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Linear output regulation of discrete-time networked systems subject to stochastic packet drops

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Abstract— **We investigate a scenario in which a given discrete-time controller communicates with a discrete-time plant via a wireless erasure channel in a linear scenario. The controller is designed for the output of the plant to track a periodic reference generated by a finite superimposition of discrete-time linear oscillators while rejecting additional disturbances, i.e., to solve an output regulation problem. Due to stochastic packet drop induced by the network, the closed-loop behavior switches between two different dynamical systems, depending on the probability of transmission. We show that, when packet drops occur on the channel from the controller to the actuator, the exogenous signal implies that the expected regulation error does not go to zero but converges to a ball centered at zero. Differently, when the packet drops are on the output channel, we provide a set of sufficient conditions such that the regulation error asymptotically shrinks to zero in expectation. Results are validated via numerical examples.**

I. Introduction

Over the past years, the rise of the so-called Internet-of-Things has led to the ubiquitous deployment of wireless networks. This is motivated by the several advantages they present with respect to wired networks, like flexibility, ease of maintenance, and so on. However, when applied to control systems, guaranteeing good behavior and performance for the closed-loop system exchanging information over the wireless channel is a major issue. This has motivated researchers to develop and study the framework of *(wireless) networked control systems* (WNCS), see [1], [2] and references therein.

A major difference with respect to wired setups is that, in the presence of a network, WNCS systems suffer from packet loss due to information transmission not being successful. As a consequence, the deterministic behavior of the closed loop is lost, and analysis and design results have to be built by relying on stochastic tools. The behavior of WCNS in the presence of packet drops has been well studied in the literature, see [3], [4] for state estimation, [5] for the stability of linear systems, and [6], [7] for the stability of nonlinear systems. Specifically for the case of discrete-time linear systems, the presence of stochastic packet drops can be modeled using the framework of Markov jump linear systems (see [8] and references therein).

While most of the works on discrete-time WCNS focus on equilibrium stabilization problems, in this paper, we consider the problem of designing a feedback controller for the output of a plant to track a reference while rejecting additional disturbances. Such a problem is commonly known as the *output regulation problem*. In a deterministic context, most results focus on continuous-time systems (see, e.g., [9], [10]), while fewer consider the discrete-time counterpart (see, e.g., [11]–[13]). Output regulation of linear stochastic systems was recently studied in [14]. It is well known that, in the linear framework and without the presence of a network, a regulation problem can be cast as a stabilization problem. Naively, one might assume that the same happens when dealing with stochastic systems, where mean-square regulation of the error is achievable. However, in this work, we show that this is not always the case.

In particular, we consider a plant modeled as a linear deterministic discrete-time system. The plant communicates with a controller via a wireless channel acting either on the input or on the output communication link. As a consequence, the actual information received by the actuators or by the sensors is a random variable associated with the probability of successful transmission. The problem is therefore discrete-time and stochastic. This reflects the novelty of our work. Indeed, existing results concerning the regulation/tracking for WNCS systems focus on a continuous-time deterministic framework (see e.g. [15]–[17] and references therein). This is our main contribution with respect to WNCS literature.

We proceed via *emulation*. We first design a dynamic output feedback controller to solve the output regulation problem without considering the presence of a wireless channel. Then, we provide a set of sufficient conditions for the asymptotic behavior of the trajectories of the regulation error for the closed loop in the presence of the network. In particular, we show that, if the network is placed in between the sensors (that is, on the output channel), the control action can still enforce an internal model property. Consequently, robust regulation can still be recast as a robust stabilization problem, and therefore, the expectation of the regulation error asymptotically shrinks to zero. In contrast, we show that, when the network is placed in between the actuators, the expectation of the regulation error is only bounded in norm, with the bound depending on the probability of transmission and on the amplitude of the exogenous signals. This aspect is a major difference with respect to WNCS stabilization problems, where there is no substantial difference between

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having input or output packet drops (see [18]).

The rest of the paper is as follows. Section II recalls the main results on stability of linear stochastic discrete-time systems. Section III presents the considered framework and the problem. The main results are presented in Section IV. A numerical example is given in Section V. Concluding remarks and future directions are discussed in Section VI.

Notation. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{C}$ the set of complex ones, $\mathbb{N}$ the set of natural numbers (including 0), and $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$. The symbol $\emptyset$ indicates the empty set. Given $x \in \mathbb{C}$, we let $|x|$ be the modulo of x. Given a square matrix A, we indicate with $\mathrm{spec}(A)$ its spectrum. We use $\Pr(\cdot)$ for the probability and $\mathbb{E}[\cdot]$ for the expectation taken over the relevant stochastic variables. For any $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ with $n_1, n_2 \in \mathbb{N}_+$, (x_1, x_2) stands for $(x_1^\top, x_2^\top)^\top \in \mathbb{R}^{n_1+n_2}$. The symbol $\otimes$ indicates the Kroeneker product and the symbol $\mathbf{1}$ the column vector full of 1 (dimension is clear from the context). The symbol I_n indicates the identity matrix of dimension $n \times n$.

II. Preliminaries

Before introducing the considered framework, and to make the paper more self-contained, we recall the main results on stability for stochastic discrete-time linear systems. For more details, we refer to [8] and references therein.

Let $q(t) \in \{0, 1\}$ be an independent and identically distributed Bernoulli variable such that $\Pr(q(t) = 1) = \pi$ and $\Pr(q(t) = 0) = 1 - \pi$ for all $t \in \mathbb{N}$, for some $\pi \in [0, 1]$. Consider a discrete-time linear system as

$$\mathbf{x}(t+1) = \begin{cases} \overline{F}\mathbf{x}(t) + \overline{G}w(t) & \text{if } q(t) = 1 \\ \underline{F}\mathbf{x}(t) + \underline{G}w(t) & \text{if } q(t) = 0, \end{cases} \tag{1}$$

with state $\mathbf{x} \in \mathbb{R}^{n_\mathbf{x}}$, $w \in \mathcal{W}$ being an external deterministic signal taking values on a compact set $\mathcal{W} \subset \mathbb{R}^{n_w}$, and $\overline{F}, \underline{F}, \overline{G}, \underline{G}$ of suitable dimension. System (1) is a stochastic system that jumps between two dynamics.

Definition 1 (Exponential (practical) mean-square stability)**.** *Consider system* (1)*. We say that the origin is exponentially practically mean-square stable (EPMSS) with respect to the input w if $\exists\, \delta_\mathbf{x} \geq 1$, $0 < \gamma_\mathbf{x} < 1$, and $\beta_\mathbf{x} \geq 0$ such that*

$$\mathbb{E}[\mathbf{x}(t)^\top \mathbf{x}(t)] \leq \delta_\mathbf{x} \gamma_\mathbf{x}^t \mathbb{E}[\mathbf{x}(0)^\top \mathbf{x}(0)] + \beta_\mathbf{x} \sup_{w \in \mathcal{W}} |w|^2 \tag{2}$$

for all $t \geq 0$ and any $\mathbf{x}(0) \in \mathbb{R}^{n_\mathbf{x}}$. Moreover, if $\beta_\mathbf{x} = 0$, the origin is said to be exponentially mean-squre stable (EMSS).

Sufficient conditions for E(P)MSS can be found in [8] and are recalled in the following.

Theorem 1 (Sufficient conditions for E(P)MSS)**.** *Consider system* (1)*. If there exist $0 < \gamma_\mathbf{x} < 1$ and two symmetric positive definite matrices $\overline{M}, \underline{M} > 0$ such that*

$$\begin{aligned} \gamma_\mathbf{x}\overline{M} &> \overline{F}^\top(\pi\overline{M} + (1-\pi)\underline{M})\overline{F}\,, \\ \gamma_\mathbf{x}\underline{M} &> \underline{F}^\top(\pi\overline{M} + (1-\pi)\underline{M})\underline{F}\,, \end{aligned} \tag{3}$$

then the origin is EPMSS. Moreover, if $w(t) = 0$ for all $t \geq 0$, then the origin is EMSS.

III. Setup and problem formulation

In this section, we describe the plant and the communication model. We then introduce the considered problem and the controller structure.

A. The system

We consider a discrete-time linear plant $\mathbf{P}$ given by

$$\mathbf{P} : \begin{cases} x_p(t+1) = A_p x_p(t) + B_p \hat{u}(t) + P w(t) & \text{(4a)} \\ e(t) = C_p x_p(t) + Q w(t) & \text{(4b)} \end{cases}$$

with $x_p \in \mathbb{R}^{n_x}$ being the plant's state, $\hat{u} \in \mathbb{R}^{n_u}$ being the control input imposed by the actuators, $e \in \mathbb{R}^{n_e}$ being an output measured via sensors, and with A_p, B_p, P, C_p, Q being matrices of suitable dimension. We assume the dimension of the input to be at least equal to the dimension of the output[1] $n_u \geq n_e$ to avoid having underactuated plants. The system is affected by the external signal w generated by the exosystem

$$\mathbf{E} : \left\{ w(t+1) = S w(t) \right. \tag{4c}$$

with $w \in \mathcal{W}$ representing an external reference signal to be tracked and/or a disturbance to be rejected taking value on a set $\mathcal{W} \subset \mathbb{R}^{n_w}$, and S a known constant matrix.

B. Communication setups

We consider a setup in which the plant $\mathbf{P}$ in (4) communicates with a feedback controller $\mathbf{C}$ having input denoted $\hat{e}$ and output u over wireless network channels $\mathbf{N}_1$ and $\mathbf{N}_2$ (see Fig. 1). The wireless channels are assumed to be erasure channels, i.e., packets are dropped with a certain probability. The control packets are perfectly communicated (without any additional noise or delay) over the wireless channels when they are not dropped. We let $q \in \mathbb{R}$ be a random variable associated with the probability of successful transmission. In particular, $q(t) = 1$ indicates a successful communication at time t, while $q(t) = 0$ indicates a packet drop. We formalize our assumptions on the wireless channel as follows.

Assumption 1 (Independent erasure channels)**.** *For any $t \in \mathbb{N}$, we have that*

$$\Pr(q(t) = 1) = \pi\,, \tag{5}$$

where $\pi \in [0, 1]$ is the probability of successful transmission.

We consider two different and separate setups. First, the case in which the wireless channel between the sensors and the controller $\mathbf{N}_1$ is unreliable, while $\mathbf{N_2}$ is reliable. As such, the output channel experiences packet drop, while the input one does not, i.e. it provides unitary probability of successful transmission. The second case is vice versa with $\mathbf{N}_1$ reliable and $\mathbf{N}_2$ experiencing packet drops. We call these two setups the *output packet dropout* (OPD) and the *input packet dropout* (IPD), respectively. As such, in both cases, the closed-loop does not hold for all $t \in \mathbb{N}$, but only when $q(t) = 1$. When $q(t) = 0$, i.e., when the packet is dropped, a zero-hold strategy is used to generate the current input/output

[1] This requirement is known to be necessary for the solution of the linear output regulation problem, see e.g. [19]

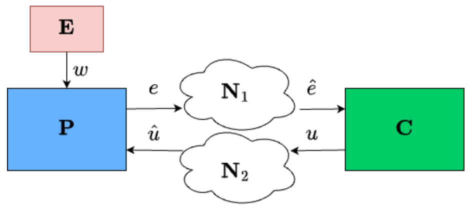

Fig. 1: Considered network setup

based on the previous measure. In other words, in the OPD case, the effect of the networks $\mathbf{N}_1$ and $\mathbf{N}_2$ is such that

$$\text{OPD}: \begin{cases} u(t) = \hat{u}(t) & \text{for all } t \in \mathbb{N} \quad (6a) \\ \hat{e}(t) = e(t) & \text{if } q(t) = 1 \quad (6b) \\ \hat{e}(t) = \hat{e}(t-1) & \text{if } q(t) = 0\,. \quad (6c) \end{cases}$$

Similarly, in the IPD case, we have

$$\text{IPD}: \begin{cases} \hat{u}(t) = u(t) & \text{if } q(t) = 1 \quad (7a) \\ \hat{u}(t) = \hat{u}(t-1) & \text{if } q(t) = 0 \quad (7b) \\ \hat{e}(t) = e & \text{for all } t \in \mathbb{N}. \quad (7c) \end{cases}$$

C. Problem statement and main assumptions

Our objective is to solve an output regulation problem in the presence of the networks in presence of packet drop. In particular, we look for a dynamic output feedback controller

$$\mathbf{C}: \begin{cases} x_c(t+1) = A_c x_c(t) + B_c \hat{e}(t) \\ u(t) = C_c x_c(t) \end{cases} \tag{8}$$

with $x_c \in \mathbb{R}^{n_c}$ and matrices A_c, B_c, C_c of suitable dimensions. We formalize our goal as follows.

Problem 1. *Consider system* (4) *paired with a communication setup described either by* (6) *or by* (7)*. If there exists a controller of the form* (8) *such that*

- *for any initial conditions the closed-loop system is EPMSS with respect to the input w and there exists $\beta_e \geq 0$ such that*

$$\lim_{t\to+\infty} \mathbb{E}[e(t)^\top e(t)] \leq \beta_e \sup_{w\in\mathcal{W}} |w|^2 \,, \tag{9}$$

the Networked Boundness Problem is solved;

- *if moreover $\beta_e = 0$, then we say that the Networked Output Regulation Problem is solved.*

D. Controller structure

Following standard output regulation results ([20, Chapter 4]), we start by assuming the following.

Assumption 2 (System's assumptions)**.** *Consider system* (4) *and assume the following.*

- *The exosystem $\mathbf{E}$ in* (4c) *is critically stable, that is, $|\lambda_S| = 1$ for all $\lambda_S \in \operatorname{spec}(S)$;*
- *The pair (A_p, B_p) is stabilizable and the pair (A_p, C_p) is detectable;*
- *For all $\lambda_S \in \operatorname{spec}(S)$, we have that*

$$\operatorname{rank}\begin{pmatrix} A_p - \lambda_S I_{n_x} & B_p \\ C_p & 0 \end{pmatrix} = n_x + n_e\,. \tag{10}$$

For the controller design, we proceed by *emulation*. We first design a (dynamic) output feedback controller $\mathbf{C}$ of the form (8) where $x_c = (x_i, x_s) \in \mathbb{R}^{n_i} \times \mathbb{R}^{n_s}$ and

$$\dot{x}_i = \Phi x_i + \Gamma \hat{e} \tag{11a}$$

$$\dot{x}_s = A_s x_s + A_i x_i + B_s \hat{e} \tag{11b}$$

that is, a controller of the form (8) where

$$A_c = \begin{pmatrix} \Phi & 0 \\ A_i & A_s \end{pmatrix}, \; B_c = \begin{pmatrix} \Gamma \\ B_s \end{pmatrix}, \tag{11c}$$

$\Phi = I_{n_e} \otimes \Phi_0$ and $\Gamma = \mathbf{1} \otimes \Gamma_0$ such that the following holds.

Assumption 3 (Controller)**.** *The following holds.*

- *The matrix Φ_0 has the internal model property, i.e. $\operatorname{spec}(\Phi_0) = \operatorname{spec}(S)$ and (Φ_0, Γ_0) is controllable;*
- *The matrices A_c, B_c, C_c are chosen such that the plant $\mathbf{P}$ in* (4) *in closed loop with the controller $\mathbf{C}$ in* (8) *has the origin asymptotically stable when the networks are reliable (that is, $u(t) = \hat{u}(t)$ and $e(t) = \hat{e}(t)$ for all $t \geq 0$) and when $w(t) = 0$ for all $t \geq 0$, i.e. the matrix*

$$A_{cl} = \begin{pmatrix} A_p & B_p C_c \\ B_c C_p & A_c \end{pmatrix} \tag{12}$$

is Schur stable[2]*.*

Namely, the design of the controller follows classical linear output regulation design strategies: a dynamical system (the internal model) processing the regulation error and a (dynamic) stabilizer for the extended closed-loop system (x_p, x_c). Design of matrices A_c, B_c, C_c can be done with any stabilization technique preserving linearity ([21]–[24]).

IV. Main results

We now present the main results of this work. We consider the OPD and the IPD cases separately.

A. Output packet dropout

Consider system (4) paired with the communication setup (6). By concatenating the state variables for any $t \in \mathbb{N}$ as

$$\mathbf{x}(t) = \begin{pmatrix} x_p^\top(t) & x_c^\top(t) & e^\top(t-1) \end{pmatrix}^\top \tag{13}$$

with arbitrarily $e(-1) \in \mathbb{R}^{n_e}$, we can rewrite the closed loop in the form (1). In the OPD scenario, in particular,

$$\overline{F} := \begin{pmatrix} A_p & B_p C_c & 0 \\ B_c C_p & A_c & 0 \\ C_p & 0 & 0 \end{pmatrix}, \overline{G} := \begin{pmatrix} P & B_c Q & Q \end{pmatrix}^\top \tag{14a}$$

$$\underline{F} := \begin{pmatrix} A_p & B_p C_c & 0 \\ 0 & A_c & B_c \\ 0 & 0 & I_{n_e} \end{pmatrix}, \quad \underline{G} := \begin{pmatrix} P & 0 & 0 \end{pmatrix}^\top. \tag{14b}$$

The closed-loop system is a linear switched system that jumps between two dynamics with a certain probability, depending on whether transmission is successful.

[2]It follows from the stabilizability of (A_p, B_p) and of (Φ, Γ), condition (10), and the detectability of (A_p, C_p) that it is always possible to define A_c, B_c, C_c of the form (11c) such that A_{cl} is Schur stable.

Theorem 2. *(Networked Output Regulation for OPD) Consider system* (4) *with the OPD setup* (6) *in closed loop with the controller* (8)*. Let Assumption 1, 2 and 3 hold. If there exists $\gamma_{\mathbf{x}} < 1$ and two symmetric positive matrices $\overline{M}, \underline{M} > 0$ such that* (3) *holds with $\underline{F}, \overline{F}$ defined as in* (14)*, then the Networked Output Regulation Problem is solved.*

Proof. By (3), we have that $V(\mathbf{x}) = \mathbf{x}^\top(\pi\overline{M} + (1-\pi)\underline{M})\mathbf{x}$ is a Lyapunov function for $\mathbf{x}(t+1) = \overline{F}\mathbf{x}(t)$, i.e. $\overline{F}$ is Schur stable. Consider the following Sylvester equation

$$\overline{\Pi}S = \overline{F}\,\overline{\Pi} + \overline{G}\,. \tag{15}$$

Since $\overline{F}$ is Schur, then $\mathrm{spec}(\overline{F}) \cap \mathrm{spec}(S) = \emptyset$. Therefore, for all $\overline{G}$, there always exists a unique solution $\overline{\Pi}$ to (15). Without loss of generality, let $\overline{\Pi}$ be partitioned as $\overline{\Pi} = (\overline{\Pi}_p, \overline{\Pi}_i, \overline{\Pi}_s, \overline{\Pi}_e)^\top$, where $\overline{\Pi}_p \in \mathbb{R}^{n_p \times n_w}$, $\overline{\Pi}_i \in \mathbb{R}^{n_e \cdot n_i \times n_w}$, $\overline{\Pi}_s \in \mathbb{R}^{n_s \times n_w}$ and $\overline{\Pi}_e \in \mathbb{R}^{n_e \times n_w}$. Using the definition of $\overline{F}$ and $\overline{G}$ in (14), equation (15) can be written as

$$\overline{\Pi}_p S = A_p\overline{\Pi}_p + B_pC_c\overline{\Pi}_s + P \tag{16a}$$
$$\overline{\Pi}_i S = \Gamma C_p\overline{\Pi}_p + \Phi\overline{\Pi}_i + \Gamma Q \tag{16b}$$
$$\overline{\Pi}_s S = B_sC_p\overline{\Pi}_p + A_i\overline{\Pi}_i + A_s\overline{\Pi}_s + B_sQ \tag{16c}$$
$$\overline{\Pi}_e S = C_p\overline{\Pi}_p + Q\,. \tag{16d}$$

Substituting (16d) in (16b) and (16c), we can rewrite (16) as

$$\overline{\Pi}_p S = A_p\overline{\Pi}_p + B_pC_c\overline{\Pi}_s + P \tag{17a}$$
$$\overline{\Pi}_i S = \Phi\overline{\Pi}_i + \Gamma\overline{\Pi}_e S \tag{17b}$$
$$\overline{\Pi}_s S = A_i\overline{\Pi}_i + A_s\overline{\Pi}_s + B_s\overline{\Pi}_e S \tag{17c}$$
$$\overline{\Pi}_e S = C_p\overline{\Pi}_p + Q\,. \tag{17d}$$

From the controllability of (Φ, Γ) and since $\mathrm{spec}(S) = \mathrm{spec}(\Phi)$ by Assumption 3, (17b) implies[3] that $\overline{\Pi}_e S = 0$ and therefore $\overline{\Pi}_e = 0$ by invertibility of S. Thus (17) becomes

$$\begin{aligned} \overline{\Pi}_p S &= A_p\overline{\Pi}_p + B_pC_c\overline{\Pi}_s + P \\ \overline{\Pi}_i S &= \Phi\overline{\Pi}_i \\ \overline{\Pi}_s S &= A_i\overline{\Pi}_i + A_s\overline{\Pi}_s \\ 0 &= C_p\overline{\Pi}_p + Q\,. \end{aligned} \tag{18}$$

Consider now the Sylvester equation

$$\underline{\Pi}S = \underline{F}\,\underline{\Pi} + \underline{G}\,. \tag{19}$$

First note that, by (3), $\widetilde{V}(\mathbf{x}) = \mathbf{x}^\top(\pi\overline{M} + (1-\pi)\underline{M})\mathbf{x}$ is a Lyapunov function for $\mathbf{x}(t+1) = \underline{F}\mathbf{x}(t)$, i.e. $\underline{F}$ is Schur stable. Therefore, since $\mathrm{spec}(S) \cap \mathrm{spec}(\underline{F}) = \emptyset$, the Sylvester equation (19) admits a unique solution for any $\underline{G}$. Following a similar partition as for (15), we rewrite (19) as

$$\begin{aligned} \underline{\Pi}_p S &= A_p\underline{\Pi}_p + B_pC_c\underline{\Pi}_s + P \\ \underline{\Pi}_i S &= \Phi\underline{\Pi}_i + \Gamma\underline{\Pi}_e \\ \underline{\Pi}_s S &= A_i\underline{\Pi}_i + A_s\underline{\Pi}_s + B_s\underline{\Pi}_e \\ \underline{\Pi}_e S &= \underline{\Pi}_e\,. \end{aligned} \tag{20}$$

Using similar arguments, since (20) has a solution $\underline{\Pi}$ and $\mathrm{spec}(S) \cap \mathrm{spec}(I_{n_e}) = \emptyset$, then necessarily $\underline{\Pi}_e = 0$. Therefore, we can rewrite (20) as

$$\begin{aligned} \underline{\Pi}_p S &= A_p\underline{\Pi}_p + B_pC_c\underline{\Pi}_s + P \\ \underline{\Pi}_i S &= \Phi\underline{\Pi}_i \\ \underline{\Pi}_s S &= A_i\underline{\Pi}_i + A_s\underline{\Pi}_s \\ 0 &= \underline{\Pi}_e\,. \end{aligned} \tag{21}$$

Comparing (18) with (21), we have that a solution to (21) is $(\underline{\Pi}_p, \underline{\Pi}_i, \underline{\Pi}_s, \underline{\Pi}_e) = (\overline{\Pi}_p, \overline{\Pi}_i, \overline{\Pi}_s, 0)$ i.e. $\underline{\Pi} = \overline{\Pi}$ is a solution to both (15) and (19). Consider now the change of coordinates $\mathbf{x} \to \widetilde{\mathbf{x}}$ with $\widetilde{\mathbf{x}} := \mathbf{x} - \overline{\Pi}w$. The closed-loop (1) can be rewritten as

$$\widetilde{\mathbf{x}}(t+1) = \begin{cases} \overline{F}\widetilde{\mathbf{x}}(t) & \text{if } q(t) = 1 \\ \underline{F}\widetilde{\mathbf{x}}(t) & \text{if } q(t) = 0. \end{cases} \tag{22}$$

Since (3) holds by assumption, then (22) is EMSS by Theorem 1. Therefore, the closed-loop system in the coordinates $\mathbf{x}$ is EPMSS. By definition of $\widetilde{\mathbf{x}}$ and by (18), we have that (9) holds. And this concludes the proof. □

B. Input packet dropout

In this Subsection, we consider system (4) with the communication setup (7). By defining

$$\mathbf{x}(t) = \begin{pmatrix} x_p^\top(t) & x_c^\top(t) & \hat{u}^\top(t-1) \end{pmatrix}^\top \tag{23}$$

with arbitrary $\hat{u}(-1) \in \mathbb{R}^{n_u}$, we get the closed loop (1) with

$$\overline{F} := \begin{pmatrix} A_p & B_pC_c & 0 \\ B_cC_p & A_c & 0 \\ 0 & C_c & 0 \end{pmatrix}, \overline{G} := \begin{pmatrix} P & B_cQ & 0 \end{pmatrix}^\top \tag{24a}$$

$$\underline{F} := \begin{pmatrix} A_p & 0 & B_p \\ B_cC_p & A_c & 0 \\ 0 & 0 & I_{n_u} \end{pmatrix}, \underline{G} := \begin{pmatrix} P & B_cQ & 0 \end{pmatrix}^\top. \tag{24b}$$

We have the following result.

Theorem 3 (Network Boundness for IPD)**.** *Consider system* (4) *with the IPD setup* (7) *in closed loop with the controller* (8)*. Let Assumption 1, 2 and 3 hold. If there exists $0 < \gamma_{\mathbf{x}} < 1$ and two symmetric positive definite matrices $\overline{M}, \underline{M} > 0$ such that* (3) *holds with $\overline{F}, \underline{F}$ defined as in* (24)*, then the Networked Boundness Problem is solved.*

Proof. By (3) and since $\mathrm{spec}(S) \cap \mathrm{spec}(\overline{F}) = \emptyset$, for every $\overline{G}$ there exists a solution Π to the Sylvester equation

$$\Pi S = \overline{F}\Pi + \overline{G}\,. \tag{25a}$$

Similarly to the proof of Theorem 2 and by partitioning Π as $\Pi = (\Pi_p, \Pi_i, \Pi_s, \Pi_u)$, by [20, Lemma 4.3] this implies

$$0 = C_p\Pi_p + Q\,. \tag{25b}$$

Consider the change of coordinates $\mathbf{x} \to \widetilde{\mathbf{x}} := \mathbf{x} - \Pi w$. The closed-loop (1) can be rewritten as

$$\widetilde{\mathbf{x}}(t+1) = \begin{cases} \overline{F}\widetilde{\mathbf{x}}(t) & \text{if } q(t) = 1 \\ \underline{F}\widetilde{\mathbf{x}}(t) + Tw(t) & \text{if } q(t) = 0\,, \end{cases} \tag{26}$$

[3] A detailed proof can be found in e.g. [20, Lemma 4.3]

where

$$T := \underline{G} - \Pi S + \underline{F}\Pi\,. \tag{27}$$

System (26) is a switched system that jumps between two dynamics depending on the probability of successful transmission $\pi \in [0,1]$. Let q be the associated random variable. Define the Lyapunov function

$$V(\widetilde{\mathbf{x}}, q) := q\widetilde{\mathbf{x}}^\top \overline{M}\widetilde{\mathbf{x}} + (1-q)\widetilde{\mathbf{x}}\top \underline{M}\widetilde{\mathbf{x}} \tag{28}$$

with $\overline{M}, \underline{M}$ solving (3). Note that $V(\widetilde{\mathbf{x}}, q) > 0$ for all $\mathbf{x} \neq 0$ and all $q \in \{0,1\}$. The Lyapunov V is a convex combination (depending on q) of the two quadratic Lyapunov functions $\widetilde{\mathbf{x}}^\top \overline{M}\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{x}}^\top \underline{M}\widetilde{\mathbf{x}}$. As such, it follows that V satisfies

$$\underline{\lambda}_V |\widetilde{\mathbf{x}}|^2 \leq V(\widetilde{\mathbf{x}}, q) \leq \overline{\lambda}_V |\widetilde{\mathbf{x}}|^2 \tag{29}$$

where

$$\begin{aligned} \underline{\lambda}_V &:= \min\left\{\min_\lambda\{\lambda \in \mathrm{spec}(\overline{M})\}, \min_\lambda\{\lambda \in \mathrm{spec}(\underline{M})\}\right\}\,, \\ \overline{\lambda}_V &:= \max\left\{\max_\lambda\{\lambda \in \mathrm{spec}(\overline{M})\}, \max_\lambda\{\lambda \in \mathrm{spec}(\underline{M})\}\right\}\,. \end{aligned} \tag{30}$$

Let $M := \pi\overline{M} + (1-\pi)\underline{M}$. By (3), the conditional expectation of $V(\widetilde{\mathbf{x}}(t+1), q(t+1))$ given $q(t) = 0$ is

$$\begin{aligned} &\mathbb{E}[V(\widetilde{\mathbf{x}}(t+1), q(t+1))|q(t) = 0] \\ &\quad = (\underline{F}\widetilde{\mathbf{x}}(t) + Tw(t))^\top M(\underline{F}\widetilde{\mathbf{x}}(t) + Tw(t)) \\ &\leq \widetilde{\mathbf{x}}^\top(t)\underline{F}^\top M\underline{F}\widetilde{\mathbf{x}}(t) + 2\widetilde{\mathbf{x}}^\top(t)\underline{F}^\top MTw(t) \\ &\qquad + w^\top(t)T^\top MTw(t) \\ &\leq \gamma_{\mathbf{x}} V(\widetilde{\mathbf{x}}(t), 0) + 2\widetilde{\mathbf{x}}^\top(t)\underline{F}^\top MTw(t) + w^\top(t)T^\top MTw(t)\,, \end{aligned}$$

where $\gamma_{\mathbf{x}}$ is defined in (3) and the last inequality comes from the fact that $q(t) = 0$. Since $\gamma_{\mathbf{x}} \in (0,1)$, define $\varepsilon^* > 0$ as

$$\varepsilon^* := \tfrac{1-\gamma_{\mathbf{x}}}{2(1-\pi)}\,. \tag{31}$$

Fix $\varepsilon \in (0, \varepsilon^\star)$. We have that $\widetilde{\gamma} := \gamma_{\mathbf{x}} + 2(1-\pi)\varepsilon < 1$. By the Young's inequality $2a^\top b \leq ca^\top a + \frac{1}{c}b^\top b$ for any $c > 0$ and any vectors a, b in which we select $c = \varepsilon, a = \widetilde{\mathbf{x}}$, and $b = \underline{F}MTw$, we obtain

$$2\widetilde{\mathbf{x}}^\top \underline{F}^\top MTw \leq \varepsilon\widetilde{\mathbf{x}}^\top\widetilde{\mathbf{x}} + \tfrac{1}{\varepsilon}(\underline{F}^\top MTw)^\top \underline{F}^\top MTw\,. \tag{32}$$

By Assumption 3, for any matrix S defining (4c) and any initial condition $w(0)$, the signal $w(t)$ is bounded for all $t \geq 0$. Thus there exists a compact set $\mathcal{W} \subset \mathbb{R}^{n_w}$ such that $w(t) \in \mathcal{W}$ for all $t \geq 0$. Let $c_0 > 0$ be defined as

$$c_0 := \tfrac{1}{\varepsilon}|\underline{F}MT|^2 + |TMT| \tag{33}$$

and note that c_0 satisfies

$$c_0|w|^2 \geq w^\top T^\top MTw + \tfrac{1}{\varepsilon}(\underline{F}^\top MTw)^\top \underline{F}^\top MTw\,.$$

Therefore,

$$\begin{aligned} \mathbb{E}[V(\widetilde{\mathbf{x}}(t+1), q(t))|q(t) = 0] &\leq (\gamma_{\mathbf{x}} + \varepsilon)V(\widetilde{\mathbf{x}}(t), 0) \\ &\quad + c_0 \sup_{w\in\mathcal{W}} |w(t)|^2\,. \end{aligned} \tag{34}$$

In case a successful transmission is experienced at time t (i.e. $q(t) = 1$), by (3) we have that

$$\begin{aligned} \mathbb{E}[V(\widetilde{\mathbf{x}}(t+1), q(t+1))|q(t) = 1] &= \widetilde{\mathbf{x}}^\top(t)\overline{F}^\top M\overline{F}\widetilde{\mathbf{x}}(t) \\ &< \gamma_{\mathbf{x}} V(\widetilde{\mathbf{x}}(t), 1)\,. \end{aligned} \tag{35}$$

Therefore, relation (35) holds with probability π and (34) with probability $1-\pi$. Thus, at any given t with an apriori unknown $q(t) \in \{0,1\}$, we have

$$\mathbb{E}[V(\widetilde{\mathbf{x}}(t{+}1), q(t{+}1))] \leq \widetilde{\gamma}V(\widetilde{\mathbf{x}}(t), q(t)) + (1-\pi)c_0 \sup_{w\in\mathcal{W}} |w|^2\,, \tag{36}$$

where we recall that $0 < \widetilde{\gamma} < 1$. Therefore,

$$\begin{aligned} \mathbb{E}[V(\widetilde{\mathbf{x}}(t), q(t))] &\leq \widetilde{\gamma}^{\,t} V(\widetilde{\mathbf{x}}(0), q(0)) \\ &\quad + \sum_{i=0}^{\infty} \widetilde{\gamma}^{\,i}(1-\pi)c_0 \sup_{w\in\mathcal{W}} |w|^2\,. \end{aligned}$$

Recalling the geometric series $\sum_{i=0}^\infty \widetilde{\gamma}^{\,i} = \frac{1}{1-\widetilde{\gamma}}$ for $\widetilde{\gamma} \in (0,1)$, and since (29) holds, it follows that

$$\mathbb{E}[\widetilde{\mathbf{x}}(t)^\top\widetilde{\mathbf{x}}(t)] \leq \frac{\overline{\lambda}_V}{\underline{\lambda}_V}\widetilde{\gamma}^{\,t}\mathbb{E}[\widetilde{\mathbf{x}}(0)^\top\widetilde{\mathbf{x}}(0)] + \frac{c_0(1-\pi)}{\underline{\lambda}_V(1-\widetilde{\gamma})} \sup_{w\in\mathcal{W}} |w|^2 \tag{37}$$

By definition of $\widetilde{\mathbf{x}}$, the closed-loop system in the coordinates $\mathbf{x}$ satisfies (2) with $\delta_{\mathbf{x}} = \frac{\overline{\lambda}_V}{\underline{\lambda}_V}, \gamma_{\mathbf{x}} = \widetilde{\gamma}$ and $\beta_{\mathbf{x}} = \frac{c_0(1-\pi)}{\underline{\lambda}_V(1-\widetilde{\gamma})}$, i.e. it is EPMSS with respect to w. Recalling the definition of $\widetilde{\mathbf{x}}$ and (25b), relation (9) follows, concluding the proof. □

Remark 1. *Recalling the definition of* c_0 *and* $\widetilde{\gamma}$ *in* (37)*, of* ε^* *in* (31)*, and of* $\underline{\lambda}_V$ *in* (29)*, relation* (9) *holds with coefficient*

$$\beta_e = \frac{1-\pi}{\underline{\lambda}_V\,|C_p|^2} \min_{\varepsilon\in(0,\varepsilon^*)}\left(\frac{|\underline{F}MT|^2 + \varepsilon|TMT|}{\varepsilon[1-\gamma_{\mathbf{x}} - 2(1-\pi)\varepsilon]}\right)\,. \tag{38}$$

As expected, we see that $\lim_{\pi\to 1}\beta_e = 0$*, namely, the expectation on the ultimate bound of the regulated output* e *reduces with the reliability of the channel.*

We recall once again the main difference between Section IV-A and Section IV-B. In the first case, since package drop is experienced between the output sensor, the control action can still enforce an internal model property on the system. Therefore, the contribution of the dynamics x_i in the control input u is always present even when communication is lost. This follows since the solution of the Sylvester equations (15) and (19) coincide. As such, the regulation problem can still be cast as a stabilization problem, and the expectation of the error exponentially shrinks to zero. Differently, in the IPD case, only an upper bound on the regulation error can be provided. The upper bound increases with the amplitude of the exogenous signal w, and decreases with the probability of successful transmission π and with the stability margin of the system $\gamma_{\mathbf{x}}$ in (3) as can be seen in (38).

V. Application Example

We illustrate our results on an electric motor model, obtained by discretizing the continuous-time system $\dot{x}_{\widetilde{p}} = A_{\widetilde{p}}x_{\widetilde{p}} + B_{\widetilde{p}}u$ with a zero-holder hold with a sampling time $T_s = 10^{-5}$. In particular, $x_{\widetilde{p}}$ is composed of the rotor speed

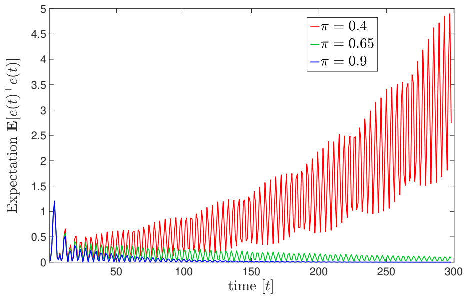

Fig. 2: OPD setup

and the induction current. The input is the voltage applied to the motor and the output is the velocity. The continuous time system is defined by $A_{\widetilde{p}} = \begin{pmatrix} -\frac{b}{J} & \frac{K}{J} \\ -\frac{K}{L} & -\frac{R}{L} \end{pmatrix}$, $B_{\widetilde{p}} = \begin{pmatrix} 0 & \frac{1}{L} \end{pmatrix}^\top$, where b is the viscous friction constant, J is the rotor inertia, K is the motor constant, L is the electrical inductance and R is the electrical resistance. We selected $b = 3.5 \cdot 10^{-6}, J = 3.223 \cdot 10^{-8}, K = 0.0274, L = 2.75 \cdot 10^{-6}$, and $R = 1$. After discretization, the discrete-time plant (4) is recovered, where we selected $P = (0, 0; 0, 1)^\top, Q = (0, 10)$. The exosystem (4c) is taken as $w(t+1) = Sw(t)$ with $S = (0, 1; -1, \sqrt{2})$. To solve our regulation problem, we first introduce the internal model (11a) with $\Phi = S$ and $\Gamma = (1, 0)^\top$. Then, we design a static stabilizer for the closed-loop to be asymptotically stable when $w = 0$ and no packet drop is experienced. Figure 2 shows some simulations for the OPD scenario with random initial conditions. Expectation is taken over 10000 simulations with the same initial conditions. In particular, Figure 2 shows the expectation of the tracking error depending on the probability of transmission in the OPD case. As it's possible to see, for $\pi = 0.9$, the average of the error asymptotically goes to zero. This is the same for $\pi = 0.65$, but with a slower decay rate. This because the convergence rate $\gamma_{\mathbf{x}}$ depends on π via (3). When the probability is very low ($\pi = 0.4$), the conditions of Theorem 1 do not hold. In this case, the closed-loop system is unstable. This is not surprising as it is known that a switched system that jumps between two stable dynamics can be unstable.

VI. Conclusions

In this work, we considered an output regulation problem for discrete-time linear systems subject to stochastic packet drop. The control law is designed via emulation, with a control action composed of an internal model processing the regulation error and a feedback stabilizer for the extended closed-loop system. We showed that, when packet drop is experienced in the output channel, then regulation can be cast as a stabilization problem and the expectation of the regulation error exponentially goes to zero. Differently, when packet drop is experienced in the input channel, the regulation error is only bounded in expectation, with the bound depending on the probability of successful transmission and on the amplitude of the exogenous signals. Future works will consider the extension of the presented results to the nonlinear scenario and/or to co-design techniques.

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