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A data-driven approach to UIO-based fault diagnosis

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***Abstract*— In this paper we propose a data-driven approach to the design of a residual generator, based on a dead-beat unknown-input observer, for linear time-invariant discrete-time state-space models, whose state equation is affected both by disturbances and by actuator faults. We first review the model-based conditions for the existence of such a residual generator, and then prove that under suitable assumptions on the collected historical data, we are both able to determine if the problem is solvable and to identify the matrices of a possible residual generator. We propose an algorithm that, based only on the collected data (and not on the system description), is able to perform both tasks. An illustrating example concludes the paper.**

***Index Terms*— Fault detection and identification, actuator fault, data-driven methods.**

I. Introduction

Data-driven approaches to the solution of control problems are quite pervasive in nowadays literature. In particular, data-driven fault detection (FD) methods have been the subject of a significant number of papers [3], [8], [9], [7], [17] Since the early works of J.F. Dong and M. Verhaegen [12] and of S.X. Ding et al. [10], FD schemes that are designed directly based on collected data, by avoiding the preliminary step of system identification, have been proposed. Most of the available results, however, focus on detection, rather than identification, [10], [9] and provide algorithms to design the matrices of a residual generator, based for instance on QR-decomposition and SVD [9]. Also, necessary and sufficient conditions for the problem solvability have typically been expressed in terms of the original system matrices, and not on the available data. This means that the assumption that fault detection and identification (FDI) can be successfully performed on the system is taken for granted and not directly checked on data.

Leveraging some recent results on data-driven unknown-input observer (UIO) design (see [19] or [11]), we propose a data-driven approach to the design of a residual generator, based on a dead-beat UIO [1], [15], for a generic linear time invariant state-space model, whose state equation is affected both by disturbances and by faults. We first provide necessary and sufficient conditions for the problem solvability, by adopting a model-based approach that relies on classic results by J. Chen and R.J. Patton [4]. Secondly, we show that under a suitable assumption on the data, that ensures that they are representative of the original system trajectories, we derive necessary and sufficient conditions for the problem solvability in terms of the available data. Such conditions are weaker than the conditions that guarantee the identifiability of the original system matrices. In fact, with the available data we would not be able to identify the original system, since we cannot measure the disturbance. Finally, we provide an algorithm to derive the matrices of a dead beat UIO-based residual generator that allows to identify any actuator fault.

As previously mentioned, the paper leverages the results about asymptotic UIO proposed in [19] or [11], but significantly advances them in three aspects: first, it provides a way to check on the data (only) the problem solvability. Secondly, it provides a practical algorithm to design the matrices of a possible dead-beat UIO (while the previous references focused only on existence conditions). Thirdly, it extends the analysis to FDI. The more restrictive choice of focusing on dead-beat UIOs, rather than asymptotic UIOs, is meant to provide a cleaner set-up, that allows for exact solutions, since one does not need to account for the contribution of the estimation error when trying to identify the fault.

The paper is organised as follows. In Section II we introduce the overall set-up and formally state the FDI problem we address in this paper, that assumes that disturbances and faults affect only the state update. Section III recalls the model-based solution to the design of a residual generator based on a dead-beat UIO, by suitably adjusting the one available in the literature for asymptotic UIOs. We also provide necessary and sufficient conditions for the existence of a residual generator, based on the dead-beat UIO, that allows to uniquely identify the fault from the residual signal. There are high chances that this specific result is already known in the literature, but since we were not able to find a reference, we provided a short proof to lay on solid ground the subsequent data-driven analysis that relies on this result. Under a rather common assumption on the data (see Assumption 2), that can be related to the persistence of excitation [20], [21] of the system inputs, in Section IV we first provide data-based necessary and sufficient conditions for the problem solvability, and then, by resorting to a couple of technical results, we provide a simple Algorithm that allows to first check on data the problem solvability conditions, and then provides matrices of a dead-beat UIO-based residual generator. An example concludes the paper. Due to page constraints, we refer to [13] for the proof of Proposition 7 as well as for a technical lemma.

Notation. Given two integers $h, k \in \mathbb{Z}$, with $h \leq k$, we let $[h, k]$ denote the set $\{h, h+1, \dots, k\}$. Given a matrix $M \in \mathbb{R}^{n \times m}$, we denote by $M^\top$ its transpose, and by $M^\dagger$ its Moore-Penrose inverse [2]. If M is of full column rank

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(FCR), then $M^\dagger = [M^\top M]^{-1} M^\top$. A similar expression can be provided in the case when M is of full row rank (FRR). Given an n-dimensional vector sequence $\{\omega(k)\}_{k\in\mathbb{Z}_+}$ taking values in $\mathbb{R}^n$ and a positive integer N, we set

$$\omega_N(k) \doteq \begin{bmatrix} \omega(k) \\ \omega(k+1) \\ \vdots \\ \omega(k+N-1) \end{bmatrix} \in \mathbb{R}^{Nn}. \quad (1)$$

II. UIO-Based Fault Detection and Identification: Problem Statement

Consider an linear time invariant discrete-time dynamical system Σ, described by the state-space model:

$$x(k+1) = Ax(k) + Bu(k) + Ed(k) + Bf(k), \quad (2a)$$
$$y(k) = Cx(k), \qquad k \in \mathbb{Z}_+, \quad (2b)$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^p$, $d(k) \in \mathbb{R}^r$ and $f(k) \in \mathbb{R}^m$ are the state, input, output, disturbance and actuator fault signals, respectively, while $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $E \in \mathbb{R}^{n\times r}$. We assume that E has rank r. Indeed, if $\operatorname{rank} E = \bar{r} < r$, we can rewrite it as $E = \bar{E}T$, where $\bar{E} \in \mathbb{R}^{n\times\bar{r}}$ is FCR and $T \in \mathbb{R}^{\bar{r}\times r}$ is FRR, and define as new unknown input $\bar{d}(t) \doteq Td(t)$.

We assume for the UIO-based residual generator the following description [4], [14]:

$$z(k+1) = A_{UIO}z(k) + B^u_{UIO}u(k) + B^y_{UIO}y(k), \quad (3a)$$
$$\hat{x}(k) = z(k) + D_{UIO}y(k), \quad (3b)$$
$$r(k) = y(k) - C\hat{x}(k), \qquad k \in \mathbb{Z}_+, \quad (3c)$$

where $z(k) \in \mathbb{R}^n$ is the residual generator state vector, $\hat{x}(k) \in \mathbb{R}^n$ denotes the state estimate and $r(k) \in \mathbb{R}^p$ the residual signal. The real matrices $A_{UIO}, B^u_{UIO}, B^y_{UIO}$ and D_{UIO} have dimensions compatible with the previously defined vectors.
If we let $e(k) \doteq x(k) - \hat{x}(k)$ denote the state estimation error at time k, then our goals are:

i) We want to ensure that there exists $k_0 > 0$ such that, when the system is not affected by actuator faults, for every initial condition $e(0)$, and every input and disturbance sequences applied to the original system Σ, the error $e(k)$ is zero for every $k \geq k_0$ (and hence so is the residual signal). This means that equations (3a) and (3b) describe a dead-beat UIO [1], [15] for Σ.

ii) If a fault affects the system from some $k_f \geq k_0$ onward, then the residual signal $r(k)$ becomes nonzero (fault detection) and the knowledge of the residual allows to uniquely identify the fault that has affected Σ (fault identification).

III. Model Based Approach

A. Dead-beat UIO

In the following we will steadily work under the following:

Assumption 1. *We assume that*

A. $\operatorname{rank}(CE) = \operatorname{rank}(E) = r$;

B. $\operatorname{rank}\left(\begin{bmatrix} zI_n - A & -E \\ C & \mathbb{0}_{p\times r}\end{bmatrix}\right) = n + r$, $\forall\, z \in \mathbb{C}, |z| \neq 0$.

Assumption 1 corresponds to what in the sequel we will call *strong* reconstructability property* of the triple (A, E, C), due to its relationship with the well known *strong* detectability property*, as defined in [5], [16].

Proposition 1. *There exists a dead-beat UIO for system Σ described as in* (3a) - (3b) *if and only if Assumption 1 holds.*

Proof: The proof is a slight modification of the one for asymptotic UIOs (see [5], [6]) and we report it here in concise form. By making use of the equations describing the system and the UIO, we deduce that the estimation error evolves according to the following difference equation:

$$\begin{aligned} e(k+1) =& [(I_n - D_{UIO}C)A - A_{UIO}(I_n - D_{UIO}C) \\ &- B^y_{UIO}C]x(k) + (I_n - D_{UIO}C)Ed(k) \\ &+ A_{UIO}e(k) + (I_n - D_{UIO}C)Bf(k) \\ &+ [(I_n - D_{UIO}C)B - B^u_{UIO}]u(k). \end{aligned} \quad (4)$$

If no fault affects the system Σ (namely $f(k) = 0$ for every $k \in \mathbb{Z}_+$), the estimation error converges to zero in a finite number of steps, independently of the initial conditions of the system and the UIO, of the input u and of the disturbance d, if and only if (*iff*) the matrices A_{UIO}, B^u_{UIO}, B^y_{UIO} and D_{UIO} satisfy the following constraints:

$$(I_n - D_{UIO}C)A - A_{UIO}(I_n - D_{UIO}C) = B^y_{UIO}C \quad (5a)$$
$$B^u_{UIO} = (I_n - D_{UIO}C)B \quad (5b)$$
$$(I_n - D_{UIO}C)E = \mathbb{0}_{n\times r} \quad (5c)$$
$$A_{UIO} \text{ nilpotent} \quad (5d)$$

By making use of the analysis in [5], [6], we can claim that equations (5) hold iff Assumption 1 holds. □

As a result of Proposition 1, under Assumption 1 there exist matrices A_{UIO}, B^u_{UIO}, B^y_{UIO} and D_{UIO} satisfying (5) and hence the error dynamics becomes

$$e(k+1) = A_{UIO}e(k) + B^u_{UIO}f(k),$$

where A_{UIO} is nilpotent. In particular, if $f(k) = 0$, the equation becomes $e(k+1) = A_{UIO}e(k)$. This implies that after a finite number of steps the estimation error becomes a function of the fault signal only.

Remark 2. *Assumption 1.A is necessary and sufficient for the solvability of* (5c)*. Once a solution to* (5c) *is obtained, conditions* (5a) *and* (5b) *immediately follow. The general solution of* (5c) *is given in [6].*
If, in addition, Assumption 1.B holds, then (see [6], Theorem 2) among the matrices D_{UIO} satisfying (5c) *there exist some for which the pair $((I_n - D_{UIO}C)A, C)$ is reconstructable. This is always the case if $\operatorname{rank}(I_n - D_{UIO}C) = n - r$ (which also means that $\ker(I_n - D_{UIO}C) = \operatorname{Im}(E)$). Lemma 11 in [13] proves that if D_{UIO} is a solution of* (5c) *of rank r (e.g., $D_{UIO} = E(CE)^\dagger$), then $\operatorname{rank}(I_n - D_{UIO}C) = n - r$. This implies that if we choose a solution D_{UIO} of rank r for* (5c)*, then the pair $((I_n - D_{UIO}C)A, C)$ is reconstructable.*

When so, a possible way to compute A_{UIO} is the following one. From (5a), we deduce that A_{UIO} can be expressed as

$$\begin{aligned} A_{UIO} &= (I_n - D_{UIO}C)A - (B^y_{UIO} - A_{UIO}D_{UIO})C \\ &= (I_n - D_{UIO}C)A - LC, \end{aligned}$$

for $L \doteq B^y_{UIO} - A_{UIO}D_{UIO}$. Consequently, we first choose L so that $A_{UIO} = (I_n - D_{UIO}C)A - LC$ is nilpotent and then deduce B^y_{UIO} as $B^y_{UIO} = L + A_{UIO}D_{UIO}$.

To summarise, by making use of Theorem 2 in [6] and the assumption that (A, E, C) is strong* reconstructable, we first find a solution D_{UIO} of (5c) of rank r. Then we determine L such that $A_{UIO} = (I_n - D_{UIO}C)A - LC$ is nilpotent and finally we set $B^y_{UIO} = L + A_{UIO}D_{UIO}$ and $B^u_{UIO} = (I_n - D_{UIO}C)B$.
In the sequel, we will always assume that $\text{rank}(D_{UIO}) = r$.

B. Fault Detection and Identification

Under the hypothesis that A_{UIO} is nilpotent, in the absence of faults, the estimation error $e(k)$ goes to zero in a finite number of steps k_0. So, if we assume that the time k_f at which the fault affects the system is greater than or equal to k_0, then $e(k_f) = \mathbb{0}_n$. Therefore, the dynamics of the estimation error for $k \geq k_f$ is

$$\begin{aligned} e(k+1) &= A_{UIO}e(k) + B^u_{UIO}f(k), \\ r(k) &= Ce(k), \end{aligned}$$

with $e(k_f) = \mathbb{0}_n$, and $\forall\ k > k_f$, the residual signal is

$$r(k) = \sum_{i=k_f}^{k-1} CA^{k-i-1}_{UIO}B^u_{UIO}f(i),$$

in particular, $r(k_f + 1) = CB^u_{UIO}f(k_f)$, and hence, it is immediate to see that we are able to uniquely identify every fault from the residual (with one step delay), iff CB^u_{UIO} is FCR. It is worth noting that the FCR condition on CB^u_{UIO} remains necessary and sufficient for the fault identification even if we consider a longer time window. Indeed, if we assume as observation window $[k_f + 1, k_f + N]$, then the family of residual signals $r(k), k \in [k_f + 1, k_f + N]$, can be described as follows:

$$r_N(k_f + 1) = M_N f_N(k_f),$$

where $r_N(k_f + 1)$ and $f_N(k_f)$ are defined as in (1), and

$$M_N \doteq \begin{bmatrix} CB^u_{UIO} & & & \\ CA_{UIO}B^u_{UIO} & CB^u_{UIO} & & \\ \vdots & \ddots & \ddots & \\ CA^{N-1}_{UIO}B^u_{UIO} & \dots & CA_{UIO}B^u_{UIO} & CB^u_{UIO} \end{bmatrix}.$$

We can identify the vector $f_N(k_f) \in \mathbb{R}^{Nm}$ from the residual vector $r_N(k_f + 1) \in \mathbb{R}^{Np}$ iff the matrix $M_N \in \mathbb{R}^{Np \times Nm}$ is FCR and this is the case iff $CB^u_{UIO} \in \mathbb{R}^{p \times m}$ is FCR.

A necessary and sufficient condition for the matrix CB^u_{UIO} to be FCR is given in Proposition 3, below.

Proposition 3. *Suppose that Assumption 1.A holds. Then the following facts are equivalent:*

i) $\text{rank}([\,CB\ CE\,]) = m + r$;
ii) $CB^u_{UIO} = C(I_n - D_{UIO}C)B$ *is FCR.*

Proof: i) $\Rightarrow$ ii) follows immediately from Lemma 11 in [13].

ii) $\Rightarrow$ i) Assume that $\text{rank}([\,CB\ CE\,])$ is smaller than $m + r$. Then there exists a nonzero vector $v = [\,v_B^\top\ v_E^\top\,]^\top$ such that

$$\begin{bmatrix} CB & CE \end{bmatrix} \begin{bmatrix} v_B \\ v_E \end{bmatrix} = \mathbb{0}_p. \tag{6}$$

Note that v_B cannot be zero, otherwise (6) would become $CEv_E = \mathbb{0}_p$ and this would contradict Assumption 1.A. Consequently,

$$\mathbb{0}_p \quad = \quad (I_p - CD_{UIO}) \begin{bmatrix} CB & CE \end{bmatrix} \begin{bmatrix} v_B \\ v_E \end{bmatrix} = CB^u_{UIO}v_B.$$

Since $v_B \neq \mathbb{0}_m$, this implies that CB^u_{UIO} is not FCR. ☐

The results so far obtained can be summarised in the following proposition.

Proposition 4. *Under Assumption 1, there exists a residual generator, based on a dead-beat UIO and described as in (3), for the discrete-time state-space model (2), that allows to uniquely identify an arbitrary fault affecting the actuators, iff* $\text{rank}\left([\,CB\ CE\,]\right) = m + r$.

As Assumption 1.A is encompassed in condition $\text{rank}\left([\,CB\ CE\,]\right) = m + r$, an alternative way to state the previous result is the following one, to which we will steadily refer for the problem solution in the model-based approach.

Proposition 5. *The following conditions are equivalent:*

i) There exists a residual generator, based on a dead-beat UIO and described as in (3), for the discrete-time state-space model (2), that allows to uniquely identify an arbitrary fault affecting the actuators;
ii) Assumption 1.B holds and $\text{rank}\left([\,CB\ CE\,]\right) = m + r$.

IV. Data-Driven Approach

We assume to have collected offline input, state and output measurements from the system (affected by disturbances but not by faults) on a finite time window of (sufficiently large) length T: $u_d \doteq \{u_d(k)\}_{k=0}^{T-2}$, $x_d \doteq \{x_d(k)\}_{k=0}^{T-1}$ and $y_d \doteq \{y_d(k)\}_{k=0}^{T-1}$. Accordingly, we set

$$\begin{aligned} U_p &= \begin{bmatrix} u_d(0), \dots, u_d(T-2) \end{bmatrix} \in \mathbb{R}^{m \times (T-1)} \\ X_p &= \begin{bmatrix} x_d(0), \dots, x_d(T-2) \end{bmatrix} \in \mathbb{R}^{n \times (T-1)} \\ Y_p &= \begin{bmatrix} y_d(0), \dots, y_d(T-2) \end{bmatrix} \in \mathbb{R}^{p \times (T-1)} \\ X_f &= \begin{bmatrix} x_d(1), \dots, x_d(T-1) \end{bmatrix} \in \mathbb{R}^{n \times (T-1)} \\ Y_f &= \begin{bmatrix} y_d(1), \dots, y_d(T-1) \end{bmatrix} \in \mathbb{R}^{p \times (T-1)} \end{aligned}$$

Note that it not uncommon to assume that the state variable is accessible (only) during the preliminary data collection process [11], [19], [18]. Moreover, if this were not the case, state estimation based on data could not be possible, as the input and output data do not provide information about the specific state realization Σ. Also, even if we assume that no direct measurements of the disturbance sequence $d_d \doteq$

$\{d_d(k)\}_{k=0}^{T-2}$ associated with the historical data are available, nonetheless, for the subsequent analysis, it is convenient to introduce the symbol $D_p = \begin{bmatrix} d_d(0), \dots, d_d(T-2) \end{bmatrix} \in \mathbb{R}^{r \times (T-1)}$.

In the sequel, we will make the following:

Assumption 2. *The dimension r of the disturbance vector d is known, and the $(m+r+n) \times (T-1)$ matrix*

$$\begin{bmatrix} U_p \\ D_p \\ X_p \end{bmatrix} \text{ is FRR.}$$

Note that this assumption holds true, in particular, if the pair $(A, [B\ E])$ is controllable and the sequence $\{u_d, d_d\}$ is persistently exciting of order $n+1$ (see [20], Theorem 1). See [11] for additional comments on this aspect.

We want to prove that, under Assumption 2, the two necessary and sufficient conditions for the existence of a residual generator, based on a dead-beat UIO, that we derived in Proposition 5 via a model-based approach, find immediate counterparts in terms of the data matrices U_p, X_p, X_f, Y_p and Y_f. This means that, under Assumption 2, the solvability conditions derived in a data-driven set-up are equivalent to those one needs to verify when relying on the system model.

Proposition 6. *Under Assumption 2, the following facts are equivalent:*

i) Assumption 1.B holds and $\operatorname{rank}([\,CB\ CE\,]) = m + r$*;*

ii) The following conditions on the data matrices hold:

iia) $\operatorname{rank}\left(\begin{bmatrix} zX_p - X_f \\ Y_p \\ U_p \end{bmatrix}\right) = n+r+m,\ \forall z \in \mathbb{C}, z \neq 0;$

iib) $\operatorname{rank}\left(\begin{bmatrix} X_p \\ Y_f \end{bmatrix}\right) = n+r+m.$

Proof: We first prove that Assumption 1.B is equivalent to iia). Since the data are derived from the state-space model (2), it follows that they satisfy the state equation and hence

$$X_f = AX_p + BU_p + ED_p. \tag{7}$$

Consequently,

$$\begin{aligned} zX_p - X_f &= (zI_n - A)X_p - BU_p - ED_p \\ &= \begin{bmatrix} -B & -E & zI_n - A \end{bmatrix} \begin{bmatrix} U_p \\ D_p \\ X_p \end{bmatrix}. \end{aligned}$$

This implies

$$\begin{bmatrix} zX_p - X_f \\ Y_p \\ U_p \end{bmatrix} = \begin{bmatrix} -B & -E & zI_n - A \\ \mathbb{0} & \mathbb{0} & C \\ I_m & \mathbb{0} & \mathbb{0} \end{bmatrix} \begin{bmatrix} U_p \\ D_p \\ X_p \end{bmatrix}.$$

By Assumption 2 the matrix of data on the right is FRR $(= m+r+n)$, therefore condition iia) holds iff

$$\operatorname{rank}\left(\begin{bmatrix} -B & -E & zI_n - A \\ \mathbb{0} & \mathbb{0} & C \\ I_m & \mathbb{0} & \mathbb{0} \end{bmatrix}\right) = m+r+n,\ \forall z \neq 0,$$

which is equivalent to Assumption 1.B. We now show that $[\,CB\ CE\,]$ is FCR iff iib) holds. Making use, again, of the fact that the data are generated by the system (2), we can write:

$$\begin{bmatrix} X_p \\ Y_f \end{bmatrix} = \begin{bmatrix} \mathbb{0}_{n\times m} & \mathbb{0}_{n\times r} & I_n \\ CB & CE & CA \end{bmatrix} \begin{bmatrix} U_p \\ D_p \\ X_p \end{bmatrix}.$$

As the last matrix is FRR by Assumption 2, condition iib) holds iff

$$\operatorname{rank}\left(\begin{bmatrix} \mathbb{0}_{n\times m} & \mathbb{0}_{n\times r} & I_n \\ CB & CE & CA \end{bmatrix}\right) = m+r+n,$$

which in turn holds true iff $[\,CB\ CE\,]$ is FCR. □

The previous proposition is extremely useful because it allows one to immediately check from data if the problem is solvable. We now provide a path to explicitly determine a solution, provided that the aforementioned conditions hold.

Proposition 7. *[13] Under Assumption 2, the following facts are equivalent:*

i) Assumption 1.B holds and $\operatorname{rank}([\,CB\ CE\,]) = m + r$*;*

ii) There exist real matrices T_1, T_3 and T_4, of suitable dimensions, such that (T_3, C) is reconstructable, $\operatorname{rank}(CT_1) = m$*, and*

$$X_f = \begin{bmatrix} T_1 & T_3 & T_4 \end{bmatrix} \begin{bmatrix} U_p \\ X_p \\ Y_f \end{bmatrix}; \tag{8}$$

iii) There exist real matrices T_1, T_2, T_3 and T_4, of suitable dimensions, with T_3 nilpotent and $\operatorname{rank}(CT_1) = m$*, such that*

$$X_f = \begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix} \begin{bmatrix} U_p \\ Y_p \\ X_p \\ Y_f \end{bmatrix}. \tag{9}$$

We now summarize the outcome of the previous analysis. In Section III we have proved (see Proposition 5) that a residual generator, based on a dead-beat UIO and described as in (3), exists iff Assumption 1.B and $\operatorname{rank}([\,CB\ CE\,]) = m+r$ hold. In Proposition 6 we showed how such conditions can be equivalently checked on the available data. Finally, Proposition 7 tells us that such conditions are equivalent to the existence of a solution (T_1, T_2, T_3, T_4) of (9), with T_3 nilpotent and $\operatorname{rank}(CT_1) = m$. By resorting to the results derived in [11] (see, also, [19]), we can immediately determine the matrices A_{UIO}, B^u_{UIO}, B^y_{UIO} and D_{UIO} of the (dead-beat) UIO (3a), (3b) as

$$\begin{aligned} A_{UIO} &= T_3, \quad D_{UIO} = T_4, \\ B^u_{UIO} &= T_1, \quad B^y_{UIO} = T_2 + T_3T_4. \end{aligned} \tag{10}$$

Also, as a consequence of Assumption 2, we can claim that X_p is FRR, and hence by exploiting the relationship $Y_p = CX_p$ we can uniquely identify C as $C = Y_pX_p^\dagger$. This allows us to also determine the expression of the residual (3c).

Proposition 7 together with identities (10) allow to derive the matrices of the desired residual generator. However, once the necessary and sufficient conditions for the problem solvability have been checked on the historical data, it is not obvious how to identify, among all the solutions (T_1, T_2, T_3, T_4) of (9), one with T_3 nilpotent and CT_1 FCR. To this end we provide an algorithm which is based on a first important observation, condensed in the following lemma.

Lemma 8. *The following facts are equivalent:*

i) *There exists a solution (T_1, T_3, T_4) of (8), for which (T_3, C) is reconstructable and $\mathrm{rank}(CT_1) = m$;*

ii) *There exists a solution (T_1, T_3, T_4) of (8), for which $\mathrm{rank}(T_4) = r$, and for every such solution the pair (T_3, C) is reconstructable and $\mathrm{rank}(CT_1) = m$.*

Proof: ii) $\Rightarrow$ i) is obvious, so we only need to prove that i) $\Rightarrow$ ii). By mimicking the proof of iii) $\Rightarrow$ i) in Proposition 7 [13], we claim that a triple (T_1, T_3, T_4) solves (8) iff it solves

$$\begin{bmatrix} T_1 & T_3 & T_4 \end{bmatrix} \begin{bmatrix} I_m & \mathbb{0}_{m\times r} & \mathbb{0}_{m\times n} \\ \mathbb{0}_{n\times m} & \mathbb{0}_{n\times r} & I_n \\ CB & CE & CA \end{bmatrix} = \begin{bmatrix} B & E & A \end{bmatrix},$$

namely iff $(T_1, T_3, T_4) = ((I_n - T_4C)B, (I_n - T_4C)A, T_4)$, with T_4 any matrix satisfying $E = T_4CE$.

Now, let $(\bar{T}_1, \bar{T}_3, \bar{T}_4)$ be any solution of (8), satisfying the hypotheses in i). This implies that $\bar{T}_4CE = E$ (and hence $\mathrm{rank}(CE) = \mathrm{rank}(E) = r$), the pair $((I_n - \bar{T}_4C)A, C) = (\bar{T}_3, C)$ is reconstructable, and $C(I_n - \bar{T}_4C)B = C\bar{T}_1$ is FCR. But by the analysis we carried out in Section III, we can claim that there exists T_4^* of rank r, satisfying $T_4^*CE = E$, and for every such T_4^* it will still be true that $((I_n - T_4^*C)A, C)$ is reconstructable, and $C(I_n - T_4^*C)B$ has FCR. This completes the proof. □

The previous Lemma 8 tells us that in order to find a solution (T_1, T_3, T_4) of (8) with the required properties we need to focus on those for which $\mathrm{rank}(T_4) = r$. On the other hand, the proof of Lemma 8 shows that the solutions of (8) are those and only those expressed as $(T_1, T_3, T_4) = ((I_n - T_4C)B, (I_n - T_4C)A, T_4)$, with T_4 any matrix satisfying $E = T_4CE$. However, the matrix E is not available, and hence we need to select such matrices T_4 by making use only of data. Let S be a $(T-1) \times (T-1)$ nonsingular square (NSS) matrix such that

$$\begin{bmatrix} U_p \\ X_p \\ Y_f \end{bmatrix} S = \begin{bmatrix} I_m & \mathbb{0}_{m\times(T-1-m-n)} & \mathbb{0}_{m\times n} \\ \mathbb{0}_{n\times m} & \mathbb{0}_{n\times(T-1-m-n)} & I_n \\ Y_B & Y_E & Y_A \end{bmatrix}, \quad (11)$$

for suitable matrices Y_A, Y_B and Y_E with p rows. Such a matrix exists because $\begin{bmatrix} U_p^\top & X_p^\top \end{bmatrix}^\top$ is FRR (as a consequence of Assumption 2). Set $\bar{X}_f \doteq X_f S$ and block-partition it, according to the block-partitioning of $Y_f S$, as $\bar{X}_f = \begin{bmatrix} X_B & X_E & X_A \end{bmatrix}$. Then, (T_1, T_3, T_4) solves (8) iff it solves

$$\begin{bmatrix} X_B & X_E & X_A \end{bmatrix} = \\ = \begin{bmatrix} T_1 & T_3 & T_4 \end{bmatrix} \begin{bmatrix} I_m & \mathbb{0}_{m\times(T-1-m-n)} & \mathbb{0}_{m\times n} \\ \mathbb{0}_{n\times m} & \mathbb{0}_{n\times(T-1-m-n)} & I_n \\ Y_B & Y_E & Y_A \end{bmatrix},$$

namely iff $(T_1, T_3, T_4) = (X_B - T_4Y_B, X_A - T_4Y_A, T_4)$, with T_4 any matrix satisfying $X_E = T_4Y_E$. This allows to say that T_4 satisfies $X_E = T_4Y_E$ iff it satisfies $E = T_4CE$.

The previous analysis and results lead to Algorithm 1 that describes a procedure to determine from data the matrices of a dead-beat UIO-based residual generator (A_{UIO}, B_{UIO}^u, B_{UIO}^y, D_{UIO}, C) for system (2) (under Assumption 2).

Algorithm 1 Data Driven UIO matrix estimation procedure

Input: r, U_p, X_p, Y_p, X_f, Y_f.

Output: A_{UIO}, B_{UIO}^u, B_{UIO}^y, D_{UIO}, C.

1. Check if conditions iia) and iib) in Proposition 6 hold. If not, the problem is not solvable. Otherwise, go to Step 2.
2. Set $C = Y_p X_p^\dagger$. Let S be a $(T-1)\times(T-1)$ NSS matrix such that (11) holds, for suitable matrices Y_A, Y_B and Y_E with p rows. Find a solution T_4^* of $X_E = T_4Y_E$ with $\mathrm{rank}(T_4^*) = r$, and set $T_1^* = X_B - T_4^*Y_B$ and $T_3^* = X_A - T_4^*Y_A$. Since at Step 1 we have checked that the problem is solvable, then necessarily $(T_3^*, C) = (X_A - T_4^*Y_A, C)$ is reconstructable and CT_1^* is FCR.
3. Let L be such that $T_3^* - LC$ is nilpotent. Then
$$X_f = \begin{bmatrix} T_1^* & L & T_3^* - LC & T_4^* \end{bmatrix} \begin{bmatrix} U_p \\ Y_p \\ X_p \\ Y_f \end{bmatrix}.$$
So, by making use of (10), the matrices of the residual generator are: $A_{UIO} = T_3^* - LC$, $D_{UIO} = T_4^*$, $C = Y_p X_p^\dagger$, $B_{UIO}^u = X_B - T_4^*Y_B$, $B_{UIO}^y = L + (T_3^* - LC)T_4^*$.

Example 9. *Consider the system Σ, of dimension $n = 5$, with describing matrices:*

$$A = \begin{bmatrix} 0.8 & 0 & 0 & 0 & 0 \\ -0.8 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1.2 & -0.5 & -1.3 \\ 2 & -0.6 & 2.6 & 1 & 2.3 \\ 0.8 & -0.9 & 0.6 & 0.1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The historical (both known and unknown) input data have been randomly generated, uniformly in the interval $(-5, 5)$ for the known input $u(k)$, and in the interval $(-2, 2)$ for the disturbance $d(k)$. The time-interval of the offline experiment has been set to $T = 150$. We have collected the data corresponding to the input/output/state trajectories and then checked assumptions iia) and iib) of Proposition 6. Clearly, from Y_p and X_p we deduce C. By making use of the

Algorithm 1, we have chosen as matrices of the UIO (3):

$$A_{UIO} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -0.8 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0 & 0 \\ -1.6 & 0 & 0 & 0 & 0 \\ 0.8 & -0.9 & 0.6 & 0.1 & 0 \end{bmatrix}, \; B^u_{UIO} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \\ 0 \end{bmatrix},$$

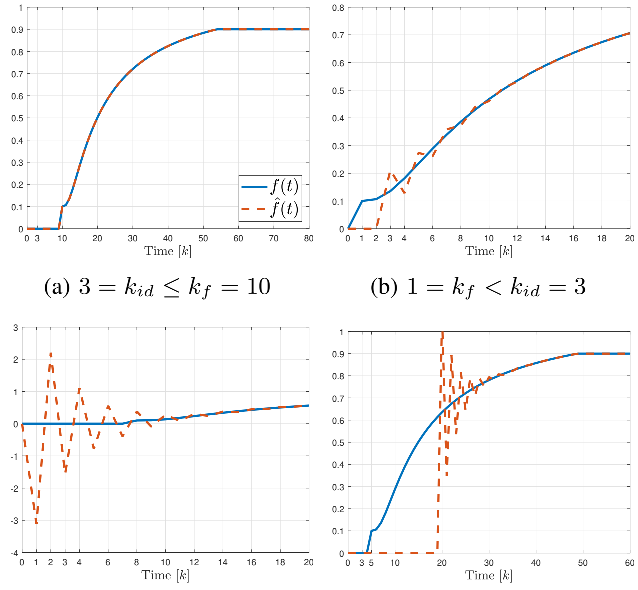
$$B^y_{UIO} = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.9 & 0.6 & 1.3 \end{bmatrix}, \; D_{UIO} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \\ 3 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to verify that A_{UIO} is nilpotent with nilpotency index $k_N = 3$. In order to test the performance of the dead-beat UIO generator, we fed the system with the (known) input $u(k) = 0.9\sin(0.4k + 3)$, $k \in \mathbb{Z}_+$, a random disturbance $d(k)$ whose first and second entries take values uniformly in the interval $(-5, 5)$ and $(-2, 2)$, respectively, and the fault signal

$$f(k) = \begin{cases} 0 & k < k_f, \\ \max\left\{0.1 + \exp\left(\frac{-10}{k-k_f+1}\right), 0.9\right\} & k \geq k_f. \end{cases}$$

We let k_{id} denote the first time at which we start the fault estimation. We have simulated four different scenarios in which we changed the relative position of k_N, k_{id} and k_f.

In Fig. 1 (a) we have the ideal case, as the identification starts only when the estimation error has become zero ($k_{id} = k_N$) and the fault starts after k_{id}. In this scenario, the fault is perfectly estimated ($\hat{f}(k) = f(k), \forall k \geq k_{id}$). In Fig. 1 (b), the fault appears before $e(k)$ becomes zero ($k_f < k_N$), and the identification starts at $k_{id} = k_N$. In Fig. 1 (c), we start the identification before k_N, when the residual is nonzero only due to the estimation error ($k_{id} < k_N$), and the fault affects the system at $k_f > k_N$. Finally, in Fig. 1 (d), the identification starts after the fault appears and the effect of $e(0)$ is extinguished ($k_N \leq k_f < k_{id}$). In the last three cases the fault estimate $\hat{f}$ oscillates around the real value of f and, in a finite number of steps, it reaches the correct value, as expected. From the third plot we deduce that starting the FDI before k_N may lead to false alarms; on the other hand, from the fourth plot we can see that late estimation leads to errors. The best choice is to start the identification exactly when $e(k)$ becomes zero, i.e. to assume $k_{id} = k_N$.



(a) $3 = k_{id} \leq k_f = 10$ (b) $1 = k_f < k_{id} = 3$

(c) $1 = k_{id} < k_N = 3$ (d) $5 = k_f < k_{id} = 20$

Fig. 1: Dynamics of the real and estimated faults.

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