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Further results on incremental input-to-state stability based on contraction-metric analysis

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***Abstract*—In this paper, we study the notion of incremental stability for continuous-time nonlinear systems forced by some external input. Thanks to the notion of "Killing vector field", we provide a set of sufficient conditions based on a metric analysis for an input-affine system to be incrementally input-to-state stable (δISS) and show that this also implies that its lifted system is transversally ISS. We conclude by providing a design achieving incremental ISS properties for the closed-loop system by means of a state-feedback control law that possesses an infinite gain margin property.**

I. Introduction

This article focuses on *incremental stability* (or contraction) properties of smooth continuous-time nonlinear systems. Roughly speaking, a system is incrementally stable if any two trajectories starting from different initial conditions that are "close" remain "close" for all their time existence. If the distance between them decreases in time, then the system is incrementally asymptotically stable (see, e.g. [1]–[12]). In this sense, we can understand incremental asymptotic stability as the property of *every solution* of the system to be asymptotically stable. The interest in incremental properties is increasing in the control community, due to the application in many control problems such as observers design [13]–[15], output regulation [11], [16]–[18], and multi-agent synchronization [19]–[22]. In order to characterize the incremental properties, many different tools have been proposed: incremental Lyapunov functions [1], [2]; Finsler-Lyapunov function [3], [4]; matrix measures based on both Euclidean and non-Euclidean norms [5]–[7]; weak-pairings [7]; and *Riemannian metrics* conditions [8]–[10]. Very similar properties have been also studied in the context of convergent theory [11], [12] raising to conditions which are mainly equivalent to those studied in the context of Riemannian metrics.

In this article, we study the notion of incremental input-to-state stability (δISS), that is, incremental properties of systems forced by an external input. In particular, we follow an approach based on Riemaniann metrics. To the author's knowledge, such a notion has been studied only by means of Lyapunov functions in [1], [2] and with weak pairings in [7]. Motivated by [1, Remark 3.5], we study *global* properties (i.e. in the whole state-space $\mathbb{R}^n$), with the awareness that our results can be generalized to systems whose trajectories remain in compact forward invariant sets. With respect to the existing literature, we provide the following results. i) Through the notion of "Killing vector field", we provide sufficient metric-based conditions for a system to be δISS for any external input. ii) We show that if a system is δISS, then the lifted system presents some ISS properties, which we refer to as "transversal input-to-state stability" (see [10]). iii) By making use of the former results, we show a simple state-feedback design achieving δISS for the closed-loop system by means of a control law with an infinite gain margin. The latter result has been also proposed in a similar conceptual form in the work on control contraction metrics [23], although here we highlight the aforementioned extra properties.

Notations: Let $\mathbb{R}_{\geq 0} = [0, +\infty)$ and $\mathbb{R}_{>0} = (0, +\infty)$. $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$. $\mathbb{S}^n$ it set of symmetric positive definite matrices of dimension $n \times n$. $\alpha : \mathbb{R} \mapsto \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ if $\alpha(0) = 0$ and it is strictly increasing. It is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and radially unbounded. Given a square matrix A, we indicate with $\mathtt{He}\{A\} := A + A^\top$. Given a vector field $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ and a 2-tensor $P : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ both C^1, we indicate with $L_f P(x)$ the Lie derivative of the tensor P along f defined as

$$\begin{aligned} L_f P(x) &:= \mathfrak{d}_f P(x) + P(x)\tfrac{\partial f}{\partial x}(x) + \tfrac{\partial f}{\partial x}^\top(x) P(x) \\ \mathfrak{d}_f P(x) &:= \lim_{h\to 0} \tfrac{P(\mathcal{X}(x,t+h)) - P(x)}{h} \end{aligned}$$

where $\mathcal{X}(x,t)$ is the solution of the initial value problem

$$\tfrac{\partial}{\partial t}\mathcal{X}(x,t) = f(\mathcal{X}(x,t)), \quad \mathcal{X}(x,0) = x, \qquad (1)$$

for all $t \geq 0$. Note that we can compute the element ij as $(L_f P(x))_{ij} = \sum_k \big[2P_{ik}\frac{\partial f_k}{\partial x_j}(x) + \frac{\partial P_{ij}}{\partial x_k}(x) f_k(x)\big]$.

II. Metric Approach for Incremental Stability

A. *Sufficient conditions for incremental stability*

This section aims to recall some of the main results and properties of incrementally stable autonomous nonlinear systems. Consider a nonlinear system of the form

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \qquad (2)$$

where f is a C^2 vector field. We denote by $\mathcal{X}(x_0, t)$ the solution of system (2) with initial condition x_0 evaluated at time $t \geq 0$ (that is, the solution of (1)). To simplify our

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analysis, we assume existence and uniqueness of trajectories in forward time.

Remark 1. *Note that most of the results of this article applies also to the case of time-varying systems of the form $\dot{x} = f(x,t)$. To ease the notation and the computation, we focus here in the case of systems of the form* (2).

Definition 1 (Incremental exponential stability)**.** *We say that system* (2) *is incrementally globally exponentially stable (δGES) if there exists two strictly positive real numbers $\lambda, k > 0$ such that*

$$|\mathcal{X}(x_1,t) - \mathcal{X}(x_2,t)| \le ke^{-\lambda t}|x_1 - x_2| \qquad (3)$$

for any initial conditions $x_1, x_2 \in \mathbb{R}^n$ and for all $t \ge 0$.

A known result is that system (2) is δGES if there exists Riemannian metric along which the vector field f generates trajectories for which the distance associated to such Riemannian metric is monotonically decreasing in forward time (i.e. the mapping $t \mapsto \mathcal{X}(x,t)$ is a contraction). Sufficient conditions are stated in the following. A proof can be found in [8], [10].

Theorem 1 (Sufficient metric conditions for δGES)**.** *Consider system* (2) *and assume there exist a C^1 function P taking values in $\mathbb{S}^n$ for any $x \in \mathbb{R}^n$, and three real numbers $\overline{p}, \underline{p}, \lambda > 0$ such that*

$$\underline{p}I \preceq P(x) \preceq \overline{p}I, \qquad (4a)$$

$$L_f P(x) \preceq -2\lambda P(x) \qquad (4b)$$

for all $x \in \mathbb{R}^n$. Then the system is δGES.

Remark 2. *A converse theorem can be found in [10, Proposition IV] in case f is a globally Lipschitz vector field.*

Note that the lower bound in (4a) is required to make sure that the whole $\mathbb{R}^n$ space endowed with the Riemannian metric P is complete. Such a condition guarantees that every geodesic (i.e. the shortest curve between (x_1, x_2)) can be maximally extended to $\mathbb{R}$ (see [13]). By Hopf-Rinow's Theorem (see [24, Theorem 1.1]) this implies that the metric is complete and hence that the minimum of the length of any curve γ connecting two point (x_1, x_2) is actually given by the length of the geodesic at any time instant. Furtheremore, it guarantees that the Lyapunov function defined as the distance associated to the norm operator is radially unbounded, and therefore incremental properties are obtained globally in the state space. On the other hand, the upper bound in (4a) is introduced for solutions to be uniformly decreasing with respect to time and to correlate the Riemaniann distance in P to the Euclidean one in (3).

B. Properties of incrementally stable systems

In this section, we recall some properties of incrementally stable systems. In particular, we will focus on the invariance with respect to diffeomorphisms. Incremental stability is a property that is preserved via diffeomorphism. This has been shown in [1, Proposition 4.6] in case convergence between trajectories is only asymptotic. Following the same lines, we provide such a result.

Lemma 1 (Invariance via diffeomorphism)**.** *Consider system* (2) *and assume that it is δGES. Let $\Phi : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a globally Lipschitz diffeomorphism with globally Lipschitz inverse Ψ, namely such that for some $L_\Phi, L_\Psi > 0$,*

$$\Phi(\Psi(z)) = z \qquad (5a)$$

$$|\Phi(x') - \Phi(x'')| \le L_\Phi |x' - x''|, \qquad (5b)$$

$$|\Psi(z') - \Psi(z'')| \le L_\Psi |z' - z''| \qquad (5c)$$

for all $z, x', x'', z', z'' \in \mathbb{R}^n$. Consider the change of coordinates $x \mapsto z := \Phi(x)$ so that system (2) *reads*

$$\dot{z} = \varphi(z) := \frac{\partial \Phi}{\partial x}(\Psi(z)) f(\Psi(z)). \qquad (6)$$

Then also (6) *is δGES.*

Remark 3. *The Lipschitz bounds* (5) *can be weakened by asking for the existence of two class-$\mathcal{K}_\infty$ functions γ_Φ, γ_Ψ such that $|\Phi(x) - \Phi(y)| \le \gamma_\Phi(|x-y|)$ and $|\Psi(x) - \Psi(y)| \le \gamma_\Psi(|x-y|)$. In such a case, however, uniformity in the difference of initial conditions in* (3) *is not preserved.*

If incremental properties are claimed using Theorem 1, then the construction of the metric in the new coordinates $z = \Phi(x)$ can be computed as follows. A proof can be found for instance in [25, Lemma 2] or [13].

Lemma 2 (Metric in different coordinates)**.** *Consider system* (2) *and assume there exists a C^1 function taking values in $\mathbb{S}^n$ for any $x \in \mathbb{R}^n$ and satisfying* (4). *Let $\Phi : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a global diffeomorphism satisfying* (5). *Then, the system* (6) *satisfies*

$$\underline{q}I \preceq Q(z) \preceq \bar{q}I, \qquad (7a)$$

$$L_\varphi Q(z) \preceq -2\lambda Q(z) \qquad (7b)$$

for all $z \in \mathbb{R}^n$ and for some real numbers $\bar{q}, \underline{q} > 0$, where

$$Q(z) = \left(\frac{\partial \Phi}{\partial x}(\Psi(z))\right)^{-\top} P(\Psi(z)) \left(\frac{\partial \Phi}{\partial x}(\Psi(z))\right)^{-1}. \qquad (8)$$

Note that such a theorem have been used for instance in the context of incremental forwarding, see [16].

III. Incremental Input-to-State Stability

A. Main definition

In this section, we study nonlinear systems whose dynamics are affected by an external input u. Such an input may represent a control action and/or a disturbance. In particular, we consider input-affine systems defined by

$$\dot{x} = f(x) + g(x)u \qquad (9)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathcal{U}$ is an exogenous signal taking values in $\mathcal{U} \subseteq \mathbb{R}^m$ and f, g are C^2 functions. We denote by $\mathcal{X}(x_0,u,t)$ the solution of system (9) at time t starting at initial condition $x_0 \in \mathbb{R}^n$ with input $u = u(t)$ and satisfying the initial value problem

$$\begin{aligned}&\tfrac{\partial}{\partial t}\mathcal{X}(x_0,u,t) = f(\mathcal{X}(x_0,u,t)) + g(\mathcal{X}(x_0,u,t)u(t),\\ &\mathcal{X}(x,u,t,t_0) = x_0.\end{aligned} \tag{10}$$

Definition 2 (δISS)**.** *System* (9) *is incrementally input-to-state stable (δISS) with global exponential rate if there exist positive real numbers $k, \lambda, \gamma > 0$ such that*

$$\begin{aligned}&|\mathcal{X}(x_1,u_1,t) - \mathcal{X}(x_2,u_2,t)|\\ &\qquad \le k\,|x_1 - x_2|\,e^{-\lambda t} + \gamma \sup_{s\in[t_0,t]} |u_1(s) - u_2(s)|\end{aligned} \tag{11}$$

for all initial conditions $x_1, x_2 \in \mathbb{R}^n$ and for all inputs u_1, u_2 taking values in $\mathcal{U}$ for all $t \ge 0$.

In this section, we aim to look for some metric-based sufficient conditions to have a δISS property. For this, we introduce the notion of Killing vector field[1].

Definition 3 (Killing vector field)**.** *Given a C^1 2-tensor $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and a C^1 vector field $g : \mathbb{R}^n \to \mathbb{R}^n$, we say that g is a Killing vector field with respect to P if*

$$L_g P(x) = 0 \qquad \forall\, x \in \mathbb{R}^n. \tag{12}$$

Remark 4. *In case g is a C^1 matrix function i.e. $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, we say that g is a Killing vector field (or it possesses the Killing vector field property), if $L_{g_i} P(x) = 0$ for all $x \in \mathbb{R}^n$ and $i = 1,\dots,m$ with g_i denoting the i-th column of g.*

In a few words, such a Killing Vector property implies that Riemannian distances between different trajectories generated by the vector field $g(x)$ are invariant. Basically, the signals entering in the directions of the vector field g do not affect the distances, in the sense that different trajectories of the differential equation $\dot{x} = g(x)u$ have a distance (associated with the norm provided by P) among them which is constant for any $t \ge 0$, i.e. the flow of the vector field g is an isometry with respect to the Riemaniann metric associated with P. Such a notion is not new in literature in control theory. For instance, it has been used in the context of control contraction metrics design in [23, Section III.A], or in output regulation and synchronization problems in [16], [19].

B. Sufficient conditions for δISS

In this section, we look for sufficient conditions for a system of the form (9) to possess an δISS property. To the best of the knowledge of the author of this manuscript, sufficient conditions for a system to be δISS have been given in a general framework only using a Lyapunov-based analysis in [1], [2] and with weak pairing in [7]. Nonetheless, similar concepts have been exploited (although not in an explicit way) in [28], [29] in the context of observer design for systems subject to noise and in [11] for output regulation. Both cases, however, are restricted to the case in which the metric is Euclidean, i.e. the matrix P in (12) is constant. The Killing Vector property can be used to derive sufficient conditions for a system of the form (9) to be δISS, as stated in the following and whose proof can be found in [25, Theorem 2].

Theorem 2 (δGES + Killing vector $\implies$ δISS)**.** *Consider system* (9) *and assume there exists a C^1 function taking values in $\mathbb{S}^n$ for any $x \in \mathbb{R}^n$ and satisfying* (4)*. Furthermore, suppose that g is a bounded and a Killing vector for P, namely*

$$|g(x)| \le \bar{g}, \quad L_g P(x) = 0, \quad \forall x \in \mathbb{R}^n. \tag{13}$$

Then, system (9) *is δISS with respect to $u \in \mathcal{U}$.*

Remark 5. *The same result hold in case system* (9) *is reads*

$$\dot{x} = f(x) + g(x)\rho(u)$$

with ρ satisfying $|\rho(u_1) - \rho(u_2)| \le \delta_\rho(|u_1 - u_2|)$ for some class-$\mathcal{K}_\infty$ function δ_ρ. In this case, inequality (11) *becomes*

$$\begin{aligned}&|\mathcal{X}(x_1,u_1,t) - \mathcal{X}(x_2,u_2,t)|\\ &\qquad \le k\,|x_1 - x_2|\,e^{-\lambda t} + \sup_{s\in[t_0,t]} \bar{\gamma}\left(|u_1(s) - u_2(s)|\right)\end{aligned}$$

for some class-$\mathcal{K}$ function $\bar{\gamma}$.

Example 1. *We now provide some trivial examples of δISS.*
1) Any linear systems of the form $\dot{x} = Ax + Bu$ with A Hurwitz.
2) Any system of the form (9) *with $u = 0$ that satisfies Theorem 1 with respect to a constant P, is also δISS with respect to any u if g is a constant matrix (in this case the the Killing vector property with respect to such a P is always verified).*
3) For any C^1 scalar vector field $g : \mathbb{R} \mapsto \mathbb{R}$ such that $g(x) \neq 0$ for all x, then $P(x) = g^{-2}(x)$ always satisfies the Killing vector property. Indeed

$$\begin{aligned}L_g P(x) &= \frac{\partial P}{\partial x}(x)g(x) + 2\frac{\partial g}{\partial x}(x)P(x)\\ &= \frac{\partial g^{-2}}{\partial x}(x)g(x) + 2\frac{\partial g}{\partial x}(x)g^{-2}(x)\\ &= -2\frac{g'(x)}{g^3(x)}g(x) + 2\frac{g'(x)}{g^2(x)} = 0\,.\end{aligned}$$

4) Any vector field g for which $\frac{\partial g}{\partial x}(x)$ is anti-symmetric is a Killing vector field for the identity matrix. An example is given by a linear oscillator (this can be seen by representing the trajectories of the system in polar coordinates).
5) In general, finding a matrix function P satisfying Theorem 2 (if it does exist) may not be a trivial task. Similarly to the problem of finding a Lyapunov function for equilibrium stability, the problem of finding a suitable P is biased by the considered framework. Indeed, existing results take advantage of the structure of the considered nonlinear system.

[1]The notion of Killing vector field takes the name from Wilhelm Killing, a German mathematician (see, for instance, [26], [27]).

Examples can be found in [15] for linear systems coupled with a monotonic nonlinearity and in [16] for systems in feedforward form.

To conclude, note that the conditions in Theorem 2 are only sufficient and not necessary. The converse result is indeed not true and follows from items 2 and 3 of Example 1. This is shown in the following (trivial) counter-example.

Example 2. *Let $x \in \mathbb{R}$ and $u, v \in \mathbb{R}$ and consider the system*

$$\dot{x} = -x + u + g(x)v\,.$$

For a certain metric P to satisfy the Killing vector property with respect to both the control gain of inputs u and v at the same time, we need $P = g^{-2}(x)$ to be constant as well, which requires $g(x)$ to be constant. Yet, this system can be δISS w.r.t. both u and v.

Anyway, even if the former condition is only sufficient and not necessary, is still possible to show that a system that is open-loop δGES is also δISS, provided that the $u(t)$ remains sufficiently small, see e.g. [30]. This property is similar to the fact that GAS implies ISS w.r.t. small inputs [31].

C. Transverse ISS

Consider now a system of the form

$$\dot{x} = f(x) + g(x)u \tag{14a}$$
$$\dot{\tilde{x}} = \left[\frac{\partial f}{\partial x}(x) + \frac{\partial g}{\partial x}(x)u\right]\tilde{x} + g(x)\tilde{u} \tag{14b}$$

We indicate with $\mathcal{X}(x, u, t)$ the solution of (10) and with $\widetilde{\mathcal{X}}(\tilde{x}, \tilde{u}, x, u, t)$ the solution of the $\tilde{x}$-dynamics. In other words, we look now at system (14a) and its *variational* (or lifted) system (14b) and we study ISS-like properties for the manifold

$$\mathcal{M} := \{(x, \tilde{x}, u, \tilde{u}) \mid (\tilde{x}, \tilde{u}) = (0, 0)\}.$$

Exponential stability properties for such a class of systems have been studied in [10] with an approach based on Riemaniann metrics. Following similar tools as in Section III-B we aim to study ISS-like properties. We have the following result, whose proof can be found in [25, Proposition 1].

Proposition 1 (δISS $\implies$ Transverse ISS)**.** *Consider system* (14) *and assume that f and g have bounded first and second derivatives. Assume moreover that the x-dynamics* (14a) *is δISS (i.e. inequality* (11) *is satisfied). Then the lifted system* (14b) *is exponentially ISS with respect to $\tilde{u}$, uniformly with respect to u and x, namely,*

$$\left|\widetilde{\mathcal{X}}(\tilde{x}, \tilde{u}, x, u, t)\right| \leq k e^{-\lambda t}|\tilde{x}| + \gamma \sup_{s\in[t_0,t]} |\tilde{u}(s)| \tag{15}$$

for all $t \geq 0$, where $k, \gamma \in \mathbb{R}_{>0}$ do not depend on (x, u) and are given in (11).

IV. δISS FEEDBACK DESIGN

Consider again a system of the form

$$\dot{x} = f(x) + g(x)u \tag{16}$$

with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$. As a concluding result, we aim to show that if a Riccati-like inequality is verified, together with a Killing vector property, then under some mild assumptions it's possible to design a feedback control law which is of *infinite gain margin* (see e.g. [32], [33]). Specifically, we say that $\psi : \mathbb{R}^n \to \mathbb{R}^m$ is an infinite-gain margin contractive feedback if the following system

$$\dot{x} = f(x) + \kappa g(x)\psi(x)$$

is δGES (according to Definition 1) for any $\kappa \geq 1$. Such a property is desirable in practical applications since closed-loop stability is preserved in presence of static and/or unmodeled fast dynamic plant uncertainties (see [33, Section 3]).

The intuition why such a Killing property allows to design an infinite gain margin control law can be seen from the proof of Theorem 2. Indeed, if g is a Killing vector field with respect to a certain P, then the flow of the vector field is an isometry with respect to such a P, meaning that, by defining the distance d_P as

$$\mathrm{d}_P(x_1, x_2) := \inf_{\gamma} \int_0^1 |\gamma(s)|_{P(\gamma(s))}\ ds$$

with $\gamma : [0, 1] \mapsto \mathbb{R}^n$ being any sufficiently smooth curve such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$ for any two points $x_1, x_2 \in \mathbb{R}^n$, we have that

$$d_P(x_1, x_2) = d_P(g(x_1), g(x_2))$$

for any x_1, x_2. This can be claimed by showing that the time-derivative of the distance is zero, which implies that d_P is constant over time along the flow induced by g and that this is true for all values of κ from the property of the Lie derivative $L_{\kappa g}P(x) = \kappa L_g P(x)$. We have then the following sufficient condition, whose proof can be found in [25, Theorem 3].

Theorem 3. *Consider system* (16) *and assume there exist C^1 functions P, M taking values in $\mathbb{S}^n$, resp. $\mathbb{S}^m$, for any $x \in \mathbb{R}^n$, a C^2 function $\psi : \mathbb{R}^n \mapsto \mathbb{R}^m$ and three positive real numbers $\underline{p}, \overline{p}, \lambda > 0$ such that* (4a), (13) *hold and moreover*

$$L_f P(x) - 2P(x)g(x)M(x)g^\top(x)P(x) \preceq -2\lambda P(x) \tag{17a}$$
$$\frac{\partial \psi}{\partial x}^\top(x) = M(x)P(x)g(x) \tag{17b}$$

for all $x \in \mathbb{R}^n$. Then, the feedback law $u = \psi(x)$ is an infinite-gain margin contractive feedback for system (16).

Note that for linear systems of the form $\dot{x} = Ax + Bu$, the assumptions of Theorem 3 boil down to the solution of the Riccati inequality $PA + A^\top P - 2PBMB^\top P \preceq -2\lambda P$, which admits a solution for any $M \succ 0$ if the pair (A, B) is stabilizable.

Remark 6. *The integrability condition* (17b)*, it is introduced since the stability analysis is performed on the Jacobian of the closed-loop system due to the fact that the control design aims to achieve incremental properties for the closed-loop. For linear systems, the existence of ψ satisfying* (17b) *is always guaranteed by linearity. As shown in [19], such an integrability condition can be relaxed at the prize of loosing the inifinite-gain margin property. In particular, by requiring*

$$\left|\frac{\partial \psi}{\partial x}^{\top}(x) - M(x)P(x)g(x)\right| \leq \varepsilon, \qquad \forall x \in \mathbb{R}^n,$$

for some $\varepsilon > 0$, one can show that the feedback law $u = \kappa\psi(x)$ is a contractive feedback for any $\kappa \in [1, \bar{\kappa})$ for some $\bar{\kappa}$ that depends on ε.

As final remark, we note that similar conditions have been also provided in [23, Section III.A]. Differently from the former work, however, our design provides additional degrees of freedom due to the fact that M is a matrix function and the design is of infinite gain margin thanks to the non-negativity of M itself.

From Theorem 2 and Theorem 3, the following result trivially holds.

Corollary 1. *Let Theorem 3 hold and let $\mathcal{V} \subset \mathbb{R}^{n_u}$ be compact. Then, the system*

$$\dot{x} = f(x) + g(x)\big(\kappa\psi(x) + v\big)$$

is δISS with respect to any $v \in \mathcal{V}$ for any $\kappa \geq 1$.

V. Conclusions

In this paper, we studied the notion of incremental input-to-state stability for input-affine nonlinear systems. We showed that it is possible to obtain a set of sufficient conditions using a Riemannian metric approach for a system to be incremental input-to-state stable with respect to an external input. This has been done through the notion of Killing vector field. Then, we showed that the lifted system an incremental ISS system possesses also some ISS-like properties, which we called transversal ISS. We then proposed a set of sufficient conditions to obtain an infinite gain margin feedback achieving δISS properties for the closed-loop. Future studies will try to derive sufficient conditions for systems that are not necessarily input affine by weakening the Killing vector assumption, and more general design tools to provide control actions making a system incremental ISS.

References

[1] D. Angeli, "A Lyapunov approach to incremental stability properties," *IEEE Transactions on Automatic Control*, vol. 47, no. 3, pp. 410–421, 2002.

[2] ——, "Further results on incremental input-to-state stability," *IEEE Transactions on Automatic Control*, vol. 54, no. 6, pp. 1386–1391, 2009.

[3] F. Forni and R. Sepulchre, "A differential Lyapunov framework for contraction analysis," *IEEE Transactions on Automatic Control*, vol. 59, no. 3, pp. 614–628, 2013.

[4] D. Wu and G. Duan, "Further geometric and Lyapunov characterizations of incrementally stable systems on Finsler manifolds," *IEEE Transactions on Automatic Control*, 2021.

[5] E. Sontag, "Contractive systems with inputs," in *Perspectives in mathematical system theory, control, and signal processing*. Springer, 2010, pp. 217–228.

[6] Z. Aminzare and E. Sontag, "Contraction methods for nonlinear systems: A brief introduction and some open problems," in *53rd IEEE Conference on Decision and Control*, 2014, pp. 3835–3847.

[7] A. Davydov, S. Jafarpour, and F. Bullo, "Non-Euclidean contraction theory for robust nonlinear stability," *IEEE Transactions on Automatic Control*, vol. 67, no. 12, pp. 6667–6681, 2022.

[8] W. Lohmiller and J. E. Slotine, "On contraction analysis for non-linear systems," *Automatica*, vol. 34, no. 6, pp. 683–696, 1998.

[9] J. Simpson-Porco and F. Bullo, "Contraction theory on Riemannian manifolds," *Systems & Control Letters*, vol. 65, pp. 74–80, 2014.

[10] V. Andrieu, B. Jayawardhana, and L. Praly, "Transverse exponential stability and applications," *IEEE Transactions on Automatic Control*, vol. 61, no. 11, pp. 3396–3411, 2016.

[11] A. Pavlov, N. Van De Wouw, and H. Nijmeijer, *Uniform output regulation of nonlinear systems: a convergent dynamics approach*. Springer, 2006, vol. 205.

[12] B. S. Rüffer, N. Van De Wouw, and M. Mueller, "Convergent systems vs. incremental stability," *Systems & Control Letters*, vol. 62, no. 3, pp. 277–285, 2013.

[13] R. Sanfelice and L. Praly, "Convergence of nonlinear observers on $\mathbb{R}^n$ with a Riemannian metric (Part I)," *IEEE Transactions on Automatic Control*, vol. 57, no. 7, pp. 1709–1722, 2011.

[14] B. Yi, R. Wang, and I. Manchester, "Reduced-order nonlinear observers via contraction analysis and convex optimization," *IEEE Transactions on Automatic Control*, vol. 67, no. 8, pp. 4045–4060, 2021.

[15] M. Giaccagli, V. Andrieu, S. Tarbouriech, and D. Astolfi, "LMI conditions for contraction, integral action, and output feedback stabilization for a class of nonlinear systems," *Automatica*, vol. 154, p. 111106, 2023.

[16] M. Giaccagli, D. Astolfi, V. Andrieu, and L. Marconi, "Sufficient conditions for global integral action via incremental forwarding for input-affine nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 67, no. 12, pp. 6537–6551, 2022.

[17] J. Reinders, M. Giaccagli, B. Hunnekens, D. Astolfi, T. Oomen, and N. van De Wouw, "Repetitive control for Lur'e-type systems: application to mechanical ventilation," *IEEE Transactions on Control Systems Technology*, 2023.

[18] P. Lorenzetti and G. Weiss, "PI control of stable nonlinear plants using projected dynamical systems," *Automatica*, vol. 146, p. 110606, 2022.

[19] M. Giaccagli, S. Zoboli, D. Astolfi, V. Andrieu, and G. Casadei, "Synchronization in networks of nonlinear systems:contraction analysis via Riemannian metrics and deep-learning for feedback estimation," *Submitted to IEEE Transaction on Automatic Control*, 2023.

[20] V. Andrieu, B. Jayawardhana, and S. Tarbouriech, "Some results on exponential synchronization of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 63, no. 4, pp. 1213–1219, 2018.

[21] Z. Aminzare and E. Sontag, "Synchronization of diffusively-connected nonlinear systems: Results based on contractions with respect to general norms," *IEEE Transactions on Network Science and Engineering*, vol. 1, no. 2, pp. 91–106, 2014.

[22] J. E. Slotine and W. Wang, "A study of synchronization and group cooperation using partial contraction theory," in *Cooperative control*. Springer, 2005, pp. 207–228.

[23] I. Manchester and J. Slotine, "Control contraction metrics: Convex and intrinsic criteria for nonlinear feedback design," *IEEE Transactions on Automatic Control*, vol. 62, no. 6, pp. 3046–3053, 2017.

[24] T. Sakai, *Riemannian geometry*. American Mathematical Soc., 1996, vol. 149.

[25] M. Giaccagli, D. Astolfi, and V. Andrieu, "Further results on incremental input-to-state stability based on contraction-metric analysis," *hal-04196098*, 2023.

[26] K. Nomizu, "On local and global existence of Killing vector fields," *Annals of Mathematics*, pp. 105–120, 1960.

[27] K. Yano, "On harmonic and Killing vector fields," *Annals of Mathematics*, pp. 38–45, 1952.

[28] A. Chakrabarty, M. Corless, G. Buzzard, S. Żak, and A. Rundell, "State and unknown input observers for nonlinear systems with bounded

exogenous inputs," *IEEE Transactions on Automatic Control*, vol. 62, no. 11, pp. 5497–5510, 2017.

[29] A. Chakrabarty and M. Corless, "Estimating unbounded unknown inputs in nonlinear systems," *Automatica*, vol. 104, pp. 57–66, 2019.

[30] S. Zoboli, S. Janny, and M. Giaccagli, "Deep learning-based output tracking via regulation and contraction theory," 22^{nd} *World Conference of the International Federation of Automatic Control (IFAC)*, 2022.

[31] E. D. Sontag and Y. Wang, "New characterizations of input-to-state stability," *IEEE transactions on automatic control*, vol. 41, no. 9, pp. 1283–1294, 1996.

[32] M. Giaccagli, V. Andrieu, S. Tarbouriech, and D. Astolfi, "Infinite gain margin, contraction and optimality: an LMI-based design," *European Journal of Control*, vol. 68, p. 100685, 2022.

[33] R. Sepulchre, M. Jankovic, and P. V. Kokotovic, *Constructive nonlinear control*. Springer Science & Business Media, 2012.