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Kalman-like Observer for Hybrid Systems with Linear Maps and Known Jump Times

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Abstract— **We propose a hybrid Kalman-like observer for general hybrid systems with linear (time-varying) dynamics and output maps, where the solutions' jump times are exactly known. After defining a hybrid observability Gramian and the corresponding hybrid uniform complete observability, we show that the estimate provided by this observer converges asymptotically to the system solution if this observability holds together with some boundedness and invertibility conditions along the considered system solution. Then, under additional uniformity and strictness of the forgetting factors, we show exponential stability of the estimation error with an arbitrarily fast rate. The robust stability of this error against input disturbances and measurement noise is also studied. The results are illustrated on several benchmark examples, including switched systems, hybrid systems with discontinuous solutions, and continuous-time systems with multi-rate sporadic outputs.**

I. INTRODUCTION

The celebrated *Kalman observer* was introduced in the early 60s by Kalman and Bucy [1] as an *optimal* filter for linear continuous-time systems. Under *uniform complete observability* and in a *stochastic* context, it was shown to minimize the covariance of the estimation error in the presence of Gaussian dynamics and measurement noise. Its appeal lies in its systematic design and easiness of tuning, which is linked to the (assumedly known) covariance of those disturbances. It was then extended to discrete-time systems [2] and multiple settings [3], thus widely used in industry.

On the other hand, in the early 90s, an alternative *Kalman-like observer* was developed [4], [5], optimizing in a *deterministic* setting the ability of the estimate to explain the past output history, with a certain forgetting factor and weighting, describing the confidence in the output measurement. The difference with the Kalman filter mainly lies in the absence of noise on the dynamics which facilitates a Lyapunov stability analysis by linking the Lyapunov matrix directly to the *observability Gramian*. This design was extended to discrete-time systems in [6], and without forgetting factor in [7].

However, surprisingly, we are not aware of any such *systematic* design for *hybrid* systems with linear maps, combining continuous (*flows*) and discrete (*jumps*) behavior, with outputs available during both flows and jumps. Indeed, observer design in this context generally goes through the resolution of LMIs with no guaranteed solvability [8], [9], or an observability decomposition isolating the part of the state that is instantaneously observable during flows [10], [11]. An exception is a particular case of constant parameter estimation with both continuous and discrete measurements, for which a hybrid gradient descent was developed [12], [13].

Otherwise, observer designs typically avoid the combination of flow/jump innovation terms, by using the output during either flows only (*flow-based*) or jumps only (*jump-based*) [9]. The latter case includes continuous-time systems with sampled measurements, for which *continuous-discrete Kalman* filters [3], [14], [9] are derived, where the estimate evolves in the open loop during flows and is corrected at the sampling instants, with a gain depending on a hybrid covariance matrix. The latter evolves either discretely, based on an equivalent discrete system describing how the error propagates during the combination of flows and jumps [9], or in a hybrid way [3], [14]. Most recent advances concern essentially the implementation of continuous-discrete *extended* Kalman filters [15] for nonlinear systems, or to find alternative LMI-based designs of the gains [16], [17].

Note that the design of a *unified* and *systematic* Kalman filter seems still open for continuous-time systems with *multi-rate* sampled outputs, namely combining *fast* (almost continuous) and *slow* measurements with different sampling rates. Designs typically include several (discrete) Kalman filters operating at different rates with *fusing* strategies [18], [19], [20], or sample-and-hold LMI-based correction terms [21], or KKL observers with inter-sample predictors [22].

In this paper, we propose a hybrid Kalman-like observer for *general* hybrid systems [23] with linear maps and known jump times, exploiting outputs available during both flows and jumps. The considered class includes linear switched systems and linear continuous-time systems with (multi-rate) sampled/sporadic measurements, and its restriction to fully continuous or discrete dynamics allows us to recover the designs of [4], [5], [6], [7]. Compared to existing hybrid designs, this one is *systematic*, automatically taking into account the observability brought by the combination of both continuous and discrete outputs and dynamics, with no need for state decomposition, unlike [10], [11], and applies to *time-varying* hybrid systems. After defining a *hybrid observability Gramian* and the corresponding *hybrid uniform complete observability* condition, we show asymptotic convergence of the estimate under some boundedness and invertibility conditions, thus extending the design of [12] to non-constant states. Then, the exponential stability of the estimation error with an arbitrarily fast convergence rate is proven under additional uniformity conditions and strictness of the forgetting factors. Finally, we show the robust stability of the estimation error (in the sense of [24]) against flow/jump input disturbances and measurement noise.

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Notations: Let $\mathbb{R}$ (resp. $\mathbb{N}$) denote the set of real numbers (resp. natural numbers, i.e., $\{0, 1, 2, \ldots\}$). We denote $\mathbb{R}^{m\times n}$ (resp. $\mathcal{S}^n_{>0}$) as the set of real $(m \times n)$- (resp. symmetric positive definite $(n \times n)$-) dimensional matrices. Let $|\cdot|$ be the Euclidean norm and $\|\cdot\|$ the induced matrix norm. Let $\phi_F(t, t')$ be the continuous-time transition matrix of $\dot{x} = Fx$ (with F possibly time-varying) from time t' to t, i.e., such that any solution verifies $x(t) = \phi_F(t, t')x(t')$. For a solution $(t, j) \mapsto x(t, j)$ of a hybrid system, we denote $\operatorname{dom} x$ its time domain [23], $\operatorname{dom}_t x$ (resp. $\operatorname{dom}_j x$) the domain's projection on the ordinary time (resp. jump) component, and for $j \in \operatorname{dom}_j x$, $t_j(x)$ the unique time such that $(t_j(x), j) \in \operatorname{dom} x$ and $(t_j(x), j-1) \in \operatorname{dom} x$, and $\mathcal{T}_j(x) := \{t \in \operatorname{dom}_t x : (t, j) \in \operatorname{dom} x\}$ (for hybrid systems with inputs, see [25]). The mention of x is omitted when no confusion is possible. A solution x to a hybrid system is *complete* if $\operatorname{dom} x$ is unbounded. In some long derivations such as (4) below, $\star$ denotes the symmetric part, i.e., $\star^\top P = P^\top P$. Woodbury matrix identity is here recalled as $(A+UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1}+VA^{-1}U)^{-1}VA^{-1}$, where A and C are square and dimensions are appropriate.

II. HYBRID KALMAN-LIKE OBSERVER

Consider a hybrid system with linear (time-varying) maps

$$\mathcal{H}\begin{cases} \dot{x} = Fx + u_c & (x, u_c) \in C \qquad y_c = H_c x \\ x^+ = Jx + u_d & (x, u_d) \in D \qquad y_d = H_d x \end{cases} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, C and D are the flow and jump sets, $y_c \in \mathbb{R}^{n_{y,c}}$ and $y_d \in \mathbb{R}^{n_{y,d}}$ are the outputs known during flows and at jumps respectively, $u_c \in \mathbb{R}^{n_x}$ and $u_d \in \mathbb{R}^{n_x}$ are known exogenous terms, as well as the dynamics matrices $F, J \in \mathbb{R}^{n_x\times n_x}$ and the output matrices $H_c \in \mathbb{R}^{n_{y,c}\times n_x}$, $H_d \in \mathbb{R}^{n_{y,d}\times n_x}$ which are all known and possibly time-varying. Denote $\mathcal{X}_0$ as a set containing the initial conditions of the trajectories to be estimated and $\mathcal{U}$ as a set of inputs (u_c, u_d) of interest. We then denote $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$ as the set of solutions of $\mathcal{H}$ initialized in $\mathcal{X}_0$ with $(u_c, u_d) \in \mathcal{U}$. Because the goal of this paper is to design an *asymptotic* observer for (1), we assume solutions $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$ are complete as stated next.

Assumption 1: All solutions $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$ are complete.

Remark 1: Models of the form (1) include not only hybrid systems with linear maps described in the setting of [23] (see Example 2), but also *switched* systems with linear maps where the active mode is seen as an exogenous signal making (F, J, H_c, H_d) time-varying (see Example 1), and continuous-time systems with sporadic or *multi-rate* sampled outputs (see Example 3). Note that in many of these systems, observability is acquired by the combination of flows with (F, H_c) and jumps with (J, H_d). Therefore, the direct coupling of classical continuous and discrete linear observers relying on the observability of each pair separately will typically not work. Here, instead, we design a single unified algorithm, automatically gathering observability from both flows and jumps via a shared covariance matrix.

A. Synchronized Hybrid Kalman-like Observer

Assuming the jump times of the solutions $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$ are exactly known or detected—for instance from discontinuities in the output, or impact sensors, or because they are triggered by the user or the sensor's availability in the sampled-data case—and exploiting the linearity of the maps of $\mathcal{H}$, we propose a *systematic* design of a synchronized *hybrid Kalman-like* observer of the form

$$\hat{\mathcal{H}}\begin{cases} \left.\begin{array}{l}\dot{\hat{x}} = F\hat{x} + u_c + PH_c^\top R_c^{-1}(y_c - H_c\hat{x}) \\ \dot{P} = \lambda P + FP + PF^\top - PH_c^\top R_c^{-1}H_cP\end{array}\right\} \text{when } \mathcal{H} \text{ flows} \\ \left.\begin{array}{l}\hat{x}^+ = J\hat{x} + u_d + JK(y_d - H_d\hat{x}) \\ P^+ = \gamma^{-1}J(I - KH_d)PJ^\top\end{array}\right\} \text{when } \mathcal{H} \text{ jumps}\end{cases} \tag{2a}$$

with

$$K = PH_d^\top(H_dPH_d^\top + R_d)^{-1}, \tag{2b}$$

where $\lambda \geq 0$ and $\gamma \in (0, 1]$ are design parameters, $R_c \in \mathcal{S}^{n_{y,c}}_{>0}$ and $R_d \in \mathcal{S}^{n_{y,d}}_{>0}$ are (possibly time-varying) weighting matrices such that there exist positive scalars $\underline{c}_{R_c}$, $\overline{c}_{R_c}$, $\underline{c}_{R_d}$, and $\overline{c}_{R_d}$ such that for all $(t, j) \in \operatorname{dom} x$, we have

$$\begin{aligned}\underline{c}_{R_c} I \leq R_c(t, j) \leq \overline{c}_{R_c} I, \\ \underline{c}_{R_d} I \leq R_d(t_{j+1}, j) \leq \overline{c}_{R_d} I.\end{aligned} \tag{2c}$$

The observer (2) gathers in a common setting the continuous and discrete Kalman-like observers of [4], [5] and [6], [7]. The difference compared to the continuous and discrete Kalman designs [1], [2] mainly lies in the absence of the Q-covariance matrices, commonly describing the confidence in the dynamics. They are here replaced by forgetting factors λ and γ, which allows us to: 1) Make the dynamics of P^{-1} *linear* and explicitly solvable, with a direct link to the so-called *observability Gramian*; and 2) Obtain a quadratic *strict* Lyapunov function. Note that in the discrete case, the computation steps of the Kalman filter [7] are gathered here into a single jump map. It combines 1) *Correction* and 2) *Prediction*, instead of the contrary, since the output available to compute $\hat{x}^+$ is its value *before* the jump, namely H_dx instead of H_dx^+. This justifies the presence of J in front of K in the discrete correction term. In the classical Kalman notations, this means that our $(\hat{x}, P)$ corresponds to $(\hat{x}, P)(k|k-1)$ instead of $(\hat{x}, P)(k|k)$, which is consistent with the use of $P(k|k-1)$ in the Lyapunov function in [7]. Note finally that adding the Kalman Q-parameters in (2) would preserve the decrease of the Lyapunov function but would make its lower-boundedness more intricate to prove.

The goal of this paper is first provide conditions ensuring asymptotic convergence of (2) without any further constraint on the forgetting factors $\lambda \geq 0$ and $\gamma \in (0, 1]$, i.e., all solutions $(x, \hat{x}, P)$ to the cascade $\mathcal{H} - \hat{\mathcal{H}}$ initialized in $\mathcal{X}_0 \times \mathbb{R}^{n_x} \times \mathcal{S}^{n_x}_{>0}$ with $(u_c, u_d) \in \mathcal{U}$ are complete and verify

$$\lim_{t+j\to+\infty} |x(t, j) - \hat{x}(t, j)| = 0, \quad (t, j) \in \operatorname{dom} x. \tag{3}$$

In a second step, conditions for exponential stability of the estimation error with an arbitrarily fast rate as well as robustness against disturbances will be derived when $\lambda > 0$ and/or $\gamma \in (0, 1)$. Classically, the asymptotic convergence of the Kalman(-like) observer is shown for continuous-time and discrete-time systems under the so-called *Uniform Complete Observability* condition [1], [5], [6]. This condition imposes *uniform* and *persistent* invertibility of the *observability Gramian*, describing the richness of the information provided by the output on a certain time window. We extend those notions and objects in the next section in the *hybrid* context.

B. Hybrid Definitions of the Observability Gramian and Uniform Complete Observability

To define the notions of Gramian and observability needed for this observer, let us assume the following.

Assumption 2: For all solutions $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0,\mathcal{U})$ and for all $j \in \mathrm{dom}_j\, x$, the map $t \mapsto F(t,j)$ is locally bounded on $\mathcal{T}_j$, and the matrix $J(t_{j+1},j)$ is invertible if $j+1 \in \mathrm{dom}_j\, x$.

Remark 2: Assuming the invertibility of each $J(t_{j+1},j)$ can be restrictive in the hybrid context. But as seen in Example 2, thanks to the non-uniqueness of representation in hybrid systems, it may be possible to rewrite J satisfying this assumption. Note though that inverting J is not necessary to implement observer (2) and is needed for analysis only, similarly to the discrete Kalman literature [7], [6]. Example 2 is a case where the observer works without this condition and the analysis might be adaptable as suggested in [26].

Under Assumption 2, solutions $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0,\mathcal{U})$ are unique in both forward and backward time, so we can define *hybrid transition matrices* of $\mathcal{H}$. More precisely, given a solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0,\mathcal{U})$ with $u_c = 0$ and $u_d = 0$, for all hybrid times $((t',j'),(t,j)) \in \mathrm{dom}\, x \times \mathrm{dom}\, x$, we have

$$x(t,j) = \Phi_{F,J}((t,j),(t',j'))x(t',j'),$$

where $\Phi_{F,J}$ is defined as $\Phi_{F,J}((t,j),(t',j')) = \phi_F(t,t_{j+1})\left(\prod_{k=j}^{j'+1}\phi_F(t_{k+1},t_k)J(t_k,k-1)\right)\phi_F(t_{j'+1},t')$ if $t \geq t'$ and $j \geq j'$, and $\Phi_{F,J}((t,j),(t',j')) = \phi_F(t,t_j)\left(\prod_{k=j+1}^{j'}\phi_F(t_{k-1},t_k)J^{-1}(t_k,k-1)\right)\phi_F(t_{j'},t')$ otherwise, with the time domain of F and J inherited from $\mathrm{dom}\, x$.

Definition 1 (Backward observability Gramian): The backward observability Gramian of the quadruple (F,J,H_c,H_d) defined on a time domain $\mathcal{D}$, from time $(t',j') \in \mathcal{D}$ to a later time $(t,j) \in \mathcal{D}$, is defined as

$$\begin{aligned}&\mathcal{G}_{(F,J,H_c,H_d)}((t',j'),(t,j)) = \textstyle\int_{t'}^{t_{j'+1}} \star^\top \Psi_c((s,j'),(t,j))ds\\ &+ \sum_{k=j'+1}^{j-1} \textstyle\int_{t_k}^{t_{k+1}} \star^\top \Psi_c((s,k),(t,j))ds\\ &+ \sum_{k=j'}^{j-1} \star^\top \Psi_d((t_{k+1},k),(t,j)) + \textstyle\int_{t_j}^{t} \star^\top \Psi_c((s,j),(t,j))ds,\end{aligned} \tag{4}$$

where $\Psi_c((s,k),(t,j)) = H_c(s,k)\Phi_{F,J}((s,k),(t,j))$ and $\Psi_d((t_{k+1},k),(t,j)) = H_d(t_{k+1},k)\Phi_{F,J}((t_{k+1},k),(t,j))$, with all the jump times determined from $\mathcal{D}$.

Remark 3: The *backward* Gramian (4) characterizes the ability to reconstruct $x(t,j)$ from the knowledge of the *past* output. This form naturally comes up in the analysis, but we could also define a *forward* Gramian, characterizing the ability to reconstruct $x(t',j')$ from the knowledge of the *future* output. They are equivalent under the capability to go forward and backward in time, namely Assumption 2.

Definition 2 (Uniform complete observability (UCO)): The quadruple (F,J,H_c,H_d) defined on a hybrid time domain $\mathcal{D}$ is uniformly completely observable (UCO) with data (Δ,μ) if there exists $\Delta > 0$ and $\mu > 0$ such that for all $((t',j'),(t,j)) \in \mathcal{D} \times \mathcal{D}$ verifying $(t-t')+(j-j') \geq \Delta$,

$$\mathcal{G}_{(F,J,H_c,H_d)}((t',j'),(t,j)) \geq \mu I. \tag{5}$$

In this paper, we show three main results: 1) The estimation error converges asymptotically to zero for any choice of $\lambda \geq 0$, $\gamma \in (0,1]$, under boundedness of the matrices and UCO *along the considered solution only* (Section III); 2) It is exponentially stable with an arbitrarily fast rate for appropriate choices of λ and γ if these requirements hold uniformly with respect to solutions (Section IV); and 3) It is robustly stable (in the sense of [24]) with respect to flow/jump input disturbances and measurement noise (Section V).

III. ASYMPTOTIC CONVERGENCE FROM UNIFORM COMPLETE OBSERVABILITY

Assumption 3: For all solutions $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0,\mathcal{U})$, assume:

- *(Boundedness)* There exist non-negative scalars c_F, c_{H_c}, and c_{H_d}, and positive scalars c_J and $c_{J^{-1}}$ such that for all $(t,j) \in \mathrm{dom}\, x$, we have (if $j+1 \in \mathrm{dom}_j\, x$) $\|F(t,j)\| \leq c_F$, $\|J(t_{j+1},j)\| \leq c_J$, $\|J^{-1}(t_{j+1},j)\| \leq c_{J^{-1}}$, $\|H_c(t,j)\| \leq c_{H_c}$, and $\|H_d(t_{j+1},j)\| \leq c_{H_d}$;
- *(Observability)* There exists a pair of positive scalars (Δ,μ) such that the quadruple (F,J,H_c,H_d) defined on the time domain of x is UCO with this data.

Note that for asymptotic *convergence* only (without stability guarantees), no uniformity with respect to solutions or time domains is required, but only along the time domain of each particular solution.

Theorem 1: Under Assumptions 1, 2, and 3, for any $\lambda \geq 0$ and any $\gamma \in (0,1]$, any solution $(x,\hat{x},P)$ of the cascade $\mathcal{H}-\hat{\mathcal{H}}$ initialized in $\mathcal{X}_0 \times \mathbb{R}^{n_x} \times \mathcal{S}_{>0}^{n_x}$ with $(u_c,u_d) \in \mathcal{U}$ and (R_c,R_d) satisfying (2c) for some $(\underline{c}_{R_c},\overline{c}_{R_c},\underline{c}_{R_d},\overline{c}_{R_d}) \in \mathbb{R}_{>0}^4$ is complete and verifies (3).

Proof: Consider a solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0,\mathcal{U})$. By Assumption 1, it is complete. In the rest of this proof, all variables are evolving on $\mathrm{dom}\, x$ and so are complete. Consider $(t,j) \mapsto \Pi(t,j)$ with $\Pi(0,0) \in \mathcal{S}_{>0}^{n_x}$ and dynamics

$$\begin{cases}\dot{\Pi} = -\lambda\Pi - \Pi F - F^\top\Pi + H_c^\top R_c^{-1} H_c\\ \Pi^+ = \gamma(J^{-1})^\top(\Pi + H_d^\top R_d^{-1} H_d)J^{-1}.\end{cases} \tag{6}$$

Because J is invertible at jumps from Assumption 2, Π is well-defined. It can be proven using mathematical induction that the closed form of $\Pi(t,j)$ for all $(t,j) \in \mathrm{dom}\, x$ is

$$\begin{aligned}\Pi(t,j) = &e^{-\lambda t}\gamma^j\Phi_{F,J}^\top((0,0),(t,j))\Pi(0,0)\Phi_{F,J}((0,0),(t,j))\\ &+ \textstyle\sum_{k=0}^{j-1}\int_{t_k}^{t_{k+1}} e^{-\lambda(t-s)}\gamma^{j-k} \star^\top \Psi_{c'}((s,k),(t,j))ds\\ &+ \textstyle\sum_{k=0}^{j-1} e^{-\lambda(t-t_{k+1})}\gamma^{j-k} \star^\top \Psi_{d'}((t_{k+1},k),(t,j))\\ &+ \textstyle\int_{t_j}^{t} e^{-\lambda(t-s)} \star^\top \Psi_{c'}((s,j),(t,j))ds,\end{aligned} \tag{7}$$

where $\Psi_{c'}((s,k),(t,j)) = R_c^{-\frac{1}{2}}(s,k)\Psi_c((s,k),(t,j))$ and $\Psi_{d'}((t_{k+1},k),(t,j)) = R_d^{-\frac{1}{2}}(t_{k+1},k)\Psi_d((t_{k+1},k),(t,j))$ (with Ψ_c and Ψ_d defined in Definition 1). Now, we show that Π is uniformly lower-bounded along $\operatorname{dom} x$. First use Gronwall's inequality to show that $\|\phi_F(t,t')\| \leq e^{c_F|t-t'|}$, then it follows that for any $((t',j'),(t,j)) \in \operatorname{dom} x \times \operatorname{dom} x$ with $t' \leq t$ and $j' \leq j$, we have $\|\Phi_{F,J}((t,j),(t',j'))\| \leq e^{c_F(t-t')}c_J^{j-j'}$. Because $\Phi_{F,J}((t,j),(t',j'))\Phi_{F,J}((t',j'),(t,j)) = I$, this implies that

$$\star^\top \Phi_{F,J}((t',j'),(t,j)) \geq e^{-2c_F(t-t')}c_J^{-2(j-j')}I.$$

Then, for any $(t,j) \in \operatorname{dom} x$ such that $t+j \leq \Delta$, we have

$$\begin{aligned}\Pi(t,j) &\geq e^{-\lambda t}\gamma^j e^{-2c_F t}c_J^{-2j}\Pi(0,0)\\ &\geq (e^{-\lambda}\gamma e^{-2c_F}\max\{1,c_J\}^{-2})^{\Delta}\Pi(0,0)\\ &\geq c_{\Pi,1}I,\end{aligned}$$

for some $c_{\Pi,1} > 0$. Next, for any $(t,j) \in \operatorname{dom} x$ such that $t+j \geq \Delta$, we can always pick $(t',j') \in \operatorname{dom} x$ (before (t,j)) such that $\Delta \leq (t-t')+(j-j') \leq \Delta+1$ and from Assumption 3, we have

$$\begin{aligned}&\Pi(t,j)\\ &\geq e^{-\lambda(t-t')}\gamma^{j-j'}\min\{\overline{c}_{R_c}^{-1},\overline{c}_{R_d}^{-1}\}\mathcal{G}_{(F,J,H_c,H_d)}((t',j'),(t,j))\\ &\geq (e^{-\lambda}\gamma)^{\Delta+1}\min\{\overline{c}_{R_c}^{-1},\overline{c}_{R_d}^{-1}\}\mathcal{G}_{(F,J,H_c,H_d)}((t',j'),(t,j))\\ &\geq \mu(e^{-\lambda}\gamma)^{\Delta+1}\min\{\overline{c}_{R_c}^{-1},\overline{c}_{R_d}^{-1}\}I := c_{\Pi,2}I.\end{aligned}$$

Therefore, for all $(t,j) \in \operatorname{dom} x$, we have

$$\Pi(t,j) \geq \min\{c_{\Pi,1},c_{\Pi,2}\}I := c_\Pi I, \tag{8}$$

which means that Π is uniformly lower-bounded and thus uniformly invertible on $\operatorname{dom} x$. Let us now study the dynamics of $W := \Pi^{-1}$, which is well-defined and belongs to $\mathcal{S}_{>0}^{n_x}$. During flows, it is straightforward to check that $\dot{W} = -W\dot{\Pi}W$ verifies

$$\dot{W} = \lambda W + FW + WF^\top - WH_c^\top R_c^{-1}H_cW.$$

At jumps, using Woodbury matrix identity, we have

$$\begin{aligned}W^+ &= (\Pi^+)^{-1} = \gamma^{-1}J(W^{-1}+H_d^\top R_d^{-1}H_d)^{-1}J^\top\\ &= \gamma^{-1}J(W - WH_d^\top(H_dWH_d^\top+R_d)^{-1}H_dW)J^\top\\ &= \gamma^{-1}J(I - WH_d^\top(H_dWH_d^\top+R_d)^{-1}H_d)WJ^\top.\end{aligned}$$

Therefore, W follows the same dynamics as P in (2). So if $W(0,0) = P(0,0)$ then $W(t,j) = P(t,j)$ for all $(t,j) \in \operatorname{dom} x$. This means that $P = \Pi^{-1}$ (with $\Pi(0,0) = (P(0,0))^{-1}$) and that P is invertible at all times. Therefore, the error $\tilde{x} := x - \hat{x}$ follows the dynamics

$$\begin{cases}\dot{\tilde{x}} = (F - \Pi^{-1}H_c^\top R_c^{-1}H_c)\tilde{x} := \tilde{F}\tilde{x}\\ \tilde{x}^+ = J(I-KH_d)\tilde{x} := \tilde{J}\tilde{x},\end{cases} \tag{9}$$

where $K = \Pi^{-1}H_d^\top(H_d\Pi^{-1}H_d^\top + R_d)^{-1}$. Consider the Lyapunov function $V(\tilde{x},\Pi) = \tilde{x}^\top\Pi\tilde{x}$. For all $(t,j) \in \operatorname{dom} x$, $V(\tilde{x}(t,j),\Pi(t,j)) \geq c_\Pi|\tilde{x}(t,j)|^2$, so Theorem 1 is proven if we show that V asymptotically converges to 0. Let us study the dynamics of V along (9) and (6). During flows, we have

$$\begin{aligned}\dot{V} &= \tilde{x}^\top[(F-\Pi^{-1}H_c^\top R_c^{-1}H_c)^\top\Pi + \dot{\Pi}\\ &\qquad + \Pi(F-\Pi^{-1}H_c^\top R_c^{-1}H_c)]\tilde{x}\\ &= -\lambda V - \tilde{x}^\top H_c^\top R_c^{-1}H_c\tilde{x} \leq -\lambda V - \overline{c}_{R_c}^{-1}\tilde{x}^\top H_c^\top H_c\tilde{x}.\end{aligned}$$

Using Woodbury matrix identity yields

$$\begin{aligned}&K\\ &= PH_d^\top(R_d^{-1} - R_d^{-1}H_d(P^{-1}+H_d^\top R_d^{-1}H_d)^{-1}H_d^\top R_d^{-1})\\ &= PH_d^\top R_d^{-1} - P((P^{-1}+H_d^\top R_d^{-1}H_d) - P^{-1})\\ &\qquad \times(P^{-1}+H_d^\top R_d^{-1}H_d)^{-1}H_d^\top R_d^{-1}\\ &= (\Pi + H_d^\top R_d^{-1}H_d)^{-1}H_d^\top R_d^{-1}.\end{aligned}$$

At jumps, thanks to the newly obtained expression of K and Woodbury matrix identity, we have

$$\begin{aligned}V^+ &= \gamma\tilde{x}^\top(I-KH_d)^\top(\Pi + H_d^\top R_d^{-1}H_d)\\ &\qquad \times(I - (\Pi+H_d^\top R_d^{-1}H_d)^{-1}H_d^\top R_d^{-1}H_d)\tilde{x}\\ &= \gamma\tilde{x}^\top(I-KH_d)^\top\Pi\tilde{x}\\ &= \gamma\tilde{x}^\top(I - \Pi^{-1}H_d^\top(H_dPH_d^\top+R_d)^{-1}H_d)^\top\Pi\tilde{x}\\ &= \gamma V - \gamma\tilde{x}^\top H_d^\top(H_d\Pi^{-1}H_d^\top + R_d)^{-1}H_d\tilde{x}\\ &\leq \gamma V - \gamma\tilde{x}^\top H_d^\top(c_{H_d}^2c_\Pi^{-1} + \overline{c}_R)^{-1}H_d\tilde{x}.\end{aligned}$$

We see that V decreases strictly and exponentially to 0 if $\lambda > 0$ and $\gamma \in (0,1)$. We next show that actually, thanks to UCO, it converges *in-the-large* even for $\lambda = 0$ and $\gamma = 1$. In this case, we have

$$\begin{aligned}\dot{V} &\leq -\overline{c}_{R_c}^{-1}\tilde{x}^\top H_c^\top H_c\tilde{x} &&:= -c_c\tilde{x}^\top H_c^\top H_c\tilde{x},\\ V^+ - V &\leq -\tfrac{c_\Pi}{c_{H_d}^2 + c_\Pi\overline{c}_R}\tilde{x}^\top H_d^\top H_d\tilde{x} &&:= -c_d\tilde{x}^\top H_d^\top H_d\tilde{x},\end{aligned}$$

and thus, for all $((t',j'),(t,j)) \in \operatorname{dom} x \times \operatorname{dom} x$, we have

$$V(t,j) \leq V(t',j') - \tilde{V}, \tag{10}$$

where

$$\begin{aligned}\tilde{V} &= c_c\int_{t'}^{t_{j'+1}}\star^\top H_c(s,j')\tilde{x}(s,j')ds + c_c\sum_{k=j'+1}^{j-1}\tilde{\mathcal{G}}_F(k)\\ &\qquad + c_d\sum_{k=j'}^{j-1}\tilde{\mathcal{G}}_J(k) + c_c\int_{t_j}^{t}\star^\top H_c(s,j)\tilde{x}(s,j)ds,\end{aligned}$$

with $\tilde{\mathcal{G}}_F$ and $\tilde{\mathcal{G}}_J$ defined as

$$\begin{aligned}\tilde{\mathcal{G}}_F(k) &= \int_{t_k}^{t_{k+1}}\star^\top H_c(s,k)\tilde{x}(s,k)ds,\\ \tilde{\mathcal{G}}_J(k) &= \star^\top H_d(t_{k+1},k)\tilde{x}(t_{k+1},k).\end{aligned} \tag{11}$$

Applying Lemma 1 in the Appendix with $\Delta_m = \Delta+1$, $K_c = \Pi^{-1}H_c^\top R_c^{-1}$, and $K_d = JK = J\Pi^{-1}H_d^\top(H_d\Pi^{-1}H_d^\top + R_d)^{-1}$, which are indeed upper-bounded by $c_{H_c}(c_\Pi\underline{c}_{R_c})^{-1}$ and $c_Jc_{H_d}(c_\Pi\underline{c}_{R_d})^{-1}$ respectively, there exists $c_\mathcal{G} > 0$ such that for all $((t',j'),(t,j)) \in \operatorname{dom} x \times \operatorname{dom} x$ such that $\Delta \leq (t-t')+(j-j') \leq \Delta+1$, we have

$$\begin{aligned}\tilde{V} &\geq \min\{c_c,c_d\}c_\mathcal{G}\tilde{x}^\top(t,j)\mathcal{G}_{(F,J,H_c,H_d)}((t',j'),(t,j))\tilde{x}(t,j)\\ &\geq \min\{c_c,c_d\}c_\mathcal{G}\mu|\tilde{x}(t,j)|^2,\end{aligned}$$

exploiting the UCO property in Assumption 3. We finally conclude that there exists $c_V > 0$ such that for any $((t',j'),(t,j)) \in \operatorname{dom} x \times \operatorname{dom} x$ verifying $\Delta \leq (t-t') + (j-j') \leq \Delta+1$, we have

$$V(t,j) \leq V(t',j') - c_V|\tilde{x}(t,j)|^2.$$

It remains to show that $\tilde{x}$ converges asymptotically to 0 using contradiction, similar to [7]. Assume that $\tilde{x}$ does not converge to 0. Then, there exists $\epsilon > 0$ such that for any $(t',j') \in \operatorname{dom} x$, we can always find (exploiting the completeness of x) $(t,j) \in \operatorname{dom} x$ such that $(t-t')+(j-j') \geq \Delta$ and $|\tilde{x}(t,j)| \geq \epsilon$. Hence we have $V(t,j) \leq V(t',j') - c_V\epsilon^2$.

By repeating this process, still thanks to the completeness of x, V becomes negative after a finite amount of time, which contradicts its definition. Therefore, by contradiction, $\tilde{x}$ converges asymptotically to 0. ∎

Example 1 (Switched system): Inspired by [10, Example 1], consider a switched system with linear maps

$$\dot{x} = A_i x, \qquad y = C_i x, \tag{12}$$

characterized by two modes $i \in \{1,2\}$ as $A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} \epsilon & 1 \\ -1 & \epsilon \end{pmatrix}$, $C_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $C_2 = \begin{pmatrix} 0 & 0 \end{pmatrix}$, and triggered such that the time between two successive switches cannot be shorter than some $\delta > 0$. As pointed out in [10], neither (A_1, C_1) nor (A_2, C_2) is observable, but the switching order $1 \to 2 \to 1$ allows us to determine the initial condition unless the times elapsed in-between switches are multiples of π, which corresponds to a *singular* switching signal. A hybrid Kalman-like observer (2) is then designed for (12), leading to a much simpler observer than in [10]. Asymptotic convergence of the error is shown in Figure 1 for $\lambda = 0$ and $\gamma = 1$. On the other hand, Figure 2 shows the observer estimate with a π-periodic switching signal for which UCO does not hold.

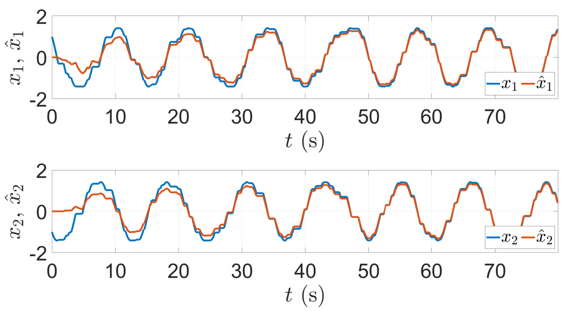

Fig. 1. State estimation in a switched system (with $\lambda = 0$ and $\gamma = 1$).

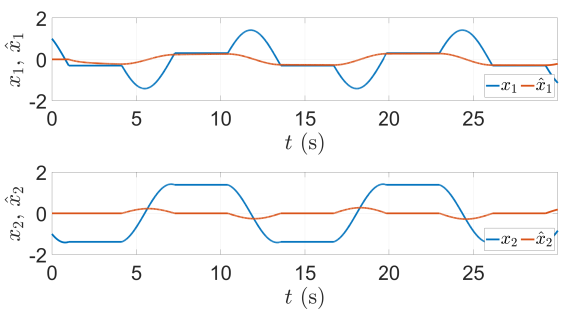

Fig. 2. State estimation in a switched system (singular switching signal).

IV. EXPONENTIAL STABILITY OF THE ERROR WITH AN ARBITRARILY FAST RATE

Assumption 4: Assume as in Assumption 3, but all scalars therein are now the same for all solutions $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$.

Theorem 2: Under Assumptions 1, 2, and 4, for any $(\underline{c}_{R_c}, \overline{c}_{R_c}, \underline{c}_{R_d}, \overline{c}_{R_d}) \in \mathbb{R}^4_{>0}$, there exists a map $c : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for any $\lambda' > 0$, the choice $\lambda = 2\lambda'$ and $\gamma = e^{-2\lambda'}$ is such that any solution $(x, \hat{x}, P)$ of the cascade $\mathcal{H} - \hat{\mathcal{H}}$ initialized in $\mathcal{X}_0 \times \mathbb{R}^{n_x} \times \mathcal{S}^{n_x}_{>0}$ with $(u_c, u_d) \in \mathcal{U}$ and (R_c, R_d) satisfying (2c) with $(\underline{c}_{R_c}, \overline{c}_{R_c}, \underline{c}_{R_d}, \overline{c}_{R_d})$, is complete and verifies for all $(t,j) \in \operatorname{dom} x$,

$$\begin{aligned} &|x(t,j) - \hat{x}(t,j)| \\ &\quad \leq c(\|\Pi(0,0)\|)|x(0,0) - \hat{x}(0,0)|e^{-\lambda'(t+j-(\Delta+1))}. \end{aligned} \tag{13}$$

Proof: First, adapting the steps leading to (8) in the proof of Theorem 1 to the particular choice of λ and γ, Π is uniformly lower-bounded by $e^{-2\lambda'(\Delta+1)}\underline{c}(\|\Pi(0,0)\|)$ for some $\underline{c} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ depending only on the uniform quantities in Assumption 4 and $\overline{c}_{R_c}, \overline{c}_{R_d}$. Second, from the proof of Theorem 1, we have $\dot{V} \leq -\lambda V$ and $V^+ \leq \gamma V$ along (9) and (6), which translates to $V(t,j) \leq e^{-\lambda t}\gamma^j V(0,0) \leq e^{-2\lambda'(t+j)}V(0,0)$. Then (13) holds. ∎

Remark 4: Note from (13) that the gain with respect to the initial error is proportional to $e^{\lambda'(\Delta+1)}$, which increases with the choice of the rate λ', characterizing the *peaking* phenomenon typically encountered in high-gain designs. While an arbitrarily fast *exponential rate* is achieved in (13) at all times, arbitrarily fast *convergence* of the error can only be achieved *after* $t + j = \Delta + 1$. This is explained by the necessity of achieving observability (see the UCO condition in Definition 2). Note finally that (13) can also easily be achieved by pushing only λ (resp. γ) under a dwell time (resp. reverse dwell time) (see [9]), by bringing stability and rate from flows to jumps and vice-versa.

Note that the asymptotic stability of the estimation error typically ensures robustness properties with respect to delays in the jump triggering of the observer, when the jump times are not perfectly known. For instance, in the autonomous context, [9, Theorem 6.4] shows the semi-global practical stability outside of the delay intervals assuming a dwell time, boundedness of solutions, and the hybrid basic conditions.

Example 2 (Spiking neuron): The spiking behavior of a neuron may be modeled with state $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ as

$$\begin{cases} \dot{\xi} = (0.04\xi_1^2 + 5\xi_1 - \xi_2 + I, a(b\xi_1 - \xi_2)) & \text{when } \xi_1 \geq v_m \\ \xi^+ = (c, \xi_2 + d) & \text{when } \xi_1 = v_m \end{cases} \tag{14}$$

where ξ_1 is the membrane potential, ξ_2 is the recovery variable, and I is a constant [27]. The parameters, characterizing the neuron type, are taken here for instance as $I = 150$, $a = 0.02$, $b = 0.2$, $c = -55$, $d = 4$, and $v_m = 30$ (all in appropriate units). The jump times of the solutions of (14) are detected from the discontinuities of the output $y_c = \xi_1$. On the other hand, we assume d is unknown and seek to estimate online (ξ_1, ξ_2, d). Note that d is not observable during flow, but it becomes observable from the combination of flows and jumps as noticed in [9]. We thus re-model (14) into the form (1) with $x = (x_1, x_2, x_3) = (\xi_1, \xi_2, d) \in \mathbb{R}^3$, matrices $F = \begin{pmatrix} 5 & -1 & 0 \\ ab & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $H_c = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, $H_d = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$, and $u_c = (0.04y_c^2 + I, 0, 0)$, $u_d = (c, 0, 0)$ known exogenous terms that can be perfectly compensated using output injection. Note that J is not invertible and does not verify Assumption 2. A possibility is to notice that because the jump map of (14) is only active when $\xi_1 = v_m$, it can be rewritten as $\xi_1^+ = \xi_1 - v_m + c$, while preserving the same hybrid system. It would then be cast in the form (1) with $u_d = (-v_m + c, 0, 0)$ and the invertible matrix $J = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, thus satisfying Assumption 2. However, for the sake of illustration, we show in Figure 3 the results of a simulation using the non-invertible formulation. This suggests that the invertibility of J might only be for theoretical analysis and it is not necessary to implement the observer (2). See [26] for more details.

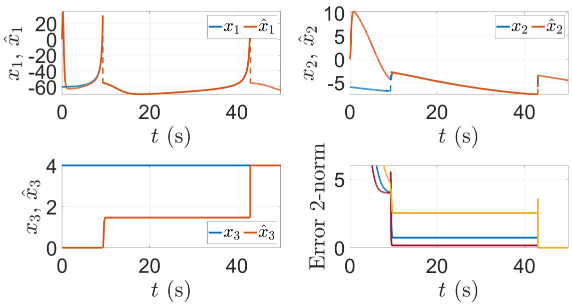

Fig. 3. State and parameter estimation in a spiking neuron. Last figure: Comparison of the 2-norm of the error for increasing λ' (yellow—nominal case, blue and red— higher values of λ' with worse peaking).

V. ROBUSTNESS OF THE ERROR AGAINST DISTURBANCES AND MEASUREMENT NOISE

Dealing with uncertainties such as input disturbances and measurement noise is an asset of Kalman(-like) observers for a robust and practical design. Consider the system (1) with flow/jump input disturbances $v_c \in \mathbb{R}^{n_x}$, $v_d \in \mathbb{R}^{n_x}$ and measurement noise $w_c \in \mathbb{R}^{n_{y,c}}$, $w_d \in \mathbb{R}^{n_{y,d}}$ as

$$\mathcal{H}_d \begin{cases} \dot{x} = Fx + u_c + v_c & (x, u_c) \in C \quad y_c = H_c x + w_c \\ x^+ = Jx + u_d + v_d & (x, u_d) \in D \quad y_d = H_d x + w_d \end{cases} \tag{15}$$

Theorem 3 shows that the estimate provided by the cascade of $\mathcal{H}_d$ in (15) with the observer $\hat{\mathcal{H}}$ in (2) is *robustly stable* with respect to the uncertainties in the sense of [24, Definition 2] (extended to hybrid systems), which is stronger than the classical Input-to-State-Stability (ISS) defined in [28] by an increasing penalty of past uncertainties.

Theorem 3: Under Assumptions 1, 2, and 4, there exist $\lambda^\star > 0$ and $\gamma^\star > 0$ such that for any $\Pi_0 \in \mathcal{S}_{>0}^{n_x}$, any $(\underline{c}_{R_c}, \overline{c}_{R_c}, \underline{c}_{R_d}, \overline{c}_{R_d}) \in \mathbb{R}_{>0}^4$, any $\lambda > \lambda^\star$, and any $0 < \gamma < \gamma^\star$, the cascade $\mathcal{H}_d - \hat{\mathcal{H}}$ initialized in $\mathcal{X}_0 \times \mathbb{R}^{n_x} \times \{\Pi_0\}$ with $(u_c, u_d) \in \mathcal{U}$ and (R_c, R_d) satisfying (2c) for $(\underline{c}_{R_c}, \overline{c}_{R_c}, \underline{c}_{R_d}, \overline{c}_{R_d})$ is complete and robustly stable with respect to the uncertainties (v_c, w_c, v_d, w_d).

Proof: Following the proof of (8) in Theorem 1, Π is uniformly lower-bounded by $(e^{-\lambda}\gamma)^{\Delta+1}\underline{c}$ with $\underline{c}$ depending only on the parameters of Assumption 4, $\overline{c}_{R_c}, \overline{c}_{R_d}$ and Π_0. On the other hand, since for any $((t', j'), (t, j)) \in \operatorname{dom} x \times \operatorname{dom} x$ with $t' < t$ and $j' < j$, we have $\|\Phi_{F,J}((t', j'), (t, j))\| \leq e^{c_F(t-t')}(c_{J^{-1}})^{j-j'}$, we get from (7) and the triangle inequality:

$$\begin{aligned} \|\Pi(t, j)\| \leq{} & e^{(2c_F-\lambda)t}(\gamma c_{J^{-1}}^2)^j \|\Pi_0\| \\ & + c_{H_c}^2 \sum_{k=0}^{j-1} (\gamma c_{J^{-1}}^2)^{j-k} \int_{t_k}^{t_{k+1}} e^{(2c_F-\lambda)(t-s)} ds \\ & + c_{H_d}^2 \sum_{k=0}^{j-1} e^{(2c_F-\lambda)(t-t_k)} (\gamma c_{J^{-1}}^2)^{j-k} \\ & + c_{H_c}^2 \int_{t_j}^{t} e^{(2c_F-\lambda)(t-s)} ds. \end{aligned}$$

Therefore, pick $\lambda_0^\star > 2c_F$ and $0 < \gamma_0^\star < c_{J^{-1}}^{-2}$ with $\gamma_0^\star \leq 1$. Then, there exists $\overline{c} > 0$ depending only on the uniform quantities in Assumption 4, Π_0, and $\lambda_0^\star$, $\gamma_0^\star$ such that for all $\lambda > \lambda_0^\star$ and $0 < \gamma < \gamma_0^\star$, $\Pi \leq \overline{c}I$. Now, in the presence of disturbances and noise, the error $\tilde{x} := x - \hat{x}$ has the dynamics

$$\begin{cases} \dot{\tilde{x}} = (F - \Pi^{-1} H_c^\top R_c^{-1} H_c)\tilde{x} + v_c - \Pi^{-1} H_c^\top R_c^{-1} w_c \\ \tilde{x}^+ = J(I - \Pi^{-1} H_d^\top (H_d \Pi^{-1} H_d^\top + R_d)^{-1} H_d)\tilde{x} \\ \qquad + v_d - J\Pi^{-1} H_d^\top (H_d \Pi^{-1} H_d^\top + R_d)^{-1} w_d. \end{cases} \tag{16}$$

Consider the Lyapunov function $V(\tilde{x}, \Pi) = \tilde{x}^\top \Pi \tilde{x}$. Let us study the dynamics of V along (16) and (6). During flows, thanks to Cauchy-Schwartz and Young's inequalities as well as the uniform upper bounds of the matrices, there exist positive scalars σ_1 and σ_2 (independent of λ and γ) such that we have for all $\lambda > \lambda_0^\star$ and $0 < \gamma < \gamma_0^\star$,

$$\begin{aligned} \dot{V} ={} & -\lambda V - \tilde{x}^\top H_c^\top R_c^{-1} H_c \tilde{x} + 2\tilde{x}^\top \Pi v_c - 2\tilde{x}^\top H_c^\top R_c^{-1} w_c \\ \leq{} & -\tfrac{\lambda}{3} V + \tfrac{\sigma_1}{\lambda}|v_c|^2 + \tfrac{\sigma_2}{\lambda(e^{-\lambda}\gamma)^{\Delta+1}}|w_c|^2. \end{aligned}$$

At jumps, in a similar way, there exist positive scalars σ_3, σ_4, and σ_5 (independent of λ and γ) such that we have for all $\lambda > \lambda_0^\star$ and $0 < \gamma < \gamma_0^\star$,

$$\begin{aligned} V^+ ={} & \gamma V - \gamma\tilde{x}^\top H_d^\top (H_d \Pi^{-1} H_d^\top + R_d)^{-1} H_d \tilde{x} \\ & + 2\gamma\tilde{x}^\top \Pi J^{-1} v_d - 2\gamma\tilde{x}^\top H_d^\top (H_d \Pi^{-1} H_d^\top + R_d)^{-1} w_d \\ & + \gamma v_d^\top (J^{-1})^\top (\Pi + H_d^\top R_d^{-1} H_d) J^{-1} v_d \\ & - 2\gamma v_d^\top (J^{-1})^\top H_d^\top R_d^{-1} w_d \\ & + \gamma w_d^\top R_d^{-1} H_d (\Pi + H_d^\top R_d^{-1} H_d)^{-1} H_d^\top R_d^{-1} w_d \\ \leq{} & 3\gamma V + \gamma\sigma_3|v_d|^2 + \gamma\left(\tfrac{\sigma_4}{(e^{-\lambda}\gamma)^{\Delta+1}} + \sigma_5\right)|w_d|^2. \end{aligned}$$

Therefore, for any $\lambda > \lambda_0^\star$ and any $0 < \gamma < \min\{\gamma_0^\star, \frac{1}{3}\}$, we have

$$\dot{V} \leq -\lambda_c V + \alpha_c |d_c|^2, \qquad V^+ \leq \gamma_d V + \alpha_d |d_d|^2,$$

where $\lambda_c = \frac{\lambda}{3} > 0$, $\gamma_d = 3\gamma \in (0, 1)$, $\alpha_c = 2\max\left\{\frac{\sigma_1}{\lambda}, \frac{\sigma_2}{\lambda(e^{-\lambda}\gamma)^{\Delta+1}}\right\}$, $\alpha_d = 2\max\left\{\gamma\sigma_3, \gamma\left(\frac{\sigma_4}{(e^{-\lambda}\gamma)^{\Delta+1}} + \sigma_5\right)\right\}$, $|d_c|^2 = \max\{|v_c|^2, |w_c|^2\}$, and $|d_d|^2 = \max\{|v_d|^2, |w_d|^2\}$. This means that $\tilde{x}$ satisfies for some positive scalars κ_1 and κ_2,

$$\begin{aligned} |\tilde{x}(t, j)|^2 \leq{} & \kappa_1 \big(e^{-\lambda_c t}\gamma_d^j \kappa_2 |\tilde{x}(0, 0)|^2 \\ & + \alpha_c \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} e^{-\lambda_c(t-s)} \gamma_d^{j-k} |d_c(s, k)|^2 ds \\ & + \alpha_d \sum_{k=0}^{j-1} e^{-\lambda_c(t-t_{k+1})} \gamma_d^{j-k} |d_d(t_{k+1}, k)|^2\big) \\ & + \int_{t_j}^{t} e^{-\lambda_c(t-s)} |d_c(s, j)|^2 ds. \end{aligned}$$

Taking the square root of both sides, we obtain robust stability according to [24] (but for a hybrid system). ■

Example 3 (Continuous system with multi-rate outputs): Consider a vehicle with position x_1, velocity x_2, and acceleration x_3. We measure x_3 with a fast rate of 50 (Hz), so this can be seen as a continuous output, which however contains a lot of high-frequency noise. We then measure x_1 thanks to a less noisy GPS at the rate of 1 (Hz). This makes the system observable already; however, to illustrate that our method covers systems with *multi-rate* sampled outputs, let us assume that we also measure x_2 sporadically from every 1.5 (s) to every 2 (s). This system is written in hybrid form, with state $x = (x_1, x_2, x_3)$, input u_c, and two additional timers τ_1, τ_2 as

$$\begin{cases} \left.\begin{array}{l} \dot{x} = (x_2, x_3, u_c) \\ \dot{\tau}_1 = -1 \\ \dot{\tau}_2 = -1 \end{array}\right\} \text{when } \begin{cases} \tau_1 \in [0,1] \\ \tau_2 \in [0,2] \end{cases} \\ \left.\begin{array}{l} x^+ = x \\ \tau_1^+ = \begin{cases} 1, & \text{if } \tau_1 = 0 \\ \tau_1, & \text{if } \tau_1 \neq 0 \end{cases} \\ \tau_2^+ \in \begin{cases} [1.5, 2], & \text{if } \tau_2 = 0 \\ \{\tau_2\}, & \text{if } \tau_2 \neq 0 \end{cases} \end{array}\right\} \text{when } \left[\begin{array}{l} \tau_1 = 0 \\ \tau_2 = 0 \end{array}\right. \end{cases} \tag{17a}$$

with the outputs

$$y_c = x_3, \quad y_d = \begin{cases} (x_1, 0), & \text{if } \tau_1 = 0 \text{ and } \tau_2 \neq 0 \\ (0, x_2), & \text{if } \tau_2 = 0 \text{ and } \tau_1 \neq 0 \\ (x_1, x_2), & \text{if } \tau_1 = \tau_2 = 0 \end{cases}, \tag{17b}$$

and initialized with $\tau_1(0,0) = 1$ and $\tau_2(0,0) \in [1.5, 2]$. Figure 4 shows a scenario where we have a fixed sampling period of 1.5 (s) for x_2, measurement noise (with high frequency and amplitude ≈ 3 in y_c, low frequency and amplitude ≈ 1 in y_d), and the input $u_c = 0.01$ (m/s^3) is an *unknown* bias (assumed 0 in the observer, so that $v_c = u_c$).

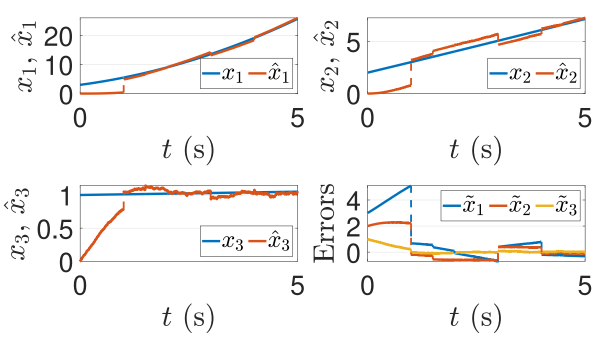

Fig. 4. State estimation in a car with multi-rate sampled outputs (with input disturbances and noise).

VI. CONCLUSION

We have provided a *systematic* hybrid Kalman-like observer for general hybrid systems with linear maps and known jump times, based on *uniform complete observability*. Its implementation is straightforward and applies directly to a wide class of systems including switched as well as continuous-time sampled systems with sporadic/multiple rates. Its complexity is the same as for a continuous or discrete Kalman filter, with dimension $n_x + n_x^2$, with the *same* covariance matrix shared among flows and jumps. Another route could be to follow [11] through an observability decomposition: we could combine a *continuous* Kalman-like observer—estimating only the part of the state that is instantaneously observable during flows from y_c—with a *discrete* Kalman-like observer for the rest of the state, thus possibly reducing the observer dimension by splitting the covariance matrix. However, the possibility of decomposition is not guaranteed for time-varying systems and may not verify the necessary decoupling conditions. Future directions include properly taking into account, in the covariance matrix, errors in the jump triggering and eventually developing a Kalman observer for hybrid systems with *unknown* jump times.

APPENDIX

Lemma 1: Consider a quadruple (F, J, H_c, H_d) defined on a hybrid time domain $\mathcal{D}$ and verifying the boundedness condition of Assumption 3. Pick (K_c, K_d) defined and uniformly upper-bounded on $\mathcal{D}$. Then, for any $\Delta_m > 1$, there exists $c_{\mathcal{G}} > 0$ such that any hybrid arc $\tilde{x}$ defined on $\mathcal{D}$ and verifying the linear dynamics $\dot{\tilde{x}} = (F - K_c H_c)\tilde{x}$ during flows and $\tilde{x}^+ = (J - K_d H_d)\tilde{x}$ at jumps, verifies for all $((t', j'), (t, j)) \in \mathcal{D} \times \mathcal{D}$ with $(t - t') + (j - j') \leq \Delta_m$,

$$\begin{aligned} &\tilde{\mathcal{G}}((t', j'), (t, j)) \\ &\quad \geq c_{\mathcal{G}}\, \tilde{x}^\top(t, j) \mathcal{G}_{(F,J,H_c,H_d)}((t', j'), (t, j)) \tilde{x}(t, j), \end{aligned} \tag{18}$$

where

$$\begin{aligned} &\tilde{\mathcal{G}}((t', j'), (t, j)) = \textstyle\int_{t'}^{t_{j'+1}} \star^\top H_c(s, j')\tilde{x}(s, j')ds \\ &+ \textstyle\sum_{k=j'+1}^{j-1} \tilde{\mathcal{G}}_F(k) + \sum_{k=j'}^{j-1} \tilde{\mathcal{G}}_J(k) + \int_{t_j}^{t} \star^\top H_c(s, j)\tilde{x}(s, j)ds, \end{aligned}$$

with $\tilde{\mathcal{G}}_F$ and $\tilde{\mathcal{G}}_J$ defined in (11).

Remark 5: Define $\tilde{F} := F - K_c H_c$ and $\tilde{J} := J - K_d H_d$. If $\tilde{J}$ is invertible at all times, then Lemma 1 is equivalent to the fact that for all $((t', j'), (t, j)) \in \mathcal{D} \times \mathcal{D}$ with $(t - t') + (j - j') \leq \Delta_m$, we have

$$\mathcal{G}_{(\tilde{F},\tilde{J},H_c,H_d)}((t', j'), (t, j)) \geq c_{\mathcal{G}}\, \mathcal{G}_{(F,J,H_c,H_d)}((t', j'), (t, j)).$$

Proof: This proof resembles that of [13, Theorem 3] but in the case where $F \neq 0$ and $J \neq 0$, and that of [7] extended to the hybrid case. The key idea is to consider the terms $-K_c H_c \tilde{x}$ and $-K_d H_d \tilde{x}$ in the dynamics of $\tilde{x}$ as flow/jump inputs respectively, so that

$$\begin{aligned} \tilde{x}(t, j) = {}& \Phi_{F,J}((t, j), (t', j'))\tilde{x}(t', j') \\ & - \textstyle\int_{t'}^{t_{j'+1}} \Psi_A((t, j), (s, j'))ds \\ & - \textstyle\sum_{k=j'+1}^{j-1} \int_{t_k}^{t_{k+1}} \Psi_A((t, j), (s, k))ds \\ & - \textstyle\sum_{k=j'}^{j-1} \Psi_B((t, j), (t_{k+1}, k)) \\ & - \textstyle\int_{t_j}^{t} \Psi_A((t, j), (s, j))ds, \end{aligned} \tag{19}$$

where $A := K_c H_c$, $B := K_d H_d$, and

$$\begin{aligned} \Psi_A((t, j), (t', j')) &= \Phi_{F,J}((t, j), (t', j'))A(t', j')\tilde{x}(t', j'), \\ \Psi_B((t, j), (t', j')) &= \Phi_{F,J}((t, j), (t', j'))B(t', j')\tilde{x}(t', j'). \end{aligned}$$

Since $(t - t') + (j - j') \leq \Delta_m$, we deduce the upper bounds on Ψ_A and Ψ_B as

$$\begin{aligned} |\Psi_A((t, j), (t', j'))| &\leq c_A |H_c(t', j')\tilde{x}(t', j')|, \\ |\Psi_B((t, j), (t', j'))| &\leq c_B |H_d(t', j')\tilde{x}(t', j')|, \end{aligned}$$

with scalars c_A and c_B independent of (t,j,t',j') given by

$$c_A = c_{K_c}(e^{c_F}\max\{c_J, c_{J^{-1}}\})^{\Delta_m},\\ c_B = c_{K_d}(e^{c_F}\max\{c_J, c_{J^{-1}}\})^{\Delta_m}.$$

Let us now lower-bound each term in $\tilde{\mathcal{G}}$. Using (19) with (s,k) replacing (t,j), and then consecutively $|a-b|^2 \geq \frac{\rho}{1+\rho}|a|^2 - \rho|b|^2$ for some $\rho > 0$, $|\sum_{i=1}^N a_i|^2 \leq N\sum_{i=1}^N |a_i|^2$, the triangle inequality, the Cauchy-Schwartz inequality, and the bounds on Ψ_A and Ψ_B (see in detail in [29]), we get

$$\tilde{\mathcal{G}}_F(k) \geq \tfrac{\rho}{1+\rho}\int_{t_k}^{t_{k+1}} \left|H_c(s,k)\Phi_{F,J}((s,k),(t,j))\tilde{x}(t,j)\right|^2 ds\\ -\rho c_{H_c}^2(2(k-j')+1)(t_{k+1}-t'+1)\max\{c_A^2, c_B^2\}\\ \times(t_{k+1}-t_k)\tilde{\mathcal{G}}.$$

Similarly to $\tilde{\mathcal{G}}_F(k)$, we get

$$\int_{t'}^{t_{j'+1}} \star^\top H_c(s,j')\tilde{x}(s,j')ds \geq\\ \tfrac{\rho}{1+\rho}\int_{t'}^{t_{j'+1}} \left|H_c(s,j')\Phi_{F,J}((s,j'),(t,j))\tilde{x}(t,j)\right|^2 ds - \rho c_{H_c}^2\\ \times (t_{j'+1}-t')\max\{c_A^2, c_B^2\}(t_{j'+1}-t')\tilde{\mathcal{G}},\\ \int_{t_j}^{t} \star^\top H_c(s,j)\tilde{x}(s,j)ds \geq\\ \tfrac{\rho}{1+\rho}\int_{t_j}^{t} \left|H_c(s,j)\Phi_{F,J}((s,j),(t,j))\tilde{x}(t,j)\right|^2 ds - \rho c_{H_c}^2\\ \times (2(j-j')+1)(t-t'+1)\max\{c_A^2, c_B^2\}(t-t_j)\tilde{\mathcal{G}},\\ \tilde{\mathcal{G}}_J(k) \geq \tfrac{\rho}{1+\rho}\left|H_d(t_{k+1},k)\Phi_{F,J}((t_{k+1},k),(t,j))\tilde{x}(t,j)\right|^2\\ -\rho c_{H_d}^2(2(k-j')+1)(t_{k+1}-t'+1)\max\{c_A^2, c_B^2\}\tilde{\mathcal{G}}.$$

Now let us lower-bound $\tilde{\mathcal{G}}$ by summing the obtained inequalities. Since $(t-t')+(j-j') \leq \Delta_m$, we get

$$\tilde{\mathcal{G}} \geq \tfrac{\rho}{1+\rho}\tilde{x}(t,j)^\top \mathcal{G}_{(F,J,H_c,H_d)}((t',j'),(t,j))\tilde{x}(t,j)\\ -\rho\max\{c_A^2, c_B^2\}\{c_{H_c}^2, c_{H_d}^2\}\tilde{\mathcal{G}}\\ \times[(j-j'-1)(2(j-j')-1)(t_j-t'+1)(t_j-t_{j'+1})\\ +(j-j')(2(j-j')-1)(t_j-t'+1)\\ +(2(j-j')+1)(t-t'+1)(t-t_j)+(t_{j'+1}-t')^2]\\ \geq \tfrac{\rho}{1+\rho}\tilde{x}(t,j)^\top \mathcal{G}_{(F,J,H_c,H_d)}((t',j'),(t,j))\tilde{x}(t,j)\\ -\rho\max\{c_A^2, c_B^2\}\{c_{H_c}^2, c_{H_d}^2\}\tilde{\mathcal{G}}\\ \times[\Delta_m^2(2\Delta_m-1)(\Delta_m+1)+\Delta_m(2\Delta_m-1)(\Delta_m+1)\\ +(2\Delta_m+1)(\Delta_m+1)\Delta_m+\Delta_m^2],$$

and thus, the result follows. ∎

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