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Bipartite Flocking Control for Multi-Agent Systems with Switching Topologies and Time Delays under Coopetition Interactions

Zhuangzhuang Ma, Bowen Li, Jinliang Shao, Yuhua Cheng, and Wei Xing Zheng, *Fellow, IEEE*

Abstract— **This paper investigates the bipartite flocking behavior of multi-agent systems with coopetition interactions, where communications between agents are described by signed digraphs. The scenario with switching topologies due to the movement of agents, and time delays caused by the limited data transmission capability, is considered comprehensively. Nonlinear weight functions are designed to describe the relationship between the communication distance of agents and the coopetition degree in real biological networks. A distributed update rule based on the neighbors' information and the designed weight functions is proposed. By the aid of the graph theory and sub-stochastic matrix properties, the effectiveness of the proposed update rule is proved theoretically, and the algebraic conditions for achieving the bipartite flocking behavior are obtained. Finally, the theoretical results are verified by numerical simulations.**

I. INTRODUCTION

In natural biological groups, simple individuals are often able to coordinate with each other to accomplish a range of complex tasks, such as ants nest-building, fishes swift-swimming, birds migrating and so on. The idea of multiple agents originated from such natural phenomena, and researchers have begun to explore engineering applications of multi-agent collective behavior since the middle of the last century. The flocking behavior, in which agents can form orderly movements with only limited information of their neighbors and simple rules, has received a lot of attention in recent years. This is due to its wide range of applications, such as mobile robots formation [1], [2], unmanned aerial vehicle flight [3], [4], and target tracking [5], [6].

In the past decades, many meaningful results have been presented for the flocking problem (see, e.g., [7]–[10]). Since agents are more susceptible to be influenced by neighboring agents, the Cucker-Smale (C-S) flocking model was proposed in [7] and a nonlinear weight function was utilized to portray this phenomenon. Along this line, the C-S flocking model in the discrete-time setting was introduced in [8]. Then in [9], the noise was added to the C-S model. Subsequently, a more general mathematical form for the weight function in the C-S model was given in [10]. It should be noted that the literature [7]–[10] always assumed that there exist only cooperative relationships between agents. However, in most biological groups, individuals often compete with others for habitat, food and reproduction in order to survive. For instance, the competition between plants for survival necessities, such as sunlight, water and nutrients, was described in [11]. In addition to this, competitions between agents can also occur due to conflicting goals as mentioned in [12]. The above analysis points out that it is necessary to consider the flocking problem under cooperative-competitive networks, which helps us to decipher the intrinsic mechanism of biological collective movements.

In cooperative-competitive networks, agents may move in opposite directions due to different opinions. This phenomenon of classifying agents as two subgroups with opposite movement trends is called the bipartite flocking [13]. In recent years, researches about bipartite flocking have been emerging. A control strategy was proposed in [14] to guarantee the appearance of bipartite flocking without collisions over structurally balanced signed graphs. The study was carried out in [15] for the fixed-time bipartite flocking behavior based on nonlinear systems. The work in [16] investigated the bipartite flocking behavior with random packet loss. In the literature [13]–[15], the coopetition degree among agents is independent of the communication distance in order to achieve the bipartite flocking. However, in biological groups, the communication distance is a significant factor affecting agents' closeness, thus constructing appropriate weight functions in the C-S model is necessary.

It is also important to note that the investigation on the bipartite flocking with switching topologies [17], [18] and time delays [19], [20] is inadequate or insufficient in the existing works. Since the communication range of sensors carried by agents is limited, an increase of the communication distance can destroy the reliability of the communication link. This can cause the neighbor set of agents to change, which makes the network topology different or switching. At the same time, limited network bandwidth and remote data transmission capabilities may cause agents to be unable to receive real-time information from others, which requires considering time delays. However, only the bipartite flocking behavior with packet dropouts and denial-of-service attacks was considered in [16] and [21], respectively.

Inspired by the above discussion, this paper concentrates

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Zhuangzhuang Ma, Bowen Li and Yuhua Cheng are with the School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu 611731, China (email: zhuangma912@126.com (Z. Ma), oulichen159lbw@gmail.com (B. Li), yhcheng@uestc.edu.cn (Y. Chen)).

Jinliang Shao is with the School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu 611731, China; with the Shenzhen Institute of Artificial Intelligence and Robotics for Society, Shenzhen 518129, China; and also with the Laboratory of Electromagnetic Space Cognition and Intelligent Control, Beijing 100089, China (email: jinliangshao@uestc.edu.cn).

Wei Xing Zheng is with the School of Computer, Data and Mathematical Sciences, Western Sydney University, Sydney, NSW 2751, Australia (email: w.zheng@westernsydney.edu.au).

on the bipartite flocking emergence mechanism with switching topologies and time delays under coopetition interactions. The contributions are reflected in research content and theoretical analysis. We first construct positive and negative weight functions to quantify the relationship between the communication distance and the coopetition degree. Secondly, the bipartite flocking behavior is generalized by considering the switching topologies and time delays scenario. Based on the specificity of our research content, we propose a method in terms of the convergence of infinite sub-stochastic matrices. We also establish an algebraic condition characterizing the coopetition degree between agents by analyzing the stability of the error system, which ensures the emergence of the bipartite flocking behavior.

The paper is constructed as follows. Section II is about the basics of graph theory. The mathematical formulation of the bipartite flocking problem is given in Section III. In Section IV, we provide an algebraic condition for achieving the bipartite flocking behavior. Computer simulations to verify the theoretical results are presented in Section V. Finally, Section VI concludes the paper.

II. Preliminaries

A. Notations

Symbols $\mathbb{R}$ and $\mathbb{N}$ denote the set of real numbers and the set of natural numbers, respectively. $\mathbb{R}^{n\times m}$ represents the set of $(n\times m)$-dimensional real matrices. Let $\mathrm{diag}(\mathrm{x})$, $x\in\mathbb{R}^{n\times 1}$ represent an $n\times n$ diagonal matrix with $[\mathrm{diag}(x)]_{ii}=x_i$. For a set S, the number of elements is represented by $|S|$. For matrix $Q=\left[Q_{ij}\right]_{n\times n}\in\mathbb{R}^{n\times m}$, let $\Lambda_i\{Q\}=\sum_{j=1}^{n}Q_{ij}$ and $|Q|$ be the matrix with element $|Q_{ij}|$, i.e., $|Q|=\left[|Q_{ij}|\right]_{n\times n}$. $\|Q\|_\infty=\max_i\{\Lambda_i\{|Q|\}\}$ denotes the infinity norm and $Q(1:s)$ denotes a sub-matrix consisting of the first s rows and s columns of matrix Q. The product of matrices $Q_i, i\in\{1,2,\ldots,n\}$ is denoted by $\prod_{i=1}^{n}Q_i=Q_nQ_{n-1}\cdots Q_1$. A nonnegative matrix Q with $\Lambda_i\{Q\}\le 1$ $(i=1,2,\ldots,n)$ is called a sub-stochastic matrix. $\mathrm{sgn}(\mathrm{x})$ denotes the signum function of a real number x.

B. Signed Digraph

Consider a multi-agent system containing $n+1$ agents (labelled by $1,2,\ldots,n+1$), where agent $n+1$ is the leader and the remaining agents are the followers. The agent, called the leader, only sends information to other agents, but does not receive any information from others. The agent, called the follower, has responsibilities of both sending and receiving information.

Information exchanges among all agents are represented by a signed digraph $\mathscr{G}=(\mathscr{E},\mathscr{V})$, where $\mathscr{E}$ and $\mathscr{V}$ are the edge set and the node set, respectively. An edge $(j,i)\in\mathscr{E}$ exists if node i can receive information from node j. It should be noted that there is no self-loop in $\mathscr{E}$, i.e., edge (i,i), $i=1,2,\ldots,n+1$, does not exist. Let $\mathscr{N}_i=\{j\in\mathscr{V}\mid(j,i)\in\mathscr{E}\}$ denote the neighbor set of agent i, and $|\mathscr{N}_i|$ denote the number of its neighbors. $\mathscr{A}=\left[a_{ij}\right]_{(n+1)\times(n+1)}$ is the adjacency matrix associated with the digraph $\mathscr{G}$, and the element a_{ij} denotes the weight coefficient of edge (j,i).

In this paper, the signed digraph $\mathscr{G}$ is structurally balanced, which means that all agents can be divided into two subgroups $\mathscr{V}_1$ and $\mathscr{V}_2$, such that $\mathscr{V}=\mathscr{V}_1\cup\mathscr{V}_2$ and $\mathscr{V}_1\cap\mathscr{V}_2=\varnothing$. If the neighboring agents i and j belong to the same subgroup, then they only have the cooperative relationship and $a_{ij}>0$; if they belong to different subgroups, then they are competitive and $a_{ij}<0$. Here, suppose that $\mathscr{V}_1=\{1,2,\ldots,m\}$ and $\mathscr{V}_2=\{m+1,m+2,\ldots,n+1\}$. The directed path from agent i_0 to agent i_r exists, if a series of non-duplicate edges $(i_0,i_1),(i_1,i_2),\ldots,(i_{r-1},i_r)\in\mathscr{E}$ can be found, and $d(i_0,i_r)$ denotes the number of edges for the shortest directed path from i_0 to i_r. If there exists a directed path from the leader to any follower, then the digraph $\mathscr{G}$ is said to contain a spanning tree rooted at the leader.

III. Problem Formulation

This paper considers the discrete-time setting. Let $t_0,t_1,t_2,\ldots$ represent the orderly communication time series, which satisfies $t_{k+1}>t_k$, $k\in\mathbb{N}$. Each agent $i\in\mathscr{V}$ is modeled by the second-order dynamics:

$$\begin{aligned}x_i(t_{k+1})&=x_i(t_k)+T\vartheta_i(t_k),\\ \vartheta_i(t_{k+1})&=\vartheta_i(t_k)+Tu_i(t_k),\end{aligned}\tag{1}$$

where $x_i(t_k)\in\mathbb{R}^{p\times 1}$, $\vartheta_i(t_k)\in\mathbb{R}^{p\times 1}$, and $u_i(t_k)\in\mathbb{R}^{p\times 1}$ denote the position, the velocity and the control input of agent i at time instant kT, respectively. $T=t_{k+1}-t_k$ is the finite time update step and $t_0=0$. To simplify notation, we usually let $p=1$, and handle the case $p>1$ with the help of Kronecker product "$\otimes$". Since the leader does not receive information from the followers, the leader's velocity is fixed, and let $\vartheta_{n+1}(t_k)=\tilde{\vartheta}$.

In cooperative networks, the cooperation (trust) degree between agents is different, which is influenced by the state of the agents. The agents are more susceptible to neighbors with a large cooperation degree. This relationship between agents is quantified by the weight coefficient $a_{ij}>0$, i.e., the more trust agent i has for agent j, the larger the value of a_{ij}. Similarly, $a_{ij}<0$ quantifies the competition degree between agents, and a larger $|a_{ij}|$ indicating a greater resistance of agent i towards agent j. In addition, the digraph $\mathscr{G}(t_k)=(\mathscr{V},\mathscr{E}(t_k))$ is used to describe the time-varying data exchange between agents, where $(j,i)\in\mathscr{E}(t_k)$ indicates that agent i can receive the message from agent j at time instant t_k. Considering coopetition interactions, the elements in the adjacency matrix $\mathscr{A}(t_k)=[a_{ij}(t_k)]_{(n+1)\times(n+1)}$ satisfy

$$a_{ij}(t_k)=\begin{cases}f^{+}(|x_i(t_k)-x_j(t_k)|),\ i\text{ and }j\text{ are cooperative},\\ f^{-}(|x_i(t_k)-x_j(t_k)|),\ i\text{ and }j\text{ are competitive},\end{cases}\tag{2}$$

where the positive weight function (denoted by $f^{+}(\cdot)$) and the negative weight function (denoted by $f^{-}(\cdot)$) are established in this paper to quantify the relationship between the communication distance and cooperation/competition degree.

Different from the traditional C-S model with the specific mathematical form for the weight function, $f^{+}(\cdot)$ and $f^{-}(\cdot)$ only need to satisfy the following assumption.

Assumption 1:

1) $f^+(\cdot)$ is a decreasing function with an upper bound ϕ_u^+ and a nonzero lower bound ϕ_l^+, e.g., $0 < \phi_l^+ \leq f^+(y_2) \leq f^+(y_1) \leq \phi_u^+$ as $0 \leq y_1 \leq y_2$.
2) $f^-(\cdot)$ is an increasing function with a nonzero upper bound ϕ_u^- and a lower bound ϕ_l^-, e.g., $\phi_l^- \leq f^-(y_1) \leq f^-(y_2) \leq \phi_u^- < 0$ as $y_1 \leq y_2 \leq 0$.

During information interactions of agents, network bandwidth and remote-data transmission may also cause that agents cannot receive information from others in a timely manner. Therefore, let τ_{ij} denote the time delay when agent i receives information from agent j and $0 \leq \tau_{ij} \leq \tau$, where $\tau > 0$ is the upper bound of the time delays τ_{ij}. The control input under switching topologies and time delays can be constructed in the following form:

$$u_i(t_k)=\sum_{j\in\mathscr{N}_i(t_k)}|a_{ij}(t_k)|\big[\mathrm{sgn}(a_{ij}(t_k))\vartheta_j(t_{k-\tau_{ij}})-\vartheta_i(t_k)\big]. \quad (3)$$

Based on the concept of bipartite flocking in the literature [13], we give the definition of the bipartite flocking behavior of the C-S model over cooperative-competitive networks.

Definition 1: The bipartite flocking behavior for the C-S model over coopetition networks is said to be achieved, if the following conditions are satisfied:

$$\begin{aligned}
&\lim_{k\to\infty}\|x_i(t_k)+x_{n+1}(t_k)\|_\infty<\infty,\ i\in\mathscr{V}_1,\\
&\lim_{k\to\infty}\|x_i(t_k)-x_{n+1}(t_k)\|_\infty<\infty,\ i\in\mathscr{V}_2,\\
&\lim_{k\to\infty}\|\vartheta_i(t_k)+\tilde{\vartheta}\|_\infty=0,\ i\in\mathscr{V}_1,\\
&\lim_{k\to\infty}\|\vartheta_i(t_k)-\tilde{\vartheta}\|_\infty=0,\ i\in\mathscr{V}_2.
\end{aligned} \quad (4)$$

IV. Main Results

In this section, the C-S model under switching topologies and time delays is analyzed based on the signed graph theory and the sub-stochastic matrix properties, and sufficient conditions for achieving the bipartite flocking are established.

Let

$$\begin{aligned}
e_x(t_k) &= [(x_1(t_k)+x_{n+1}(t_k))^T,(x_2(t_k)+x_{n+1}(t_k))^T,\ldots,\\
&\quad (x_{n-1}(t_k)-x_{n+1}(t_k))^T,(x_n(t_k)-x_{n+1}(t_k))^T]^T,\\
e_\vartheta(t_k) &= [(\vartheta_1(t_k)+\tilde{\vartheta})^T,(\vartheta_2(t_k)+\tilde{\vartheta})^T,\ldots,\\
&\quad (\vartheta_{n-1}(t_k)-\tilde{\vartheta})^T,(\vartheta_n(t_k)-\tilde{\vartheta})^T]^T.
\end{aligned}$$

Then the error system of system (1) can be expressed as

$$\begin{aligned}
e_x(t_{k+1}) &= e_x(t_k)+Te_\vartheta(t_k),\\
e_\vartheta(t_{k+1}) &= (\boldsymbol{I}_n-T\mathscr{D}(t_k))\otimes\boldsymbol{I}_p\times e_\vartheta(t_k)\\
&\quad +T\sum_{j=1}^{\tau}|\mathscr{A}_j(t_k)|\otimes\boldsymbol{I}_p\times e_\vartheta(t_{k-j+1}),
\end{aligned} \quad (5)$$

where

$$\mathscr{D}(t_k)=\mathrm{diag}\left\{\sum_{j=1}^{n+1}|a_{1j}(t_k)|,\ldots,\sum_{j=1}^{n+1}|a_{nj}(t_k)|\right\},$$

and $\mathscr{A}_j(t_k) \in \mathbb{R}^{n\times n}$, $j = 1,2,\ldots,\tau$ are utilized to indicate the information interactions between agents under time delays. Specifically, if there exists the time delay $\tau_{im} = j-1$, $j \in \{1,2,\ldots,\tau\}$ between agent i and agent m, i.e., the follower i receives information of agent m at time instant t_{k-j+1}, then $[\mathscr{A}_j(t_k)]_{im} = [\mathscr{A}(t_k)]_{im}$; otherwise $[\mathscr{A}_j(t_k)]_{im} = 0$. Since the time delays are bounded, we can get $\sum_{j=1}^{\tau}\mathscr{A}_j(t_k) = \mathscr{A}(t_k)(1:n)$. Let ϕ_l and ϕ_u denote the lower and upper bounds of element $|a_{ij}(t_k)|$, respectively, where $\phi_l = \min\{\phi_l^+, -\phi_u^-\}$ and $\phi_u = \max\{\phi_u^+, -\phi_l^-\}$.

In order to analyze the convergence of the error system (5), let $\eta(t_k) = \left[e_\vartheta^T(t_k), e_\vartheta^T(t_{k-1}),\ldots,e_\vartheta^T(t_{k-\tau+1})\right]^T$. Then the error system (5) is transformed into the vector form:

$$\begin{aligned}
e_x(t_{k+1}) &= e_x(t_k)+Te_\vartheta(t_k),\\
\eta(t_{k+1}) &= \Xi(t_k)\eta(t_k),
\end{aligned} \quad (6)$$

where

$$\Xi(t_k)=\begin{bmatrix}
\Xi_1(t_k) & \Xi_2(t_k) & \cdots & \Xi_{\tau-1}(t_k) & \Xi_\tau(t_k)\\
\boldsymbol{I}_{np} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0}\\
\boldsymbol{0} & \boldsymbol{I}_{np} & \cdots & \boldsymbol{0} & \boldsymbol{0}\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
\boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{I}_{np} & \boldsymbol{0}
\end{bmatrix}_{np\tau\times np\tau} \quad (7)$$

with $\Xi_1(t_k) = [\boldsymbol{I}_n - T(\mathscr{D}(t_k) - |\mathscr{A}_1(t_k)|)]\otimes\boldsymbol{I}_p$ and $\Xi_j(t_k) = T|\mathscr{A}_j(t_k)|\otimes\boldsymbol{I}_p$, $j = 2,\ldots,\tau$.

It is worth to underline that $\Xi(t_k)$ is an important factor to guarantee the convergence of the error system (6), so the matrix $\Xi(t_k)$ needs to satisfy the following lemma.

Lemma 1: For all $k \in \mathbb{N}$, if the time step T satisfies

$$T < \frac{1}{\mathscr{N}_{\max}\phi_u}, \quad (8)$$

where $\mathscr{N}_{\max} = \max\{|\mathscr{N}_i(t_k)| \mid i = 1,\ldots,n\}$, then the matrix $\Xi(t_k)$ is a sub-stochastic matrix.

Proof: First, we make it clear that $\Xi(t_k), k \in \mathbb{N}$ should be a nonnegative matrix. From the construction of the matrix $\Xi(t_k)$, it follows that $[\Xi(t_k)]_{ij} = 0$ or $[\Xi(t_k)]_{ij} = 1$ if $i,j \in \{np+1, np+2,\ldots,np\tau\}$. Therefore, it is only necessary to ensure that $\Xi_\mu(t_k), \mu = 1,\ldots,\tau$ are nonnegative matrices. The expression (7) indicates that the nonzero elements in the matrix $\Xi(t_k)$ are $T|a_{ij}(t_k)| > 0$ or

$$\Xi_1(t_k)_{zz}=1-T\sum_{j\in\mathscr{N}_i(t_k)}|a_{ij}(t_k)|, z=1,\ldots,np, i\in\{1,\ldots,n\}.$$

Based on the assumptions that $0 < |\mathscr{N}_i(t_k)| \leq \mathscr{N}_{\max}$ and $0 < |a_{ij}(t_k)| \leq \phi_u$, the diagonal elements of $\Xi_1(t_k)$ satisfy the inequality $\Xi_1(t_k)_{zz} \geq 1 - T\mathscr{N}_{\max}\phi_u$. The restriction of the time step T by condition (8) ensures that $\Xi_1(t_k)_{zz} > 0$. This means that the matrix $\Xi(t_k)$ is a nonnegative matrix.

Next, we only need to prove that the row sum of the matrix $\Xi(t_k)$ is not larger than 1. Similarly, based on the construction of the matrix $\Xi(t_k)$, we know that $\Lambda_i\{\Xi(t_k)\} = 1$, $i \in \{np+1, np+2,\ldots,np\tau\}$. If $i \in \{1,2,\ldots,np\}$, then the row sum of $\Xi(t_k)$ is associated with the element $a_{m,n+1}(t_k)$, $m \in \{1,2,\ldots,n\}$. Specifically, for $i \in \{(m-1)p+1,(m-1)p+2,\ldots,mp\}$, if $a_{m,n+1}(t_k) \neq 0$, then

$$\Lambda_i\{\Xi(t_k)\} = 1 - T|a_{m,n+1}(t_k)| \leq 1 - T\phi_l < 1, \quad (9)$$

otherwise $\Lambda_i\{\Xi(t_k)\} = 1$.

The above analysis proves that condition (8) can ensure that $\Xi(t_k)$ is a nonnegative matrix with a maximum row sum of 1, i.e., $\Xi(t_k)$ is a sub-stochastic matrix. ∎

Based on the matrix $\Xi(t_k)$, we construct the matrix $\hat{\Xi}(t_k)$:

$$\hat{\Xi}(t_k) = \begin{bmatrix} \Xi(t_k) & \mathbf{b}(t_k) \\ \mathbf{0}_{1\times np\tau} & 1 \end{bmatrix}_{(np\tau+1)\times(np\tau+1)},$$

where $\mathbf{b}(t_k) = [b_1(t_k), b_2(t_k), \ldots, b_{np\tau}(t_k)]^T$ with $b_i(t_k) = 1 - \Lambda_i[\Xi(t_k)]$, $i = 1, 2, \ldots, np\tau$. Meanwhile, the digraph $\hat{\mathscr{G}}(t_k) = (\hat{\mathscr{E}}(t_k), \hat{\mathscr{V}})$ with $\hat{\Xi}(t_k)$ as the weighted adjacency matrix is constructed, where $\hat{\mathscr{V}} = \{1, 2, \ldots, np\tau+1\}$.

The next assumption provides the basic topological conditions to solve the bipartite flocking problem in the switching topologies and time delays environment.

***Assumption** 2:* The time series $\{t_0, t_1, t_2, \ldots\}$ is divided into infinite uniformly bounded time intervals $[t_{k_j}, t_{k_{j+1}})$, $j = 0, 1, 2, \ldots$, starting at $k_0 = 0$. In each time interval $[t_{k_j}, t_{k_{j+1}})$, the union of digraphs $\mathscr{G}(t_{k_j})$, $\mathscr{G}(t_{k_j+1})$, $\ldots, \mathscr{G}(t_{k_{j+1}-1})$ contains a spanning tree rooted at $n+1$.

Suppose that there are at most H instants in the time interval $[t_{k_j}, t_{k_{j+1}})$, i.e., $\max(t_{k_{j+1}} - t_{k_j}) = HT$. Before proceeding, we provide a lemma to guarantee the topological properties of the digraph $\hat{\mathscr{G}}(t_k)$.

***Lemma** 2:* If Assumption 2 holds, then the union of digraphs $\hat{\mathscr{G}}(t_{k_j}), \hat{\mathscr{G}}(t_{k_j+1}), \ldots, \hat{\mathscr{G}}(t_{k_{j+1}-1})$ contains a spanning tree rooted at the node $np\tau+1$.

***Proof**:* To simplify the notation of this lemma, we let $p = 1$. We below consider the directed path from node $n\tau+1$ to node $r_z \in \{1, 2, \ldots, n\tau\}$ in two cases.

Case I: $r_z \in \{1, 2, \ldots, n\}$. Since Assumption 2 is satisfied, there exists a directed path $(n+1, r_1), \ldots, (r_{z-1}, r_z)$ from the leader $n+1$ to node r_z, where $(r_k, r_{k+1}) \in \mathscr{E}(t), t \in [t_{k_j}, t_{k_{j+1}})$. It follows from the structure of the matrix $\Xi_1(t_k)$ that the directed path $(n\tau+1, r_1), \ldots, (r_{z-1}, r_z)$ from $nh+1$ to r_z can be found without time delays at the time interval $[t_{k_j}, t_{k_{j+1}})$. If there exists an upper bound of the time delays, then we can still find an associated directed path.

Without loss of generality, assume the edge (r_m, r_{m+1}) cannot be found in the time interval $[t_{k_j}, t_{k_{j+1}})$ with time delays. Based on $\sum_{j=1}^{\tau} \mathscr{A}_j(t) = \mathscr{A}(t)(1:n)$ and Assumption 2, it is not hard to find the directed path $(r_m, r_m+n), (r_m+n, r_m+2n), \ldots, (r_m+\beta n, r_{m+1})$, where $t \in [t_{k_j}, t_{k_{j+1}})$, $\beta \in \{1, 2, \ldots, (\tau-1)\}$. Therefore, we can always find the directed path from node $n\tau+1$ to node $r_z \in \{1, 2, \ldots, n\}$.

Case II: $r_z \in \{n+1, n+2, \ldots, n\tau\}$. According to the structure of $\Xi(t)$, there is always a directed path $(i, i+n), (i+n, i+2n), \ldots, (i+n(\tau-2), i+n(\tau-1))$ from node i to node $i+n(\tau-1)$, where $i \in \{1, 2\ldots, n\}$, $t \in [t_{k_j}, t_{k_{j+1}})$.

Case I and Case II show that for all $r_z \in \{1, 2, \ldots, n\tau\}$, we can find a directed path from $n\tau+1$ to r_z at the time interval $[t_{k_j}, t_{k_{j+1}})$, which means that there is a spanning tree of the union of digraphs $\hat{\mathscr{G}}(t_{k_j})$, $\hat{\mathscr{G}}(t_{k_j+1}), \ldots$, $\hat{\mathscr{G}}(t_{k_{j+1}-1})$ with the root node $n\tau+1$. The proof is complete. ∎

Let $D = \max\{d(np\tau+1, j) | j = 1, 2, \ldots, np\tau\}$ and $P = DH$. Then we divide $\{t_k\}_{k=0}^{\infty}$ into a series of continuous, nonempty, uniformly bounded time intervals $[0, t_P)$, $[t_P, t_{2P}), \ldots$, $[t_{\alpha P}, t_{\alpha P+P}), \ldots, \alpha \in \mathbb{N}$. Now the following theorem provides the properties of the matrix $\Xi(t_k)$.

***Theorem** 1:* If system (1) satisfies Assumption 1, Assumption 2 and condition (8), then for each time interval $[t_{\alpha P}, t_{\alpha P+P})$, there holds

$$\|\eta(t_{\alpha P+P})\|_\infty \le (1 - \beta^D T\phi_l)\|\eta(t_{\alpha P})\|_\infty, \qquad (10)$$

where $\beta = (1 - T\mathscr{N}_{\max}\phi_u)^H$.

***Proof**:* According to the error system (5), we can get

$$\eta(t_{\alpha P+P}) = \prod_{\omega=1}^{D}\left\{\prod_{\psi=1}^{H} \Xi(t_{\alpha P+(\omega-1)H+\psi-1})\right\} \cdot \eta(t_{\alpha P}).$$

We first analyze $\left\| \prod_{\psi=1}^{H} \Xi(t_{\alpha P+\psi-1}) \right\|_\infty$ in the time interval $[t_{\alpha P}, t_{\alpha P+H-1})$. For the sake of notational simplicity, let $t_{k_j} = t_{\alpha P+(j-1)H}$ and $t_{k_{j+1}-1} = t_{\alpha P+jH-1}$. Due to $\max(t_{k_{j+1}} - t_{k_j}) = HT$, Lemma 2 holds in the time interval $[t_{\alpha P}, t_{\alpha P+H-1})$. From Assumption 2 and Lemma 2, there exists an agent s_1 that communicates directly with the leader $np\tau+1$ and $(np\tau+1, s_1) \in \hat{\mathscr{E}}(t_{k_{j+1}-1}) \cup \hat{\mathscr{E}}(t_{k_{j+1}-2}) \cup \cdots \cup \hat{\mathscr{E}}(t_{k_j})$. Assume that $(np\tau+1, s_1) \in \hat{\mathscr{E}}(t_{k_j+l_j})$, $0 \le l_j < H$, which means that $a_{m,n+1}(t_{k_j+l_j}) \neq 0$ with $s_1 \in \{(m-1)p+1, (m-1)p+2, \ldots, mp\}$. According to (7), we know that $s_1 \in \{1, 2, \ldots, np\}$ and $\Lambda_{s_1}\{\Xi(t_{k_j+l_j})\} = 1 - T|a_{m,n+1}(t_{k_j+l_j})| \le 1 - T\phi_l < 1$.

Since $\Xi(t_{k_j})$ is a sub-stochastic matrix, there holds $\Lambda_i\{\Xi(t_{k_j})\} \le 1$ for all $i \in \{1, \ldots, np\tau\}$. It is further concluded that $\Lambda_i\{\Xi(t_{k_j+1})\Xi(t_{k_j})\} \le 1$, $i \in \{1, \ldots, np\tau\}$. As $\Xi(t_{k_j+1})\Xi(t_{k_j})$ is a sub-stochastic matrix, we also know that $\prod_{t=t_{k_j}}^{t_{k_j+l_j-1}} \Xi(t)$ is a sub-stochastic matrix and $\Lambda_i\left\{\prod_{t=t_{k_j}}^{t_{k_j+l_j-1}} \Xi(t)\right\} \le 1$. Based on the above facts, we get

$$\begin{aligned}\Lambda_{s_1}\left\{\prod_{t=t_{k_j}}^{t_{k_j+l_j}} \Xi(t)\right\} &= \sum_{i=1}^{np\tau} \left[\Xi(t_{k_j+l_j})\right]_{s_1 i} \Lambda_i\left\{\prod_{t=t_{k_j}}^{t_{k_j+l_j-1}} \Xi(t)\right\} \\ &\le 1 - T\phi_l. \end{aligned} \qquad (11)$$

Considering (11) and the following fact:

$$\left[\prod_{t=t_{k_j+l_j+1}}^{t_{k_{j+1}-1}} \Xi(t)\right]_{s_1 s_1} \ge (1 - T\mathscr{N}_{\max}\phi_u)^H, \qquad (12)$$

we can obtain

$$\begin{aligned}\Lambda_{s_1}\left\{\prod_{t=t_{k_j}}^{t_{k_{j+1}-1}} \Xi(t)\right\} &= \sum_{i=1, i\neq s_1}^{np\tau} \left[\prod_{t=t_{k_j+l_j+1}}^{t_{k_{j+1}-1}} \Xi(t)\right]_{s_1 i} \Lambda_i\left\{\prod_{t=t_{k_j}}^{t_{k_j+l_j}} \Xi(t)\right\} \\ &\quad + \left[\prod_{t=t_{k_j+l_j+1}}^{t_{k_{j+1}-1}} \Xi(t)\right]_{s_1 s_1} \Lambda_{s_1}\left\{\prod_{t=t_{k_j}}^{t_{k_j+l_j}} \Xi(t)\right\} \\ &\le \Lambda_{s_1}\left\{\prod_{t=t_{k_j+l_j+1}}^{t_{k_{j+1}-1}} \Xi(t)\right\} - \left[\prod_{t=t_{k_j+l_j+1}}^{t_{k_{j+1}-1}} \Xi(t)\right]_{s_1 s_1} \\ &\quad + (1 - T\mathscr{N}_{\max}\phi_u)^H(1 - T\phi_l) \\ &\le 1 - (1 - T\mathscr{N}_{\max}\phi_u)^H T\phi_l < 1. \end{aligned}$$

Next, we analyze the matrix $\prod_{t=t_{k_{j+1}}}^{t_{k_{j+2}-1}}\Xi(t)$ in the time interval $[t_{k_{j+1}}, t_{k_{j+2}})$. From Assumption 2, the union of digraphs $\hat{\mathscr{G}}(t_{k_{j+1}}), \hat{\mathscr{G}}(t_{k_{j+1}+1}), \ldots, \hat{\mathscr{G}}(t_{k_{j+2}-1})$ contains a spanning tree, which is denoted by $\mathscr{F}(t_{k_{j+1}})$. Then we consider two ways for the followers to communicate with the leader $np\tau+1$ in $\mathscr{F}(t_{k_{j+1}})$.

Case I: The node s_1 communicates indirectly with the leader $np\tau+1$ or there exist two different nodes s_1 and s_2, both of which communicate directly with the leader. We assume that $(np\tau+1, s_2) \in \hat{\mathscr{E}}(t_{k_{j+2}-1}) \cup \hat{\mathscr{E}}(t_{k_{j+2}-2}) \cup \cdots \cup \hat{\mathscr{E}}(t_{k_{j+1}})$. Repeating the analysis process with the matrix $\prod_{t=t_{k_j}}^{t_{k_{j+1}-1}}\Xi(t)$, it is easy to obtain that

$$\Lambda_{s_2}\left\{\prod_{t=t_{k_{j+1}}}^{t_{k_{j+2}-1}}\Xi(t)\right\} \le 1-(1-T\mathscr{N}_{\max}\phi_u)^H T\phi_l,$$

$$\Lambda_j\left\{\prod_{t=t_{k_{j+1}}}^{t_{k_{j+2}-1}}\Xi(t)\right\} \le 1,\ j \ne s_2.$$

Considering the row sum of the matrix $\prod_{t=t_{k_j}}^{t_{k_{j+2}-1}}\Xi(t)$, we can get

$$\Lambda_{s_1}\left\{\prod_{t=t_{k_j}}^{t_{k_{j+2}-1}}\Xi(t)\right\}\le 1-\beta^2 T\phi_l,\ \Lambda_{s_2}\left\{\prod_{t=t_{k_j}}^{t_{k_{j+2}-1}}\Xi(t)\right\}\le 1-\beta^2 T\phi_l. \quad (13)$$

Case II: If only s_1 is directly connected to the leader in $\mathscr{F}(t_{k_{j+1}})$, then there exists a node $s_2 \ne s_1$ such that $(s_1, s_2) \in \mathscr{F}(t_{k_{j+1}})$. It may be assumed that $(s_1, s_2) \in \hat{\mathscr{E}}(t_{k_{j+1}+l_{j+1}})$. Then $[\Xi(t_{k_{j+1}+l_{j+1}})]_{s_2 s_1} \ge T\phi_l$ and

$$\left[\prod_{t=t_{k_{j+1}}}^{t_{k_{j+2}-1}}\Xi(t)\right]_{s_2 s_1} > (1-T\mathscr{N}_{\max}\phi_u)^H.$$

With the help of the analysis of Case I, we can also obtain the equation (13)

The above analysis shows that we can find different nodes $s_1, s_2, \ldots, s_D$ in the time interval $[t_{k_j}, t_{k_{j+D}})$ such that

$$\Lambda_j\left\{\prod_{t=t_{k_j}}^{t_{k_{j+D}-1}}\Xi(t)\right\} \le 1-\beta^D T\phi_l,\ j \in \{s_1, s_2, \ldots, s_D\}. \quad (14)$$

It is worth noting that for any directed path, we can still use the same analysis to reach the conclusion (14), that is, (14) is valid for all node $j \in \{1, 2, \ldots, np\tau\}$. Therefore,

$$\left\|\prod_{t=t_{k_j}}^{t_{k_{j+D}-1}}\Xi(t)\right\|_\infty \le 1-\beta^D T\phi_l < 1. \quad (15)$$

Furthermore, we can infer that

$$\|\eta(t_{\alpha P+P})\|_\infty \le (1-\beta^D T\phi_l)\|\eta(t_{\alpha P})\|_\infty.$$

This completes the proof. ■

Following the result of Theorem 1, we next state the conclusion of the bipartite flocking with switching topologies and time delays.

***Theorem** 2:* If Assumption 1, Assumption 2 and condition (8) are satisfied, then the bipartite flocking behavior of system (1) is realized with the control input (3), i.e.,

$$\lim_{k\to\infty}\|e_x(t_k)\|_\infty \le \xi,\ \ \lim_{k\to\infty}\|e_\vartheta(t_k)\|_\infty = 0,$$

where

$$\xi = \|e_x(0)\|_\infty + \frac{P}{\beta^D\phi_l}\|e_v(0)\|_\infty.$$

***Proof**:* Based on the result of Theorem 1, we attain

$$\lim_{k\to\infty}\|\eta(t_k)\|_\infty \le \lim_{\alpha\to\infty}(1-\beta^D T\phi_l)^{\alpha+1}\|\eta(0)\|_\infty = 0,$$

which means that $\lim_{k\to\infty}\|e_\vartheta(t_k)\|_\infty = 0$.

Observing the error system (6), we can get

$$\|e_x(t_{\alpha P+P})\|_\infty = \|e_x(t_{\alpha P+P-1})\|_\infty + T\|e_\vartheta(t_{\alpha P+P-1})\|_\infty \le \|e_x(0)\|_\infty + PT\sum_{i=0}^{\alpha}\|e_\vartheta(t_{iP})\|_\infty.$$

Further, we can obtain

$$\lim_{k\to\infty}\|e_x(t_k)\|_\infty \le \|e_x(0)\|_\infty + PT\lim_{\alpha\to\infty}\sum_{i=0}^{\alpha}\|e_\vartheta(t_{iP})\|_\infty.$$

According to (15), we know

$$\|e_\vartheta(t_{iP})\|_\infty \le (1-\beta^D T\phi_l)^i\|e_\vartheta(0)\|_\infty,\ i = 0, 1, \ldots, \alpha.$$

Now, a bounded geometric sequence $\{\|e_\vartheta(t_{iP})\|_\infty\}_{i=0}^{\alpha}$ with a ratio of $1-\beta^D T\phi_l$ is obtained. According to the sum formula of the geometric sequence, we arrive at

$$\lim_{\alpha\to\infty}\sum_{i=0}^{\alpha}\|e_\vartheta(t_{iP})\|_\infty = \frac{1}{\beta^D T\phi_l}\|e_\vartheta(0)\|_\infty.$$

This means that

$$\lim_{k\to\infty}\|e_x(t_k)\|_\infty = \lim_{\alpha\to\infty}\|e_x(t_{\alpha P+P})\|_\infty \le \xi.$$

The proof is complete. ■

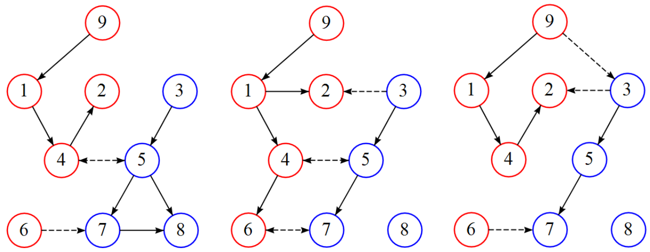
V. Simulation Results

This section provides simulation results to verify that the control input (3) can achieve the bipartite flocking in multi-agent systems with switching topologies and time delays, if Assumption 1, Assumption 2 and condition (8) are satisfied.

Consider a multi-agent system with one leader (labeled by 9) and eight followers (labeled by $1, 2, \ldots, 8$). The communication topology of agents at time instant $t_k \in \{t_0, t_1, t_2, \ldots\}$ is chosen randomly among $\mathscr{G}_a, \mathscr{G}_b, \mathscr{G}_c$ as shown in Fig. 1. Let the time interval $[t_{k_j}, t_{k_{j+1}})$, $j = 0, 1, 2, \ldots$, contain at most $H = 6$ time instants. Since the union of digraphs $\mathscr{G}_a, \mathscr{G}_b, \mathscr{G}_c$ contains a spanning tree rooted at the leader 9, it is important to ensure that the topology in Fig. 1 is selected at least once during time intervals $[t_k, t_{k+6}), k \in \mathbb{N}$. Assume that $\tau = 3$, which means that the time delays of all agents is chosen randomly in $\{0, T, \ldots, 3T\}$.

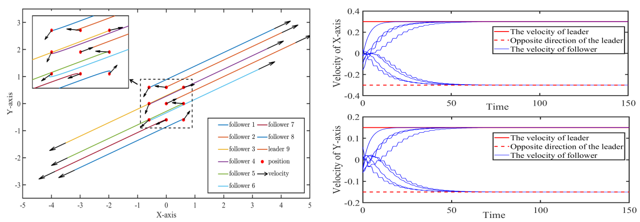
In the simulation, the positive and negative weight functions are chosen respectively as follows:

$$f^+(y) = \frac{1}{1+0.2y} + 1.5,\ f^-(y) = -\frac{1}{1+0.2y} - 1.5,$$



(a) $\mathscr{G}_a$ (b) $\mathscr{G}_b$ (c) $\mathscr{G}_c$

Fig. 1. Switching communication topologies, where circles denote agents and circles of the same color belong to the same subgroup. The solid and dotted lines indicate the cooperative and competitive relationships between agents, respectively.



(a) Position (b) Velocity

Fig. 2. Position and velocity evolutions of all agents.

where $f^+(y)$ and $f^-(y)$ satisfy Assumption 1. Based on the settings of the communication topology and the weight functions, it is known that $\phi_u = 2.5$ and $\mathscr{N}_{\max} = 2$. For this reason, let $T = 0.1$ in order to satisfy condition (8).

From Fig. 2(a), we can see that the position evolutions of all agents, where the positions and velocities are marked by red "•" and black "→", respectively. The agents eventually split into two subgroups based on the control protocol (3) and the position error vector $\lim_{k\to\infty} ||e_x(t_k)||_\infty$ is bounded. Fig. 2(b) shows that the velocities of the followers, who cooperate with the leader, are consistent with the leader, while the velocities of the followers, who compete with the leader, are opposite to the leader, i.e.,

$$\lim_{k\to\infty} ||e_\vartheta(t_k)||_\infty = 0.$$

In other words, the bipartite flocking behavior of multi-agent systems is achieved under switching topologies and time delays, which means that Theorem 2 is validated by the simulation results.

VI. Conclusion

This paper has studied the bipartite flocking behavior of the C-S model over coopetition networks for multi-agent systems. The stability problem has been transformed into a convergence problem of infinite products with sub-stochastic matrices by constructing the error system. With the help of the theories of sub-stochastic matrix and signed digraph, we have systematically analyzed this convergence problem and obtained the algebraic conditions for achieving the bipartite flocking. Lastly, the presented simulation results have effectively supported our theory. The algorithm in this paper can facilitate the application of flocking in real systems due to the relaxation of the constraints on the communication conditions. However, the volume of the agents and collision avoidance issues are ignored. In the future work, the flocking behavior under safety constraints is a worthwhile topic.

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