nu(0) = 0$, $\frac{\partial^2}{\partial z^2}\nu(0) = H < 0$. Note that the reduced system (36) has the same structure as (15) except the different constant parameters. Therefore, we can perform averaging analysis and stability analysis in Step 3 and 4 of the proof of Theorem 1, respectively, for the reduced system (36). Then, we conclude that there exists δ such that for all $\delta \in (0, \bar{\delta})$, the system (36) has a unique exponentially stable periodic solution $\zeta_{f,r}^\Pi(\tau) = \begin{bmatrix} \tilde{\theta}_{f,r}^\Pi(\tau), \hat{G}_{f,r}^\Pi(\tau), \tilde{\eta}_{f,r}^\Pi(\tau), \alpha^\Pi(\tau) \end{bmatrix}^T$ such that

$$\frac{d\zeta_{f,r}^\Pi(\tau)}{d\tau} = \delta E(\tau, L(\tau, \zeta_{f,r}^\Pi(\tau)), \zeta_{f,r}^\Pi(\tau)). \tag{37}$$

Step 4: Singular perturbation analysis. To convert the system (34) and (35) into the standard singular perturbation form, we shift the states ζ_f and x using the transformations $\tilde{\zeta}_f = \zeta_f - \zeta^{\Pi}_{f,r}(\tau)$ and $\tilde{x} = x - L(\tau, \zeta_f)$ such that

$$\frac{d\tilde{\zeta}_f}{d\tau} = \delta\tilde{E}(\tau, \tilde{x}, \tilde{\zeta}_f), \tag{38}$$

$$\omega\frac{d\tilde{x}}{d\tau} = \tilde{F}(\tau, \tilde{x}, \tilde{\zeta}_f), \tag{39}$$

where $\tilde{E}(\tau, \tilde{x}, \tilde{\zeta}_f) = E\Big(\tau, \tilde{x} + L(\tau, \tilde{\zeta}_f + \zeta^{\Pi}_{f,r}(\tau)), \tilde{\zeta}_f + \zeta^{\Pi}_{f,r}(\tau)\Big) - E\Big(\tau, L(\tau, \zeta^{\Pi}_{f,r}(\tau)), \zeta^{\Pi}_{f,r}(\tau)\Big)$, and $\tilde{F}(\tau, \tilde{x}, \tilde{\zeta}_f) = f\Big(\tilde{x} + L(\tau, \tilde{\zeta}_f + \zeta^{\Pi}_{f,r}(\tau)), \phi(\tilde{x} + L(\tau, \tilde{\zeta}_f + \zeta^{\Pi}_{f,r}(\tau)), \theta^* + \tilde{\zeta}_{f_1}\alpha + \zeta^{\Pi}_{f_1,r}(\tau)\alpha + \bar{S}(\tau)\alpha)\Big)$,with $\tilde{\zeta}_{f_1} = \tilde{\theta}_f - \tilde{\theta}^{\Pi}_{f,r}(\tau)$ and $\zeta^{\Pi}_{f_1,r}(\tau) = \tilde{\theta}^{\Pi}_{f,r}(\tau)$. Note that $\tilde{x} = 0$ is the quasi-steady state. By substituting the quasi-steady state into (38), we obtain the following reduced model

$$\frac{d\tilde{\zeta}_{f,r}}{d\tau} = \delta\tilde{E}(\tau, 0, \tilde{\zeta}_{f,r}), \tag{40}$$

which has an equilibrium at the origin $\tilde{\zeta}_{f,r} = 0$. We prove in Step 3 that this equilibrium is exponentially stable. The next step in the singular perturbation analysis is to examine the boundary layer model in the time scale $t = \tau/\omega$ as follows

$$\frac{\mathrm{d}x_{\mathrm{b}}}{\mathrm{d}t} = \tilde{F}(\tau, x_b, \tilde{\zeta}_f) = f(x_{\mathrm{b}} + l(\theta), \phi(x_{\mathrm{b}} + l(\theta), \theta)). \tag{41}$$

Recalling $f(l(\theta), \phi(l(\theta), \theta)) \equiv 0$ from Assumption 2, we deduce that $x_{\mathrm{b}} \equiv 0$ is an equilibrium of (41). According to Assumption 3, this equilibrium is locally exponentially stable uniformly in θ. By combining exponential stability of the reduced model (40) with the exponential stability of the boundary layer model (41), and noting that $\tilde{E}(\tau, 0, 0) = 0, \tilde{F}(\tau, 0, 0) = 0$, we conclude from Theorem 11.4 of [9] that $\tilde{\zeta}_f \to 0$ and $\tilde{x} \to 0$, i.e., $\zeta_f \to \zeta^{\Pi}_{f,r}$ and $x \to l(\theta) = L(\tau, \zeta_f)$ exponentially as $\tau \to \infty$.

Step 5: Convergence to extremum. Note that $\tilde{\theta}_f(\tau) \to \tilde{\theta}^{\Pi}_f(\tau)$ and $\alpha \to 0$ exponentially. It follows then that $\theta(\tau) = \theta^* + \tilde{\theta}_f(\tau)\alpha + \bar{S}(\tau)\alpha \to \theta^*$ exponentially and $l(\theta) = l(\theta^* + \tilde{\theta}_f\alpha + \bar{S}(\tau)\alpha) \to l(\theta^*)$ exponentially. Consequently, $y = h(x)$ exponentially converges to $h \circ l(\theta^*)$. ■

IV. Source Seeking by a 2D Point Mass

In this section, we investigate the problem of source localization using an autonomous vehicle modeled as a point mass in a two-dimensional plane

$$\dot{x}_1 = v_{x_1}, \qquad \dot{x}_2 = v_{x_2}, \tag{42}$$

in which the vehicle's position is represented by the vector $[x_1, x_2]^T$ and its velocity is controlled by inputs v_{x_1} and v_{x_2}. The objective of this problem is to guide the vehicle towards the static source of a scalar signal in an environment where the vehicle's position data is not available. The only information provided to the vehicle at its current location is the strength of the signal, which is assumed to decrease as the distance from the source increases. Our specific goal is to detect the source while continuously measuring the source signal, ultimately bringing the vehicle to a complete stop at the exact location of the source. We give a block diagram in Fig. 2, in which we apply our exponential ES design. For simplicity, but without loss of generality, we assume that the nonlinear map is quadratic with diagonal Hessian matrix

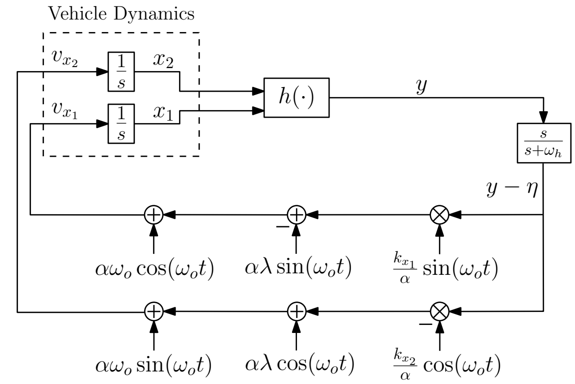

Fig. 2: The developed ES scheme for velocity-actuated point mass.

$$h(x_1, x_2) = h^* - q_{x_1}(x_1 - x_1^*)^2 - q_{x_2}(x_2 - x_2^*)^2, \tag{43}$$

where (x_1^*, x_2^*) is the unknown maximizer, $h^* = h(x_1^*, x_2^*)$ is the unknown maximum, and q_{x_1}, q_{x_2} are some unknown positive constants. Before presenting our results, let us first introduce the new coordinates

$$\tilde{x}_1 = x_1 - x_1^* - \alpha\sin(\omega_o t), \tag{44}$$

$$\tilde{x}_2 = x_2 - x_2^* + \alpha\cos(\omega_o t), \tag{45}$$

$$\tilde{\eta} = \eta - h(x_1^*, x_2^*), \tag{46}$$

where the signal η is defined in (11). Then, we summarize the system in Fig. 2 as follows

$$\frac{d}{dt}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\\ \tilde{\eta}\\ \alpha\end{bmatrix} = \begin{bmatrix}+k_{x_1}\sin(\omega_o t)(y - h^* - \tilde{\eta})(1/\alpha)\\ -k_{x_2}\cos(\omega_o t)(y - h^* - \tilde{\eta})(1/\alpha)\\ -\omega_h\tilde{\eta} + \omega_h(y - h^*)\\ -\lambda\alpha\end{bmatrix} \tag{47}$$

with the parameters chosen as

$$\omega_h > 2\lambda, \qquad k_{x_i} > \lambda/q_{x_i}, \qquad i = 1, 2. \tag{48}$$

An extension of the exponential ES result in Theorem 1 to the system (47) can be easily done. For convenience, we give the exponential ES result for this source seeking problem below without a proof.

Theorem 3: Consider the feedback system (47) with the parameters that satisfy (48) and with the nonlinear map of the form (43). There exists $\bar{\omega}_o$ and for any $\omega_o > \bar{\omega}_o$, there exists a neighborhood of the point $(x_1, x_2, \eta, \alpha) = (x_1^*, x_2^*, h^*, 0)$ such that any solution of system (47) from the neighborhood exponentially converges to that point. Hence, $y(t)$ exponentially converges to $h(x_1^*, x_2^*)$. Furthermore, the velocity inputs remain bounded for all $t \geq t_0$.

V. Application to Source Seeking Problem

We consider the application of the developed ES technique to the problem of source seeking by a velocity-actuated point mass as outlined in Section IV. In order to make a comparison, we utilize the following ES controllers to control the velocity of the vehicle depicted in Fig. 2:

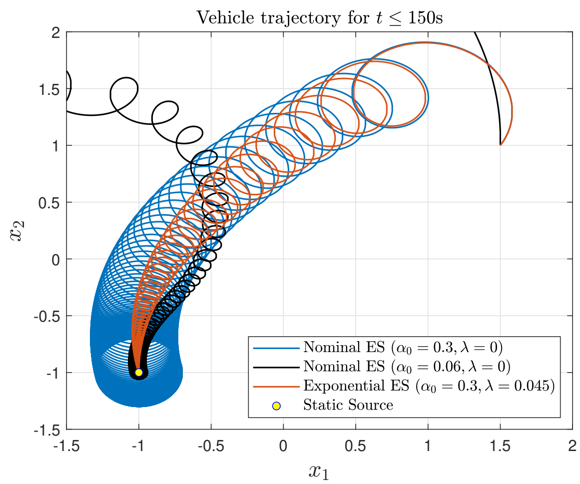

Fig. 3: Static source seeking by an autonomous vehicle. The nominal ES with low amplitude α_0 approaches the source more closely but requires high initial velocity, leading to initial deviation from the source. The exponential ES, with its exponentially decaying amplitude, avoids this issue.

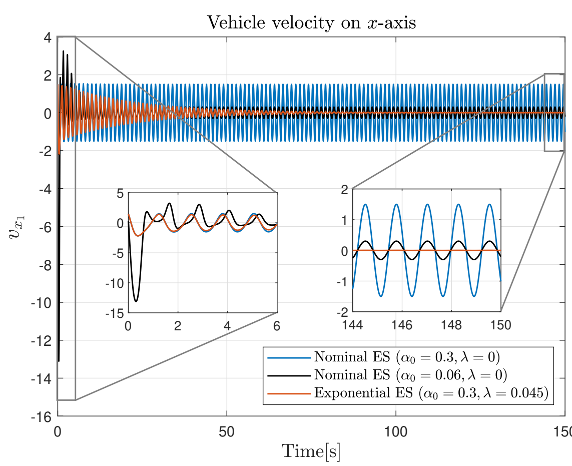

Fig. 4: The evolution of the vehicle velocity v_{x_1} over $t \in [0, 150]$ in seconds with the first and last 6 seconds zoomed in. The velocity input of nominal ES with low amplitude α_0 exhibits rapid growth due to the high demodulation amplitude and oscillate around zero, whereas the velocity of our design does not exhibit this behavior and slows down as the vehicle approaches the source.

- Nominal ES with $\alpha(t) \equiv \alpha_0$, (i.e., $\lambda = 0$) for all $t \geq 0$. This design boils down to [21], in which the vehicle asymptotically converges to a neighborhood of the source and shows steady-state oscillations around it.
- Exponential ES with α-function dynamics (6). This design is a modified version of Fig. 1.

The real-time measurement is defined as $y(t) = h(x_1(t), x_2(t))$, where the function $h(\cdot)$ is described in (43) with parameters $(x_1^*, x_2^*) = (-1, -1), h^* = 1, q_{x_1} = 1, q_{x_2} = 0.5$. The parameters used in Fig. 2 are selected as $\omega_h = 1$, $k_{x_1} = k_{x_2} = 0.1, \lambda = 0.045$ and $\omega_o = 5$. We present the comparison between the nominal ES and exponential ES in Fig. 3 and Fig. 4. In Fig. 3, the exponential ES exhibits exponential convergence to the source at $(-1, -1)$ with circular trajectories and exponentially decaying amplitude. On the other hand, the nominal ES with constant amplitude asymptotically converges to the vicinity of the source and shows steady-state oscillation around it. The nominal design with $\alpha_0 = 0.06$ converges closer to the source than the one with $\alpha_0 = 0.3$. However, it initially moves away from the source due to its high initial velocity. As illustrated in Fig. 4, the velocity of the vehicle on x-axis grows rapidly from 0.3 to -13.1 when using the nominal ES with $\alpha_0 = 0.06$. Low initial velocity and perfect convergence are achieved through our design.

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