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The First Achievement of a Given Level by a Random Process

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***Abstract*— We estimate the probability that the first achievement of a given level by the component $y_1(x)$ of n-dimensional continuous process $\mathbf{y}(x)=\{y_1(x),...,y_n(x)\}$ occurs at some moment x^* from a given interval (x',x'') and, at this moment x^*, the condition $(y_2(x^*),...,y_n(x^*))\in D$ holds, where D is a given domain of $(n-1)$-dimensional Euclidean space R^{n-1}. The need to calculate the above-mentioned probability arises in the problems of aircraft control during landing.**

I. INTRODUCTION AND PROBLEM STATEMENT

Consider a point M moving in the n-dimensional Euclidean space R^n; changing the position of point M is described by an n-dimensional continuous random process $\mathbf{y}(x)$. At the initial moment $x = x_0$, the point is located at position $\mathbf{y}(x_0) = \mathbf{y}^0$; in general case, this position may be unknown. The independent variable is time or one of the components of the process $\mathbf{y}(x)$ (provided that this component changes continuously and monotonously). We denote by Q a given domain in R^n; by ∂Q, the boundary of domain Q; and by $\partial_1 Q$, some part of boundary ∂Q. It is known that $\mathbf{y}^0\in Q$. We are to find the probability

$$\mathbf{P}\left\{\begin{array}{c}\text{there exists some value } x^* \text{ from a given interval } (x',x'')\\ \text{such that } M(x^*)\in\partial_1 Q \text{ and } M(x)\in Q\ \ \forall x<x^*\end{array}\right\}$$

of the event that point M will reach the boundary of domain Q for the first time at some moment x^* from a given interval (x',x'') and it will reach specifically the part $\partial_1 Q$ of boundary ∂Q (see Fig. 1).

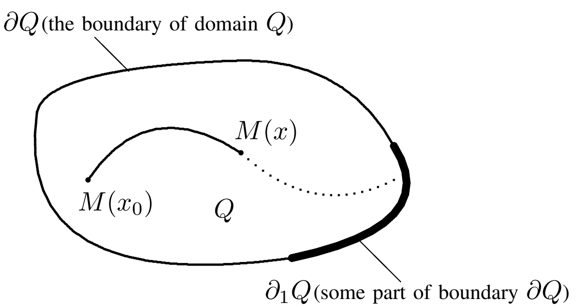

Fig. 1. The problem statement

This mathematical problem was posed in [1]. *This problem appears in studying stochastic systems in situations when a normal system operation corresponds to a position of a point depicting the system in a certain domain Q of the system state space and consequences of the point leaving this domain depend on what part of the domain boundary it will leave through.*

We consider the particular case of domain Q; namely, when Q is a half-space:

$$Q = \{(v_1,\ldots,v_n)\in R^n:\ v_1>h\}, \tag{1}$$

where h is a given number; in this case (see Fig. 2),

$$\partial Q = \{(v_1,\ldots,v_n)\in R^n:\ v_1=h\},$$
$$\partial_1 Q = \{(v_1,\ldots,v_n)\in R^n:\ v_1=h,\ (v_2,\ldots,v_n)\in D\},$$

where D is a given subset of the $(n-1)$-dimensional Euclidean space R^{n-1}.

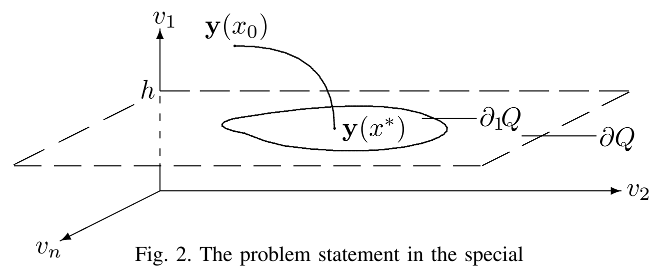

Fig. 2. The problem statement in the special case when domain Q is a half-space

In this important special case, the mathematical statement of the general problem described above is formulated as follows. Let $\mathbf{y}(x)=\{y_1(x),\ldots,y_n(x)\}$ be an n-dimensional continuous random process and let h be a given number. We will consider process $\mathbf{y}(x)$ defined on the semiinterval $(x_0,x'']$, $x_0\geq-\infty$, and satisfying the condition

$$\lim_{x\to x_0}\mathbf{P}\{y_1(x)>h\}=1. \tag{2}$$

The physical meaning of condition (2) is that, at the initial time moment, the stochastic system whose behavior is described by process $\mathbf{y}(x)$ is located inside a known domain of system's state space; in our case (see Fig. 2), this domain is defined by equality (1).

Choose arbitrary $x'\in(x_0,x'')$ and D, a subset of R^{n-1}. Define the events L and Z_D as follows:

$$L=\left\{\exists\,\hat{x}\in(x_0,x'']\ \ \forall x\in(x_0,\hat{x})\ \ y_1(x)>h\right\}$$

and

$$Z_D=\left\{\begin{array}{c}\exists\,x^*\in(x',x'')\ \ \forall x\in(x_0,x^*)\ \ y_1(x)>h;\\ y_1(x^*)=h;\ (y_2(x^*),\ldots,y_n(x^*))\in D\end{array}\right\}. \tag{3}$$

Event Z_D consists in the fact that the first achievement of a given level h by the component $y_1(x)$ occurs at some moment x^* from a given interval (x',x'') and, at this moment x^*, the condition $(y_2(x^*),\ldots,y_n(x^*))\in D$ holds.

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We are to find the conditional probability $\mathbf{P}\{Z_D|L\}$ of event Z_D given that event L has occurred[1].

As shown in [2], [3], and [4], the problem of calculating the probability of an aircraft safe landing is the problem of calculating the probability $\mathbf{P}\{Z_D\}$ if the condition $(y_2(x^*),\ldots,y_n(x^*))\in D$ from (3) is specified as follows:

$$y_{i,min} \le y_i(x^*) \le y_{i,max},\; i=2,\ldots,n, \qquad (4)$$

where $y_{i,min}$ and $y_{i,max}$ are given numbers. In this case (see Fig. 3), process $\mathbf{y}(x)$ describes an aircraft's behavior during landing, component $y_1(x)$ is the flight altitude, level h equals zero, independent variable x is the flight length, and event Z_D denotes safe landing, i.e., the fact

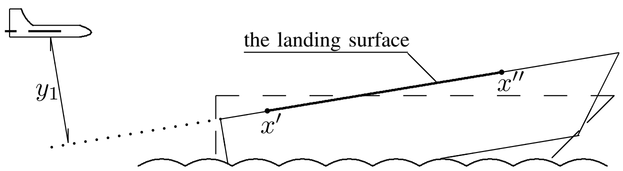

Fig. 3. Landing an aircraft

that the aircraft first touches the landing surface on a given interval (x',x'') and, at the moment x^* of this touching, the aircraft's phase coordinates (elevation angle, banking angle, vertical velocity, and so on), which represent components of vector $\mathbf{y}(x)$, remain inside admissible ranges that exclude an emergency. Numbers $y_{i,min}$ and $y_{i,max}$ from (4) define the range of safe values of phase coordinates at the moment x^* of landing.

II. PRELIMINARIES

A. General Description of the Situation

There are a large number of publications devoted to problems about achievements and crossings of a given level by a random process; see, e.g., survey publications [5]–[9]. The systematic study of these problems was started in [10]. Using heuristic methods, a number of important results, in particular a formula for the average number of crossings of a fixed level, were obtained in [10]. Decades and efforts of many researchers were required to find the weakest conditions for the correctness of these results and the mathematically faultless formulations and proofs of the corresponding statements and their further generalizations; see, e.g., [11]–[35].

For a certain class of continuous Markov processes, the problem described in Section I can be reduced to solving a boundary value problem for a partial differential equation (see [1]). For smooth differentiable processes, the above mathematical problem has not been solved. It is those processes that describe the changes of the aircraft's phase coordinates during landing. These processes are non-Markovian processes. All (known to the authors) results for the probability of reaching the given boundaries by a non-Markovian process refer to the one-dimensional case and to the situation when a random process under consideration is a stationary Gaussian process with a correlation function of a special type (see, e.g., [23] and [28]).

We also note that there is an approach to studying the safe functioning of continuous-time stochastic systems based on finding the so-called barrier certificates (see, e.g., [36], [37], and [38]): using a special technique, an upper estimate is found for the probability that the system will be in an unsafe domain of the state space. However the corresponding technique assumes that the behavior of the system is described by a Markov process.

In this paper, we present the results concerning the estimates of the probability of event (3) for non-Markovian processes.

B. Previously Established Result

Following [5], we denote by $G_h(x_0,x'')$ the set of scalar functions continuous on $[x_0,x'']$ or $(x_0,x'']$ (depending on the set of values for variable x that we consider) that do not identically equal h on any subinterval inside interval (x_0,x''). For functions from $G_h(x_0,x'')$, we use the concepts "crossing of level h", "touching of level h", "upcrossing of level h", and "downcrossing of level h" in accordance with their definitions given in [5].

We will assume that with probability 1 sample functions $y_1(x)$ belong to the set $G_h(x_0,x'')$ and do not touch the level h, and the average[2] number of crossings $N(x_0,x'')$ of level h by process $y_1(x)$ on the interval (x_0,x'') is finite. Then, taking into account condition (2), it is easy to see that $\mathbf{P}\{Z_D|L\}=\mathbf{P}\{Z_D\}$, i.e., instead of bounds on the conditional probability $\mathbf{P}\{Z_D|L\}$ we can prove bounds on the unconditional probability $\mathbf{P}\{Z_D\}$.

By $N^+(x_1,x_2)$ denote the average number of upcrossings of level h by the component $y_1(x)$ on the interval (x_1,x_2), by $N^-(x_1,x_2)$ denote the average number of downcrossings of level h by the component $y_1(x)$ on the interval (x_1,x_2), and by $N_D^-(x_1,x_2)$ denote the average number of downcrossings of level h by the component $y_1(x)$ on the interval (x_1,x_2) such that at moments of these downcrossings the condition $(y_2,\ldots,y_n)\in D$ is satisfied for other components $y_2,\ldots,y_n$ of process $\mathbf{y}(x)$. The following result takes place (see [39]).

Theorem 1. *Suppose that* 1) *with probability* 1 *sample functions* $y_1(x)$ *belong to the set* $G_h(x_0,x'')$ *and do not touch the level h on the interval* (x_0,x''), $N(x_0,x'')<\infty$; 2) $\mathbf{P}\{y_1(x')=h\}=0$; 3) *condition* (2) *holds. Then at any* $l=2,3,\ldots$ *and any partition of the interval* (x_0,x'')

$$x_0 < x_1 < \ldots < x_{l-1} < x_l = x'',$$

where for every $i=1,2,\ldots,l-1$ $\quad \mathbf{P}\{y_1(x_i)=h\}=0$, *the inequalities*

$$N_D^-(x',x'') - N^+(x_0,x'') + \Delta \le \mathbf{P}\{Z_D\} \le N_D^-(x',x'')$$

hold, where

[1]We note that if x_0 is finite then all results shown below also hold if instead of processes defined on the semiinterval $(x_0,x'']$ and satisfying condition (2) we consider processes defined on the segment $[x_0,x'']$ and satisfying condition $\mathbf{P}\{y_1(x_0)>h\}=1$; in this case, event L will be defined as $L=\{y_1(x_0)>h\}$.

[2]The term "average" means the mathematical expectation of the corresponding random variable.

$$\Delta = \Delta(x_1,\dots,x_l) = \sum_{i=1}^{l-1} \mathbf{P}\Big\{\Big(y_1(x_i)<h\Big)\cap\Big[\bigcap_{j=i+1}^{l}\Big(y_1(x_j)>h\Big)\Big]\Big\}.$$

And if we add to existing points x_i, $i=1,2,\dots,l-1$, new points x_k, for which $\mathbf{P}\{y_1(x_k)=h\}=0$, the value of Δ can only increase as a result.

In order for sample functions $y_1(x)$ to belong to the set $G_h(x_0,x'')$, it suffices to require that sample functions are continuous with probability 1 on the corresponding interval and that equality $\mathbf{P}\{y_1(x_r)=h\}=0$ holds for every rational point x_r from this interval. In order to ensure that there are no touchings of level h with probability 1 and the average number of crossings $N(x_0,x'')$ is finite, it suffices to require that for every fixed x the one-dimensional distribution density of process $y_1(x)$ is bounded and that sample functions $y_1(x)$ are continuously differentiable with probability 1 on the corresponding interval (see, e.g., [13]).

If the process $y_1(x)$ is differentiable in mean-square, then the number $N^+(x_0,x'')$ can be calculated by the Rice formula (see, e.g., [18])

$$N^+(x_0,x'') = \int_{x_0}^{x''} dx \int_0^{\infty} v f_x(h,v)dv.$$

Here $f_x(h,v) = f_x(u_1,v)\Big|_{u_1=h}$, $f_x(u_1,v)$ is the joint distribution density of random values $y_1(x)$ and $\zeta(x)$, where $\zeta(x)\equiv y_1'(x)$ is the derivative in mean-square of process $y_1(x)$. The number $N_D^-(x',x'')$ can be calculated as

$$-\int_{x'}^{x''} dx \int\dots\int_D du_2\dots du_n \int_{-\infty}^{0} v f_x(h,v,u_2,\dots,u_n)dv,$$

where $f_x(u_1,v,u_2,\dots,u_n)$ is the joint distribution density of random values $\xi_1(x),\zeta(x),\xi_2(x),\dots,\xi_n(x)$. This result is obtained by generalizing the Rice formula.

Examples of numerical estimates of probability $\mathbf{P}\{Z_D\}$ by Theorem 1 can be found in [2], [3], [4], [40], and [41].

III. EXACT FORMULA FOR THE REQUIRED PROBABILITY IN THE ONE-DIMENSIONAL CASE

We denote by $A_j^-(x_1,x_2)$, $j=1,2,\dots$, the event that the number of crossings of level h by process $y_1(x)$ on interval (x_1,x_2) is equal to j and the first crossing is a downcrossing. Based on the intermediate results obtained in the course of the proof of Theorem 1, it is not difficult to establish the following result concerning a question about the existence of $\lim\Delta(x_1,\dots,x_n)$ as a cross-partition of the interval (x_0,x'').

Theorem 2. *Suppose that the conditions of Theorem 1 are satisfied, $x_0>-\infty$. Let $x_0<x_1<\dots<x_i\dots<x_n=x''$ and $x_0=\tilde{x}_0<\tilde{x}_1<\dots<\tilde{x}_j<\dots<\tilde{x}_m=x'$ be any partitions of segments $[x_0,x'']$ and $[x_0,x']$; $P\{y_1(x_i)=h\}=0$ for every $i=1,\dots,n-1$ and $P\{y_1(\tilde{x}_j)=h\}=0$ for every $j=1,\dots,m$. In addition, suppose that*

$$\lim_{\max\limits_{i=0,1,\dots,n-1}(x_{i+1}-x_i)\to 0} \sum_{i=0}^{n-1} \mathbf{P}\Big\{\bigcup_{k=1}^{\infty} A_{2k}^-(x_i,x_{i+1})\Big\}=0. \quad (5)$$

Then there exist limits

$$\lim_{\max\limits_{i=0,1,\dots,n-1}(x_{i+1}-x_i)\to 0} \Delta(x_1,\dots,x_n) = \Delta_{lim}(x_0,x'')$$

and

$$\lim_{\max\limits_{j=0,1,\dots,m-1}(\tilde{x}_{j+1}-\tilde{x}_j)\to 0} \Delta(\tilde{x}_1,\dots,\tilde{x}_m) = \Delta_{lim}(x_0,x');$$

and the exact equality takes place for the probability $\mathbf{P}\{Z\}$:

$$\mathbf{P}\{Z\}=N^-(x',x'')-N^+(x',x'')+\Delta_{lim}(x_0,x'')-\Delta_{lim}(x_0,x'),$$

where symbol Z denotes event Z_D when $D=R^{n-1}$. The numbers $\Delta_{lim}(x_0,x'')$ and $\Delta_{lim}(x_0,x')$ have the following sense:

$$\Delta_{lim}(x_0,x'') = \mathbf{P}\Big\{\bigcup_{k=1}^{\infty} A_{2k}^-(x_0,x'')\Big\},$$

$$\Delta_{lim}(x_0,x') = \mathbf{P}\Big\{\bigcup_{k=1}^{\infty} A_{2k}^-(x_0,x')\Big\}.$$

The following lemma gives sufficient conditions for holding assumption (5).

Lemma. *Let with probability 1 sample functions $y_1(x)$ be continuous on the segment $[x_I,x_{II}]$ and belong to the set $G_h(x_I,x_{II})$. Let $\mathbf{P}\{y_1(x)=h\}=0$ for every point $x\in[x_I,x_{II}]$ except, perhaps, a finite number of points and there exists a positive constant C such that the condition*

$$\mathbf{P}\Big\{\Big(y_1(x_i)>h\Big)\cap\Big(y_1(x_i+\frac{x_{i+1}-x_i}{2})<h\Big)\cap\Big(y_1(x_{i+1})>h\Big)\Big\} \le C\varepsilon(x_{i+1}-x_i), \quad i=0,1,\dots,n-1, \quad (6)$$

is satisfied for every rather small partition $x_I=x_0<x_1<\dots<x_{n-1}<x_n=x_{II}$, where the function $\varepsilon(\tau)$ satisfies the condition

$$\lim_{\max\limits_{i=0,1,\dots,n-1}(x_{i+1}-x_i)\to 0} \sum_{i=0}^{n-1}\sum_{m=0}^{\infty} 2^m\varepsilon\Big(\frac{x_{i+1}-x_i}{2^m}\Big)=0. \quad (7)$$

Then equality (5) holds.

Proof. Consider the sample functions $y_1(x)$ such that at every $m=0,1,2,\dots$, every $l=1,2,3,\dots,2^m$, and every $i=0,1,\dots,n-1$

$$y_1\Big(x_i+\frac{x_{i+1}-x_i}{2^m}l\Big)\neq h.$$

Under the conditions of the Lemma, the probability of such sample functions is equal to 1. We introduce the event

$$B_{m,k}(x_i,x_{i+1}) = \Big\{y_1\Big(x_i+\frac{k-1}{2^m}(x_{i+1}-x_i)\Big)>h\Big\} \cap\Big\{y_1\Big(x_i+\frac{k-\frac{1}{2}}{2^m}(x_{i+1}-x_i)\Big)<h\Big\}\cap\Big\{y_1\Big(x_i+\frac{k}{2^m}(x_{i+1}-x_i)\Big)>h\Big\}.$$

If the event $\bigcup_{k=1}^{\infty} A_{2k}^-(x_i,x_{i+1})$ occurs, then either at least one of a denumerable number of events

$$B_{0,1}(x_i,x_{i+1}),$$

$$B_{1,1}(x_i,x_{i+1}), B_{1,2}(x_i,x_{i+1}),$$

$$B_{2,1}(x_i,x_{i+1}), B_{2,2}(x_i,x_{i+1}), B_{2,3}(x_i,x_{i+1}), B_{2,4}(x_i,x_{i+1}),$$

$$\dots$$

$$B_{m,1}(x_i,x_{i+1}),B_{m,2}(x_i,x_{i+1}),\dots,B_{m,2^m}(x_i,x_{i+1}),$$
$$\dots$$

occurred or the equality $y_1\big(x_i+\frac{x_{i+1}-x_i}{2^m}l\big)=h$ was satisfied at some m (from the set of $0,1,2,\dots$) and some l (from the set of $1,2,3,\dots,2^m$). Since $\mathbf{P}\Big\{y_1\big(x_i+\frac{x_{i+1}-x_i}{2^m}l\big)=h\Big\}=0$, we have

$$\mathbf{P}\Big\{\bigcup_{j=1}^{\infty}A_{2j}^{-}(x_i,x_{i+1})\Big\}\le\sum_{m=0}^{\infty}\sum_{k=1}^{2^m}\mathbf{P}\Big\{B_{m,k}(x_i,x_{i+1})\Big\}.$$

Under the conditions of the Lemma, at every $m=0,1,2,\dots$ and every $k=1,2,3,\dots,2^m$

$$\mathbf{P}\Big\{B_{m,k}(x_i,x_{i+1})\Big\}\le C\varepsilon\Big(\frac{x_{i+1}-x_i}{2^m}\Big).$$

Therefore,

$$\sum_{i=0}^{n-1}\mathbf{P}\Big\{\bigcup_{j=1}^{\infty}A_{2j}^{-}(x_i,x_{i+1})\Big\}\le\sum_{i=0}^{n-1}\sum_{m=0}^{\infty}2^mC\varepsilon\Big(\frac{x_{i+1}-x_i}{2^m}\Big).$$

Passing this inequality to a limit as $\max\limits_{i=0,1,\dots,n-1}(x_{i+1}-x_i)\to 0$ and using condition (7), we obtain result (5). The Lemma is proved.

We note one important case of function $\varepsilon(\tau)$ when condition (7) is satisfied.

Remark. Condition (7) is satisfied if $\varepsilon(\tau)=\tau^{1+\alpha}$, where $\alpha>0$.

Proof. We have

$$\sum_{i=0}^{n-1}\sum_{m=0}^{\infty}2^m\varepsilon\Big(\frac{x_{i+1}-x_i}{2^m}\Big)=\sum_{i=0}^{n-1}\sum_{m=0}^{\infty}\frac{2^m(x_{i+1}-x_i)^{1+\alpha}}{2^m2^{m\alpha}}$$
$$=\sum_{i=0}^{n-1}(x_{i+1}-x_i)^{1+\alpha}\sum_{m=0}^{\infty}\frac{1}{2^{m\alpha}}=\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^{1+\alpha}}{1-(1/2)^{\alpha}}$$
$$\le\frac{\Big(\max\limits_{i=1,\dots,n-1}(x_{i+1}-x_i)\Big)^{\alpha}}{1-(1/2)^{\alpha}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)$$
$$=\frac{\Big(\max\limits_{i=1,\dots,n-1}(x_{i+1}-x_i)\Big)^{\alpha}}{1-(1/2)^{\alpha}}(x_{II}-x_I).$$

The last expression tends to zero as $\max\limits_{i=0,1,\dots,n-1}(x_{i+1}-x_i)\to 0$. The Remark is proved.

Now suppose that $y_1(x)$ is a Gaussian process. We represent it as

$$y_1(x)=m(x)+\overset{\circ}{y}_1(x),$$

where $m(x)=\mathbf{E}\{y_1(x)\}$ is a mathematical expectation of process $y_1(x)$ and $\overset{\circ}{y}_1(x)$ is a centered process. We find conditions for $m(x)$ and $\overset{\circ}{y}_1(x)$ when the Lemma can be used. For this purpose we consider the probability

$$p(x,\tau)=\mathbf{P}\Big\{\Big(y_1(x)>h\Big)\cap\Big(y_1(x+\tau)<h\Big)\cap\Big(y_1(x+2\tau)>h\Big)\Big\};$$

this probability is similar to the probability on the left side of inequality (6).

Let us introduce the following notation:

$$\sigma_1=\sigma(x)=\sqrt{\mathbf{E}\{\overset{\circ}{y}_1(x)\}^2},\ \sigma_2=\sigma(x+\tau)=\sqrt{\mathbf{E}\{\overset{\circ}{y}_1(x+\tau)\}^2},$$
$$\sigma_3=\sigma(x+2\tau)=\sqrt{\mathbf{E}\{\overset{\circ}{y}_1(x+2\tau)\}^2},\ r_{11}=r_{22}=r_{33}=1,$$

$$r_{12}=r(x,x+\tau)=\frac{\mathbf{E}\{\overset{\circ}{y}_1(x)\ \overset{\circ}{y}_1(x+\tau)\}}{\sigma_1\sigma_2},$$
$$r_{13}=r(x,x+2\tau)=\frac{\mathbf{E}\{\overset{\circ}{y}_1(x)\ \overset{\circ}{y}_1(x+2\tau)\}}{\sigma_1\sigma_3},$$
$$r_{23}=r(x+\tau,t+2\tau)=\frac{\mathbf{E}\{\overset{\circ}{y}_1(x+\tau)\ \overset{\circ}{y}_1(x+2\tau)\}}{\sigma_1\sigma_3},$$
$$r_{21}=r_{12},\quad r_{31}=r_{13},\quad r_{32}=r_{23};$$
$$m_1=m(x),\quad m_2=m(x+\tau),\quad m_3=m(x+2\tau);$$
$$R=\begin{vmatrix}r_{11}&r_{12}&r_{13}\\r_{21}&r_{22}&r_{23}\\r_{31}&r_{32}&r_{33}\end{vmatrix}=1+2r_{12}r_{13}r_{23}-r_{12}^2-r_{13}^2-r_{23}^2;$$

R_{ij} is the algebraic addition of the element r_{ij} of the determinant R. Without loss of generality it can be assumed that $h=0$. Then

$$p(x,\tau)=\frac{1}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3\sqrt{R}}$$
$$\cdot\int_0^{\infty}dz_1\int_{-\infty}^{0}dz_2\int_0^{\infty}\exp\Big\{-\frac{1}{2R}\sum_{i,j=1}^{3}R_{ij}\frac{(z_i-m_i)(z_j-m_j)}{\sigma_i\sigma_j}\Big\}dz_3.$$

By means of the replacement of variables

$$\tilde z_1=\frac{z_1-m_1}{\sigma_1},\quad \tilde z_2=\frac{z_2-m_2}{\sigma_2},\quad \tilde z_3=\frac{z_3-m_3}{\sigma_3}$$

we obtain

$$p(x,\tau)=\frac{1}{(2\pi)^{3/2}\sqrt{R}}$$
$$\cdot\int_{-\frac{m_1}{\sigma_1}}^{\infty}d\tilde z_1\int_{-\infty}^{-\frac{m_2}{\sigma_2}}d\tilde z_2\int_{-\frac{m_3}{\sigma_3}}^{\infty}\exp\Big\{-\frac{1}{2R}\sum_{i,j=1}^{3}R_{ij}\tilde z_i\tilde z_j\Big\}d\tilde z_3.$$

Taking into account that

$$R_{11}=1-r_{23}^2,\ R_{12}=r_{13}r_{23}-r_{12},\ R_{13}=r_{12}r_{23}-r_{13},$$
$$R_{21}=r_{13}r_{23}-r_{12},\ R_{22}=1-r_{13}^2,\ R_{23}=r_{12}r_{13}-r_{23},$$
$$R_{31}=r_{12}r_{23}-r_{13},\ R_{32}=r_{12}r_{13}-r_{23},\ R_{33}=1-r_{12}^2,$$

we have

$$\frac{1}{R}\sum_{i,j=1}^{3}R_{ij}\tilde z_i\tilde z_j=\frac{1-r_{23}^2}{R}\tilde z_1^2+\frac{1-r_{13}^2}{R}\tilde z_2^2+\frac{1-r_{12}^2}{R}\tilde z_3^2$$
$$+\frac{2(r_{13}r_{23}-r_{12})}{R}\tilde z_1\tilde z_2+\frac{2(r_{12}r_{23}-r_{13})}{R}\tilde z_1\tilde z_3$$
$$+\frac{2(r_{12}r_{13}-r_{23})}{R}\tilde z_2\tilde z_3$$
$$=\Big(\frac{\sqrt{1-r_{23}^2}}{\sqrt{R}}\tilde z_1+\frac{r_{13}r_{23}-r_{12}}{\sqrt{R(1-r_{23}^2)}}\tilde z_2+\frac{r_{12}r_{23}-r_{13}}{\sqrt{R(1-r_{23}^2)}}\tilde z_3\Big)^2$$
$$+\Big(\frac{1}{\sqrt{1-r_{23}^2}}\tilde z_2-\frac{r_{23}}{\sqrt{1-r_{23}^2}}\tilde z_3\Big)^2+\tilde z_3^2.$$

By definition, put

$$v_1=\frac{\sqrt{1-r_{23}^2}}{\sqrt{R}}\tilde z_1+\frac{r_{13}r_{23}-r_{12}}{\sqrt{R(1-r_{23}^2)}}\tilde z_2+\frac{r_{12}r_{23}-r_{13}}{\sqrt{R(1-r_{23}^2)}}\tilde z_3,\quad (8)$$

$$v_2=\frac{1}{\sqrt{1-r_{23}^2}}\tilde{z}_2-\frac{r_{23}}{\sqrt{1-r_{23}^2}}\tilde{z}_3, \quad v_3=\sqrt{R}\tilde{z}_3. \tag{9}$$

Then

$$p(x,\tau)=\frac{1}{(2\pi)^{3/2}\sqrt{R}} \cdot \int_{-\frac{m_3}{\sigma_3}}^{\infty}\left\{\int_{-\infty}^{-\frac{m_2}{\sigma_2}}\left(\int_{-\frac{m_1}{\sigma_1}}^{\infty}e^{-v_1^2/2}d\tilde{z}_1\right)e^{-v_2^2/2}d\tilde{z}_2\right\}e^{-v_3^2/2R}d\tilde{z}_3.$$

We will make a change of variables in the inner integral

$$\int_{-\frac{m_1}{\sigma_1}}^{\infty}e^{-v_1^2/2}d\tilde{z}_1;$$

namely, instead of integrating over the variable $\tilde{z}_1$, we will integrate over the variable v_1. When we integrate over $\tilde{z}_1$, variables $\tilde{z}_2$ and $\tilde{z}_3$ are fixed. Therefore v_2 and v_3 (see (9)) are also fixed. It follows from (8) and (9) that

$$\tilde{z}_3=\frac{v_3}{\sqrt{R}}, \quad \tilde{z}_2=\sqrt{1-r_{23}^2}v_2+\frac{r_{23}}{\sqrt{R}}v_3,$$

and

$$\begin{aligned}v_1&=\frac{\sqrt{1-r_{23}^2}}{\sqrt{R}}\tilde{z}_1+\frac{r_{13}r_{23}-r_{12}}{\sqrt{R(1-r_{23}^2)}}\left(\sqrt{1-r_{23}^2}v_2+\frac{r_{23}}{\sqrt{R}}v_3\right)\\&\quad+\frac{r_{12}r_{23}-r_{13}}{\sqrt{R(1-r_{23}^2)}}\frac{v_3}{\sqrt{R}}\\&=\frac{\sqrt{1-r_{23}^2}}{\sqrt{R}}\tilde{z}_1+\frac{r_{13}r_{23}-r_{12}}{\sqrt{R}}v_2-\frac{r_{13}\sqrt{1-r_{23}^2}}{R}v_3,\end{aligned}$$

i.e., when the variable $\tilde{z}_1$ is changed from $-\frac{m_1}{\sigma_1}$ to $+\infty$, the variable v_1 is changed from $-\eta(v_2,v_3)$ to $+\infty$, where

$$\eta(v_2,v_3)=\frac{\sqrt{1-r_{23}^2}}{\sqrt{R}}\frac{m_1}{\sigma_1}-\frac{r_{13}r_{23}-r_{12}}{\sqrt{R}}v_2+\frac{r_{13}\sqrt{1-r_{23}^2}}{R}v_3. \tag{10}$$

By definition, put

$$\Phi(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y}e^{-u^2/2}du.$$

Then

$$\int_{-\frac{m_1}{\sigma_1}}^{\infty}e^{-v_1^2/2}d\tilde{z}_1=\frac{\sqrt{R}}{\sqrt{1-r_{23}^2}}\int_{-\eta(v_2,v_3)}^{\infty}e^{-v_1^2/2}dv_1$$

$$=\frac{\sqrt{R}}{\sqrt{1-r_{23}^2}}\int_{-\infty}^{\eta(v_2,v_3)}e^{-v_1^2/2}dv_1=\sqrt{2\pi}\frac{\sqrt{R}}{\sqrt{1-r_{23}^2}}\Phi\Big(\eta(v_2,v_3)\Big)$$

and

$$p(x,\tau)=\frac{1}{2\pi\sqrt{R}} \cdot \int_{-\frac{m_3}{\sigma_3}}^{\infty}\left\{\int_{-\infty}^{-\frac{m_2}{\sigma_2}}\frac{\sqrt{R}}{\sqrt{1-r_{23}^2}}\Phi\Big(\eta(v_2,v_3)\Big)e^{-v_2^2/2}d\tilde{z}_2\right\}e^{-v_3^2/2R}d\tilde{z}_3.$$

Let us consider the inner integral on the right side of the last equality. Integration is carried out over variable $\tilde{z}_2$ when variable $\tilde{z}_3$ is fixed. If variable $\tilde{z}_3$ is fixed, then variable v_3 is also fixed. It follows from (9) that

$$v_2=\frac{1}{\sqrt{1-r_{23}^2}}\tilde{z}_2-\frac{r_{23}}{\sqrt{1-r_{23}^2}\sqrt{R}}v_3,$$

i.e., when the variable $\tilde{z}_2$ is changed from $-\infty$ to $-\frac{m_2}{\sigma_2}$, the variable v_2 is changed from $-\infty$ to $\mu(v_3)$, where

$$\mu(v_3)=-\frac{m_2}{\sigma_2}\frac{1}{\sqrt{1-r_{23}^2}}-\frac{r_{23}}{\sqrt{1-r_{23}^2}\sqrt{R}}v_3. \tag{11}$$

Passing in the inner integral from integrating over $\tilde{z}_2$ to integrating over v_2, we obtain

$$p(x,\tau)=\frac{1}{2\pi\sqrt{R}} \cdot \int_{-\frac{m_3}{\sigma_3}}^{\infty}\left\{\int_{-\infty}^{\mu(v_3)}\sqrt{R}\Phi\Big(\eta(v_2,v_3)\Big)e^{-v_2^2/2}dv_2\right\}e^{-v_3^2/2R}d\tilde{z}_3.$$

Passing in the outer integral from integrating over $\tilde{z}_3$ to integrating over v_3, we get

$$p(x,\tau)=\frac{1}{\sqrt{2\pi}\sqrt{R}} \cdot \int_{-\frac{m_3\sqrt{R}}{\sigma_3}}^{\infty}\left\{\int_{-\infty}^{\mu(v_3)}\frac{1}{\sqrt{2\pi}}\Phi\Big(\eta(v_2,v_3)\Big)e^{-v_2^2/2}dv_2\right\}e^{-v_3^2/2R}dv_3, \tag{12}$$

where $\eta(v_2,v_3)$ is determined by formula (10) and $\mu(v_3)$ is determined by formula (11).

By assumption of stationarity of the process $\overset{\circ}{y}_1(x)$, the following theorem establishes conditions for $m(x)$ and $\overset{\circ}{y}_1(x)$ when the Lemma can be used and equality (5) holds.

Theorem 3. *Let $y_1(x)=m(x)+\overset{\circ}{y}_1(x)$, where $m(x)=\mathbf{E}\{y_1(x)\}$ is a mathematical expectation of process $y_1(x)$ and $\overset{\circ}{y}_1(x)$ is a stationary Gaussian process with variance $\sigma^2=\mathbf{E}\{\overset{\circ}{y}_1(x)\}^2$ and normalized correlation function $r(\tau)=\mathbf{E}\{\overset{\circ}{y}_1(x)\,\overset{\circ}{y}_1(x+\tau)\}/\sigma^2$. Let I be a finite open or closed interval from domain of definition of process $y_1(x)$, and there exist positive constants $\Delta,\delta,\Theta,\theta$, and τ_0 such that*

1) *$r(\tau)$ is twice differentiated on the segment $[0,\tau_0]$, $r'(0)=0$, $r''(0)<0$, and*
$$|r''(\tau_1)-r''(\tau_2)|\le\Delta|\tau_1-\tau_2|^{\delta} \text{ for all } \tau_1,\tau_2\in(0,\tau_0);$$
2) *$m(x)$ is continuous on I and differentiated at every internal point $x\in I$, and*
$$|m'(x_1)-m'(x_2)|\le\Theta|x_1-x_2|^{\theta} \text{ when } |x_1-x_2|<\tau_0.$$

Then there exist positive constants τ^*,Γ, and γ such that the inequality

$$p(x,\tau)\le\Gamma\tau^{1+\gamma}$$

holds for every $\tau\in(0,\tau^*)$ and every $x\in I$, where $x+2\tau\in I$.

Proof is based on a detailed analysis of formula (12).[3]

Now we can state that if the process $y_1(x)$ satisfies the conditions of Theorem 3, then condition (7) holds. Therefore, by the Lemma, it follows that equality (5) and the conclusions of Theorem 2 hold.

[3]The publication of the proof takes up a lot of space and for this reason cannot be given in this conference paper.

IV. CONCLUSION

For continuous random processes, we continued the research related to estimating the probability of the event Z_D, which consists in the fact that the first achievement of a given level h by the component $y_1(x)$ of process $\mathbf{y}(x)=\{y_1(x),\ldots,y_n(x)\}$ occurs at some moment x^* from a given interval (x',x'') and, at this moment x^*, the condition $(y_2(x^*),\ldots,y_n(x^*)) \in D$ holds. The ability to estimate the probability $\mathbf{P}\{Z_D\}$ is required in problems related to ensuring the safety of an aircraft landing (see, e.g., [2], [3], [4], [41], and [42]).

For the case $D=R^{n-1}$, we have obtained the exact result for the probability $\mathbf{P}\{Z_D\}$ (see Theorem 2) if equality (5) holds. For stationary Gaussian processes, we have found sufficient conditions (see Theorem 3) for the fulfillment of equality (5).

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