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Regularized Continuation Method For Motion Planning

Zhengping Ji, Yacine Chitour and Emmanuel Trélat

***Abstract*— In this article, we investigate the motion planning problem for control-affine systems with nontrivial drifts using a regularized homotopy continuation method. We prove that when there exists a nonsingular solution of the original motion planning problem, the regularized solution converges to it almost everywhere, and the endpoints derived from the regularized solutions converge to the desired target point, may the solution of the classical continuation method is well defined or not. This provides a way to design the steering control in the presence of singular controls when the classical continuation method is not applicable. The effectiveness of the regularization is illustrated by numerical experiments on the rolling systems.**

I. Introduction

In control theory, robotics and autonomous vehicles, the problem of motion planning is fundamental. Given a controlled dynamical system, this problem consists in designing algorithms that derive controls steering the system to preassigned destinations. Over the past forty years, with the development of intelligent transportation systems, many methods have been designed for motion planning, ranging from feedback techniques [17], sampling-based roadmaps [18], navigation functions [23] to computational algebraic geometry techniques [21].

In particular, for the models with dynamics described by finite-dimensional nonlinear control systems, the most important and commonly investigated case for motion planning is the control-affine system [5]

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} u^i(t) g_i(x(t)) \tag{1}$$

on an n-dimensional smooth connected Riemannian manifold M, with smooth vector fields f, $\{g_i\}_{i=1}^m$ and L^2-bounded controls $\{u^i(t)\}_{i=1}^m$, $t \geqslant 0$. The motion planning problem for (1) is formalized as: given any pair $x, x_1 \in M$, design a control $u = (u^1, \cdots, u^m) \in L^2([0,T], \mathbb{R}^m)$ with $T > 0$, that yields a solution of (1), denoted by $x_u(t)$, satisfying $x_u(0) = x$, $x_u(T) = x_1$.

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Z. Ji is with the Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science & School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100190, P. R. China, and Laboratoire Jacques-Louis Lions (LJLL), Sorbonne Université, F-75005 Paris, France, jizhengping@amss.ac.cn

Y. Chitour is with Laboratoire des Signaux et Systèmes (L2S), Université Paris-Saclay, CNRS, CentraleSupélec, 91190, Gif-sur-Yvette, France, yacine.chitour@l2s.centralesupelec.fr

E. Trélat is with Laboratoire Jacques-Louis Lions (LJLL), Sorbonne Université, CNRS, Université Paris Cité, Inria, F-75005 Paris, France, emmanuel.trelat@sorbonne-universite.fr

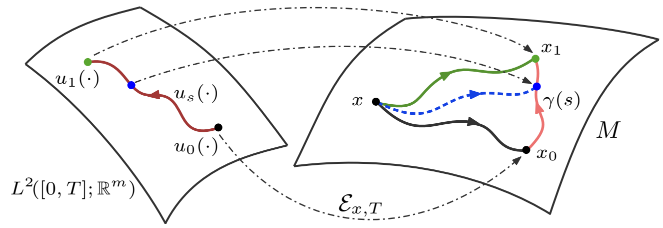

Fig. 1: HCM for motion planning

When the drift vector field f in (1) is zero, the above motion planning problem has been widely considered. Existing methods include the nilpotent approximation [14], the Murray-Sastry method [22], [8], the loop method [26], optimal control method [30] and sinusoidal controllers [20]. However, there have been fewer investigations on the motion planning of control-affine systems with a nontrivial drift, because the drift vector field breaks the symmetric nonholonomic constraints on the tangent bundle [32] and forbids the time-rescaling invariance [15]. Approaches proposed for motion planning of control-affine systems with nontrivial drifts or of nonlinear systems in more general forms include adaptive extremum seeking control [11], sample-based differential fast marching tree algorithm [24], affine geometric heat flow method [19], and convex spline optimization [31].

Since the 1990s, the homotopy continuation method (HCM) introduced in [27] has been applied to motion planning of nonlinear systems. For the control-affine system (1), let $\mathcal{U} = L^2([0,T], \mathbb{R}^m)$ be the space of controls, define the endpoint map $\mathcal{E}_{x,T} : \mathcal{U} \to M$, $u \mapsto x_u(x;T)$, where $t \mapsto x_u(x;t)$ is the solution of (1) corresponding to u starting from $x \in M$ in time $t \geqslant 0$. Then the motion planning from x to x_1 in time T aims at constructing a control $u \in \mathcal{E}_{x,T}^{-1}(x_1)$. The basic idea of HCM is to reformulate this problem as a differential equation by searching for the preimage of a curve in M as a parametrized control in the control space $\mathcal{U}$. To be precise, let $x_0 \in M$ be a point different from x and x_1, if there is a smooth path $\gamma : [0,1] \to M$, $s \mapsto \gamma(s)$ satisfying $\gamma(0) = x_0$, $\gamma(1) = x_1$ which can be lifted through $\mathcal{E}_{x,T}$ to a path u_s in $\mathcal{U}$ starting from u_0, satisfying $\mathcal{E}_{x,T}(u_0) = x_0$, that is,

$$\begin{aligned} &\exists u_s : [0,1] \to \mathcal{U}, \quad s \mapsto u_s(\cdot), \\ &\text{s.t. } \gamma(s) = \mathcal{E}_{x,T}(u_s(\cdot)), \end{aligned} \tag{2}$$

then $u_s(\cdot)|_{s=1} = (u_1^1(\cdot), \cdots, u_1^m(\cdot)) \in \mathcal{U}$ will be the desired control that steers the system (1) from x to x_1 in time T. This is illustrated on Fig. 1.

Differentiating (2), we infer a sufficient condition for the curve γ to be lifted, namely, the existence of the solution of

$$\mathrm{d}\mathcal{E}_{x,T}(u_s)\frac{\partial u_s}{\partial s} = \dot{\gamma}(s), \quad s \in [0,1] \tag{3}$$

with $u_s|_{s=0} = u_0$, where $\mathrm{d}\mathcal{E}_{x,T}(u) : L^2([0,T],\mathbb{R}^m) \to T_xM$ is the Fréchet differential of the endpoint map $\mathcal{E}_{x,T}$ at $u \in \mathcal{U}$, $\dot{\gamma}(s)$ is the differential of γ with respect to s. When the solution of (3) exists, the least L^2-norm solution of (3) satisfies

$$\frac{\partial u_s}{\partial s} = P(u_s)\dot{\gamma}(s), \quad s \in [0,1] \tag{4}$$

with $u_s|_{s=0} = u_0$, where $P(u_s)$ is the Moore-Penrose pseudo-inverse of $\mathrm{d}\mathcal{E}_{x,T}$ at u_s. When $\mathrm{d}\mathcal{E}_{x,T}$ is surjective at $u \in \mathcal{U}$, $P(u)$ has the following expression:

$$P(u) := \mathrm{d}\mathcal{E}^*_{x,T}(u)(\mathrm{d}\mathcal{E}_{x,T}(u)\mathrm{d}\mathcal{E}^*_{x,T}(u))^{-1}, \tag{5}$$

where $\mathrm{d}\mathcal{E}^*_{x,T}(u) : T^*_xM \to L^2([0,T],\mathbb{R}^m)$ is the adjoint map of $\mathrm{d}\mathcal{E}_{x,T}(u)$.

The equation (4) is called the *path lifting equation* (PLE). If (4) admits a global solution on $s \in [0,1]$, then the motion planning problem is solved by $u_s|_{s=1}$. Thus, the problem is reduced to finding conditions for the PLE to be globally well-posed on $s \in [0,1]$.

If $\mathrm{d}\mathcal{E}_{x,T}(u)$ is not surjective, i.e. $\mathrm{rank}(\mathrm{d}\mathcal{E}^*_{x,T}(u)\mathrm{d}\mathcal{E}_{x,T}(u)) < n$, then u is called a singular control, and $\mathcal{E}_{x,T}(u)$ is called a critical value of $\mathcal{E}_{x,T}$. Since the pseudo-inverse $P(u)$ considered in (4) is equal to (5) only when u is nonsingular, one cannot design the path in the domain where $\mathrm{d}\mathcal{E}_{x,T}$ is not of full rank [4], [6], otherwise there might be a blow-up at the right-hand side of (5). For the motion planning outside of these critical values, it was proven [7] that (4) is globally well-posed on $[0,1]$ if on any compact subset of M, the norm of P as defined in (5) has at most linear growth in $\|u\|$ at infinity; this condition is satisfied for special types of dynamics under strong assumptions on the Lie configuration of the system [7], [9], [28]. The HCM has been generalized to constrained motion planning [10], [12] and control-affine systems with drifts [16], showing good numerical performance [2], [13]. But so far there are no theoretical results on the application of HCM to control-affine systems in the presence of singular controls.

In the PhD thesis [25], the author proposed a regularized continuation method to deal with the above difficulty. If we replace the operator P in the PLE (4) by an operator P_δ, which is the Tikhonov regularized pseudo-inverse of $\mathrm{d}\mathcal{E}_{x,T}(u)$, defined as

$$P_\delta(u) := \mathrm{d}\mathcal{E}^*_{x,T}(u)\big(\mathrm{d}\mathcal{E}_{x,T}(u)\mathrm{d}\mathcal{E}^*_{x,T}(u) + \delta I_n\big)^{-1} \tag{6}$$

where the constant $\delta > 0$ is a regularization parameter. Compared with (5), this operator P_δ always exists for any $u \in L^2([0,T],\mathbb{R}^m)$ and $\delta > 0$, no matter $\mathrm{d}\mathcal{E}_{x,T}$ is singular at u or not. The PLE (4) with P replaced by P_δ is called the *regularized path lifting equation* (RPLE). If as δ tends to zero, the solutions of the RPLE converge to those of the PLE under the same boundary condition, then the motion planning will be solved for singular controls. However, [25] did not establish such convergence, although the numerical simulations showed good performance of the regularized method.

In this paper we consider the motion planning of control-affine systems defined on smooth manifolds with nontrivial drifts, and for the first time provide theoretical results on the regularized continuation method. Based on the analysis performed in Section II on the operator P in the PLE, in Section III we prove the convergence of the solutions and endpoints corresponding to the RPLE, and in Section IV numerical simulations on the rolling systems show the effectiveness of our method.

II. Locally Lipschitz Property of the Operator P

In this section, we show that the operator P in the PLE (4) is locally Lipschitz over nonsingular controls. This fact is crucial for proving the convergence of the solutions of the RPLE. Let $x \in M$. Throughout, we make the following assumption on the completeness of the system (1):

(A) For any $u \in L^2([0,T],\mathbb{R}^m)$, $\mathcal{E}_{x,t}(u)$ is well defined for all $t \in [0,T]$.

From now on, assume that the system (1) is globally controllable at x in time T. We denote $\mathcal{E}_{x,T}$ briefly by $\mathcal{E}$, and denote the integral curve of the system (1) corresponding to u starting from x briefly by $x_u(t)$, $t \in [0,T]$.

The Fréchet differential of $\mathcal{E}$ is given by (see, e.g., [1])

$$\mathrm{d}\mathcal{E}(u)(v) = \int_0^T \sum_{i=1}^m v^i(t)R_u(T,t)g_i(x_u(t))\mathrm{d}t \tag{7}$$

for any $v \in L^2([0,T],\mathbb{R}^m)$, where $R_u(T,t)$ is the transition matrix of the system (1) under control u from time t to time T, satisfying

$$\frac{\partial R_u(T,t)}{\partial t} = -R_u(T,t)\Big(\mathrm{d}f(x_u(t)) + \sum_{i=1}^m u^i(t)\mathrm{d}g_i(x_u(t))\Big)$$

with $R_u(T,T) = \mathrm{id}$, where $\mathrm{d}f := \frac{\partial f}{\partial x}$, $\mathrm{d}g_i := \frac{\partial g_i}{\partial x}$, $i = 1,\cdots,m$ are the differentials of the vector fields. The adjoint $\mathrm{d}\mathcal{E}^*(u)$ of the differential of the endpoint map at u is given by (see, e.g., [7])

$$\mathrm{d}\mathcal{E}^*(u)(z) = (\langle p_z(t), g_1(x_u(t))\rangle, \cdots, \langle p_z(t), g_m(x_u(t))\rangle)^\top$$

for any $z \in T^*_{\mathcal{E}(u)}M$, where $p_z(t)$ is the solution of

$$\dot{p}_z(t) = -\Big(\mathrm{d}f(x_u(t)) + \sum_{i=1}^m u^i(t)\mathrm{d}g_i(x_u(t))\Big)^\top p_z(t), \tag{8}$$

with $p_z(T) = z$.

Denote by $\tilde{S}$ the set of singular controls and by S the set of critical values, i.e., $S := \mathcal{E}(\tilde{S})$. Denote by $G_u := \mathrm{d}\mathcal{E}(u)\mathrm{d}\mathcal{E}^*(u) \in \mathbb{R}^{n\times n}$ the Gramian matrix of (1) at u. When $u \notin \tilde{S}$, (5) gives $P(u) := \mathrm{d}\mathcal{E}^*(u)G_u^{-1}$, and by definition we have [7]

$$\|P(u)\| = \frac{1}{\min\limits_{\|z\|=1} \|\mathrm{d}\mathcal{E}^*(u)(z)\|} = \frac{1}{\sqrt{\min \mathrm{Spec}(G_u)}}. \tag{9}$$

Denote by d the distance derived from the metric on the Riemannian manifold M. We start with the following basic and classical property for the endpoint map $\mathcal{E}$.

Lemma 2.1: Let $U \subset L^2([0,T],\mathbb{R}^m)$ be an L^2-bounded set with $d(\mathcal{E}(U),S) > 0$. If **(A)** holds, then there exists a compact set $V \subset M$, such that $\mathcal{E}(U) \subset V$; both $\{\|\mathrm{d}\mathcal{E}^*(u)\| \mid u \in U\}$ and $\{\|P(u)\| \mid u \in U\}$ are bounded.

Proof: By Proposition 3.5 in [29], on a weakly bounded subset $U \subset L^2([0,T],\mathbb{R}^m)$, for any two controls $u_1, u_2 \in U$, $d(\mathcal{E}(u_1),\mathcal{E}(u_2))$ is bounded by $\|u_1 - u_2\|_{L^2}$, where d is the distance derived from the metric of the Riemannian manifold M. Therefore $\mathcal{E}(U)$ is bounded, and V can be chosen as its closure.

By Proposition 3.7 in [29], $\mathrm{d}\mathcal{E}(u)$ is weakly continuous with respect to u, hence $\sup_{u\in V}\|\mathrm{d}\mathcal{E}^*(u)\|$ is bounded on the compact set V.

Assume that $d(\mathcal{E}(U),S) = \alpha > 0$. Then for any sequence $\{u_n\} \subset U$, $d(\mathcal{E}(u_n),S) \geqslant d(\mathcal{E}(U),S) \geqslant \alpha$, $\forall n$, and hence $\inf_n d(\mathcal{E}(u_n),S) \geqslant \alpha > 0$, therefore if $\inf_{u\in U}(\min \operatorname{Spec} G_u) = 0$, then there exists a sequence $\{\bar{u}_n\} \subset U$, such that $\lim_{n\to\infty} d(\mathcal{E}(\bar{u}_n),S) = 0$, which yields a contradiction. Hence we have $\inf_{u\in U}(\min \operatorname{Spec} G_u) > 0$, and by (9), $\|P(u)\|$ is uniformly bounded on U. ∎

Proposition 2.2: Let $U \subset L^2([0,T],\mathbb{R}^m)$ be a weakly bounded set. If $d(\mathcal{E}(U),S) > 0$, then $\exists M_U > 0$, s.t. $\forall u_1$, $u_2 \in U$, $\|P(u_1) - P(u_2)\| \leqslant M_U\|u_1 - u_2\|$.

Proof: Since $P(u) = \mathrm{d}\mathcal{E}^*(u)G_u^{-1}$, for any $u_1, u_2 \in U$

$$\begin{aligned}P(u_1) - P(u_2) =&(\mathrm{d}\mathcal{E}^*(u_1) - \mathrm{d}\mathcal{E}^*(u_2))G_{u_1}^{-1} + \\ &\mathrm{d}\mathcal{E}^*(u_1)(G_{u_1}^{-1} - G_{u_2}^{-1}).\end{aligned}$$

As $d(\mathcal{E}(U),S) > 0$, there exists $\alpha > 0$ such that $\min \operatorname{Spec}(G_u) > \alpha$, $\forall u \in U$. Hence $\|G_{u_1}^{-1}\|$ is uniformly bounded. Since $G_{u_1}^{-1} - G_{u_2}^{-1} = G_{u_2}^{-1}(G_{u_2} - G_{u_1})G_{u_1}^{-1}$, and $G_{u_1} - G_{u_2} = \mathrm{d}\mathcal{E}(u_1)(\mathrm{d}\mathcal{E}^*(u_1) - \mathrm{d}\mathcal{E}^*(u_2)) + (\mathrm{d}\mathcal{E}(u_1) - \mathrm{d}\mathcal{E}(u_2))\mathrm{d}\mathcal{E}^*(u_2)$, applying Lemma 2.1, the estimate is further transformed to

$$\|P(u_1) - P(u_2)\| \leqslant \beta\|\mathrm{d}\mathcal{E}^*(u_1) - \mathrm{d}\mathcal{E}^*(u_2)\|$$

where β is a constant only depending on U, obtained from the boundedness of the operators $G_{u_1}^{-1}$, $G_{u_2}^{-1}$, $\mathrm{d}\mathcal{E}^*(u_1)$, $\mathrm{d}\mathcal{E}^*(u_2)$, according to Lemma 2.1 and the fact that $\|\mathrm{d}\mathcal{E}(u)\| = \|\mathrm{d}\mathcal{E}^*(u)\|$.

Therefore, define the switching function as $\psi_u^z(t) := \mathrm{d}\mathcal{E}^*(u)(z) \in L^2([0,T],\mathbb{R}^m)$, $\forall z \in T^*_{\mathcal{E}(u)}M$, it suffices to prove the property in the lemma for $\psi_u^z(t)$, given any z of unit length. For all u_1, $u_2 \in U$, denote briefly $g_i^1(t) := g_i(x_{u_1}(t))$, $g_i^2(t) := g_i(x_{u_2}(t))$, $i = 1,\cdots,m$. Then

$$\psi_{u_1}^z(t) - \psi_{u_2}^z(t) = \begin{pmatrix} \langle p_z^1(t), g_1^1(t)\rangle - \langle p_z^2(t), g_1^2(t)\rangle \\ \vdots \\ \langle p_z^1(t), g_m^1(t)\rangle - \langle p_z^2(t), g_m^2(t)\rangle \end{pmatrix}$$

where $p_z^1(t)$, $p_z^2(t)$ are adjoint vectors corresponding to u_1, u_2 respectively. Since on a bounded subset $\mathcal{E}(U) \subset M$ and finite time interval $[0,T]$, the adjoint curves and the vector fields $\{g_i\}_{i=1}^m$ together with their differentials are all bounded, there exist $C_1, C_2, C_3 > 0$, such that $\|g_j^1(t)\| < C_1$, $\|p_z^2(t)\| < C_2$, $\|g_j^1(t) - g_j^2(t)\| < C_3\|x_{u_1}(t) - x_{u_2}(t)\|$, $\forall j = 1,\cdots,m$, therefore we have

$$\begin{aligned}&\|\psi_{u_1}^z(t) - \psi_{u_2}^z(t)\| \\ \leqslant&\left\|\left(\langle p_z^1(t) - p_z^2(t), g_1^1(t)\rangle, \cdots, \langle p_z^1(t) - p_z^2(t), g_m^1(t)\rangle\right)^\top\right\| + \\ &\left\|\left(\langle p_z^2(t), g_1^1(t) - g_1^2(t)\rangle, \cdots, \langle p_z^2(t), g_m^1(t) - g_m^2(t)\rangle\right)^\top\right\| \\ \leqslant& C_4\|p_z^1(t) - p_z^2(t)\| + C_5\|x_{u_1}(t) - x_{u_2}(t)\|, \end{aligned} \tag{10}$$

where the positive constants C_4, C_5 can be constructed from the constants C_1, C_2, C_3.

By Proposition 3.5 in [29], the second part in (10) is bounded as

$$\|x_{u_1}(t) - x_{u_2}(t)\| \leqslant A\|u_1 - u_2\| \tag{11}$$

where $A > 0$ is a constant. Since $\mathcal{E}(U)$ and U are both bounded and all vector fields are smooth, the constant A are uniform for all $u_1, u_2 \in U$.

Next, we estimate the first part in (10). Denote briefly $F(u,t) := [\mathrm{d}f(x_u(t)) + \sum_{i=1}^m u^i(t)\mathrm{d}g_i(x_u(t))]^\top$, then over U we have

$$\int_0^t \left\|F(u_2,s) - F(u_1,s)\right\|\mathrm{d}s \leqslant M\|u_1 - u_2\| \tag{12}$$

where the positive constant M is obtained by the weak boundedness of the controls, the smoothness of vector fields and the estimate (11).

Since the adjoint curve satisfies equation (8), we have

$$\begin{aligned}\left\|p_z^1(t) - p_z^2(t)\right\| =& \left\|\int_t^T F(u_2,s)\big(p_z^2(s) - p_z^1(s)\big) + \right. \\ &\left.\Big(F(u_2,s) - F(u_1,s)\Big)p_z^1(s)\mathrm{d}s\right\| \\ \leqslant& B_0\int_t^T \left\|p_z^1(s) - p_z^2(s)\right\|\mathrm{d}s + B_1\|u_1 - u_2\|,\end{aligned}$$

where the constants B_0, $B_1 > 0$ are constructed in a similar way as C_4, C_5, A were, due to smoothness of vector fields, boundedness of adjoint curves (by Lemma 2.1), and weak boundedness of U, together with the estimates (11)(12). By the Grönwall lemma, integrating the above inequality yields

$$\|p_1^z(t) - p_2^z(t)\| \leqslant B_1\|u_1 - u_2\|\exp(B_0 t). \tag{13}$$

By the similar reason as in (11), B_0, B_1 are also uniform with respect to any $u_1, u_2 \in U$ and any z of unit length.

Plugging (11)(13) into (10), we have the estimate of the difference of switching functions, and further, of the operator P stated in the lemma. ∎

III. Regularized Continuation Method

We first introduce the notion of regularized pseudo-inverse, then we show how to regularize the HCM in the presence of singular controls.

Proposition 3.1 ([3]): $\forall A \in \mathbb{R}^{m\times n}$, the matrix

$$A^+ := \lim_{\delta\to 0} A^*(AA^* + \delta I_m)^{-1} \tag{14}$$

is a Moore-Penrose pseudo inverse of A, i.e. it satisfies the following axioms: (1) $AA^+A = A$; (2) $A^+AA^+ = A^+$; (3) $AA^+ \in \mathbb{R}^{m\times m}$ and $A^+A \in \mathbb{R}^{n\times n}$ are self-adjoint.

The procedure of adding a regularization term δI_m, with $\delta > 0$, as in (14), is called *Tikhonov regularization*. Recall that $A^+ = A^*(AA^*)^{-1}$ when A is of full rank. This regularization procedure allows to calculate the pseudo-inverse of matrices that are not of full rank. In particular, this idea can be applied to the operator $\mathrm{d}\mathcal{E}_{x,T}$ in the PLE (4).

Now we propose the regularized continuation method. Consider the following regularized path-lifting equation

$$\frac{\partial u_{s,\delta}}{\partial s} = \mathrm{d}\mathcal{E}^*(u_{s,\delta})\big(\mathrm{d}\mathcal{E}(u_{s,\delta})\mathrm{d}\mathcal{E}^*(u_{s,\delta}) + \delta I_n\big)^{-1}\dot{\gamma}(s), \tag{15}$$

where $\gamma : [0,1] \to M$ is the same smooth curve as in (4), and $\delta > 0$ is a constant.

The following questions emerge:

(Q1) Is the regularized solution, i.e. solution of (15), well-posed when the solution of (4) is singular?

(Q2) As $\delta \to 0$, does the regularized solution converge to the original one when the latter is well-posed?

(Q3) As $\delta \to 0$, does the endpoint of the regularized solution converge to the desired target point x_1?

If all the above questions can be answered positively, then fix an initial $u_{0,\delta} = u_0$, $\forall\delta$, and draw a curve γ starting from $\mathcal{E}(u_0)$, when $\gamma(1) = x_1$, the solution $u_{s,\delta}$ to (15) will give the desired control driving the system from x to x_1 as $\lim\limits_{\delta\to 0} u_{s,\delta}|_{s=1}$, even if the solution of the PLE (4) is not well defined for every $s \in [0,1]$. The motion planning will thus be solved.

We first give a positive answer to (Q1).

Proposition 3.2: For any $u \in L^2([0,T],\mathbb{R}^m)$, the operator P_δ defined by (6) satisfies $\|P_\delta(u)\| \leqslant \frac{1}{2\sqrt{\delta}}$; when $P(u)$ defined by (5) exists, we have $\lim\limits_{\delta\to 0}\|P_\delta(u)\| = \|P(u)\|$.

Proof: By definition,

$$\begin{aligned}\|P_\delta(u)\|^2 &= \max_{\|z\|=1} z^*\big(G_u + \delta I_n\big)^{-1} G_u \big(G_u + \delta I_n\big)^{-1} z \\ &= \max_{i=1,\dots,n} \frac{\lambda_u^i}{(\lambda_u^i + \delta)^2},\end{aligned}$$

where λ_u^i, $i = 1, \cdots, n$ are the eigenvalues of G_u. Therefore

$$\|P_\delta(u)\| = \max_{\lambda_u^i \neq 0} \frac{1}{\sqrt{\lambda_u^i} + \frac{\delta}{\sqrt{\lambda_u^i}}} \leqslant \frac{1}{2\sqrt{\delta}},$$

and when $P(u)$ is well defined, λ_u^i, $i = 1, \cdots, n$ are all nonzero; then $\lim\limits_{\delta\to 0}\|P_\delta(u)\| = \max\limits_{i=1,\dots,n} \frac{1}{\sqrt{\lambda_u^i}} = \|P(u)\|$. ∎

Remark 3.3: The above proposition shows that the RPLE (15) is well defined for all $s \in [0,1]$ when $\delta > 0$, making it possible to lift the path through $P_\delta(u)$ even if the control u is singular.

Based on Proposition 2.2, we answer the question (Q2) by proving the convergence of the regularized solutions to the un-regularized ones as $\delta \to 0$.

Theorem 3.4: Assume that the solution $s \mapsto u_s$ to (4) is nonsingular and well-defined for all $s \in [0,1]$. Then there exists $\mu > 0$ such that $u_{s,\delta} \notin \tilde{S}$, $\forall s \in [0,1]$, $\forall\delta \in [0,\mu]$, where $u_{s,\delta}$ is the solution of (15) for $s \in [0,1]$ such that $u_{0,\delta} = u_0$ for every $\delta \in [0,\mu]$. Moreover, for any $s \in [0,1]$, we have $u_{s,\delta}(t) \xrightarrow{L^2} u_s(t)$ as $\delta \to 0$.

Proof: As u_s is globally well defined, there exists $\alpha > 0$ such that $d(\mathcal{E}(u_s), S) \geqslant \alpha$ for all $s \in [0,1]$. By the continuity of $\mathcal{E}$ and $d(\cdot, S)$, together with Lemma 2.1, there exists $\tilde{\alpha} > 0$ such that $\tilde{d}(u_s, \tilde{S}) \geqslant \tilde{\alpha}$ for all $s \in [0,1]$, where $\tilde{d}$ is the distance on $L^2([0,T],\mathbb{R}^m)$ induced by the L^2-norm.

For any $\mu > 0$, since $u_{0,\delta} = u_0$ for all $\delta \in [0,\mu]$ and $\tilde{d}(u_0, \tilde{S}) > 0$, there exists $s_\mu \leqslant 1$, such that $u_{s,\delta} \notin \tilde{S}$, $\forall s \in [0, s_\mu]$, $\forall\delta \in [0,\mu]$ (we will later prove that, actually, $s_\mu = 1$ for some μ specially chosen).

Define $e_{s,\delta} := u_s - u_{s,\delta}$, and denote $G_{u_{s,\delta}}$ briefly by $G_{s,\delta}$, G_{u_s} by G_s. Then for any $s \leqslant s_\mu$, we have

$$\begin{aligned}\|e_{s,\delta}\| = &\Big\| \int_0^s \Big[\mathrm{d}\mathcal{E}^*(u_\tau)G_\tau^{-1} - \mathrm{d}\mathcal{E}^*(u_{\tau,\delta})G_{\tau,\delta}^{-1} + \\ &\qquad \mathrm{d}\mathcal{E}^*(u_{\tau,\delta})^*\big(G_{\tau,\delta}^{-1} - (G_{\tau,\delta} + \delta I)^{-1}\big)\Big]\dot{\gamma}(\tau)\mathrm{d}\tau\Big\| \\ \leqslant &\int_0^s \Big[\|P(u_\tau) - P(u_{\tau,\delta})\| + \\ &\qquad \delta\|\mathrm{d}\mathcal{E}^*(u_\tau)\|\|G_{\tau,\delta}^{-1}\|\|(G_{\tau,\delta} + \delta I)^{-1}\|\Big]\|\dot{\gamma}(\tau)\|\mathrm{d}\tau\end{aligned}$$

Since $u_{0,\delta} = u_0$, there exists a bounded set $W \subset L^2([0,T],\mathbb{R}^m)$ satisfying $\tilde{d}(W, \tilde{S}) > 0$, such that u_s, $u_{s,\delta} \in W$, $\forall s \in [0, s_\mu]$, $\forall\delta \in [0,\mu]$. As $\|P(u)\|$ is (by Proposition 2.2) Lipschitz and (by Lemma 2.1) uniformly bounded over W, we have the boundedness of $\|\mathrm{d}\mathcal{E}^*(u_s)\|\|G_{s,\delta}^{-1}\|\|(G_{s,\delta} + \delta I)^{-1}\|$ over $s \in [0,1]$, $\delta \in [0, s_\mu]$. The above inequality can be therefore reduced to

$$\|e_{s,\delta}\| \leqslant C_1 \int_0^s \|e_{\tau,\delta}\|\mathrm{d}\tau + C_2\delta, \quad \forall s \in [0, s_\mu], \tag{16}$$

where C_1, $C_2 > 0$ are constants only depending on μ, and hence by the Grönwall lemma, $\lim\limits_{\delta\to 0}\|e_{s,\delta}\|_{L^2} = 0$ for all $s \in [0, s_\mu]$.

It remains to prove that, actually, we can take $s_\mu = 1$. For any $\mu \geqslant 0$, let

$$\sigma(\mu) := \sup\Big\{s' \in [0,1] \;\Big|\; u_{s,\delta} \notin \tilde{S},\ \forall (s,\delta) \in [0,s'] \times [0,\mu]\Big\}.$$

Let us prove that there exists $\mu_0 > 0$ such that $\sigma(\mu_0) = 1$. By contradiction, assume that there exists a decreasing sequence of positive real numbers $\{\mu_n\}_{n\in\mathbb{N}}$ satisfying $\sigma(\mu_n) < 1$ and $\lim\limits_{n\to 0}\mu_n = 0$. Moreover, there exist two sequences $\{s_n\}_{n\in\mathbb{N}}$ and $\{\delta_n\}_{n\in\mathbb{N}}$ such that $\sigma(\mu_n) < s_n < 1$, $\delta_n \leqslant \mu_n$, $u_{s_n,\delta_n} \in \tilde{S}$, and considering a subsequence if necessary, $\lim\limits_{n\to\infty}\sigma(\mu_n) = \lim\limits_{n\to\infty} s_n = \bar{s} \leqslant 1$. By continuity of $u_{s,\delta}$ with respect to s, applying the convergence derived from (16) of $u_{s,\delta}$ with respect to δ yields $\lim\limits_{n\to\infty} u_{s_n,\delta_n} = u_{\bar{s}}$, and hence $u_{\bar{s}} \in \tilde{S}$, which contradicts the fact that $\tilde{d}(u_s, \tilde{S}) \geqslant \tilde{\alpha}$ for every $s \in [0,1]$.

Therefore, there exists $\mu > 0$ such that $u_{s,\delta} \notin \tilde{S}$ for all $\delta \in [0,\mu]$ and $s \in [0,1]$. The theorem is proven. ∎

Next, we characterize the discrepancy between the endpoint derived from the solution of the RPLE and the desired target for the motion planning in terms of the regularization parameter δ, answering positively the question (Q3).

Theorem 3.5: Consider a smooth curve $\gamma : [0,1] \to M$ and $u_0 \in L^2([0,T],\mathbb{R}^m)$ so that $\mathcal{E}(u_0) = \gamma(0)$. Assume that $\gamma(s) \notin S$ for every $s \in [0,1)$ and that the solution $s \mapsto u_s$ of the PLE (4) starting from u_0 along γ is well defined on $[0,1)$. Let $u_{s,\delta}$ be the solution of the corresponding RPLE (15) and let $x_1^\delta := \mathcal{E}(u_{s,\delta})|_{s=1}$. Then $\lim_{\delta\to 0} x_1^\delta = \gamma(1)$.

Proof: The assumption on the PLE means that, for every $\varepsilon \in (0,1)$, the solution u_s of (4) starting from u_0 along γ is defined over $s \in [0,\varepsilon]$. Then, $\mathcal{E}(u_s(t)) = \gamma(s)$ for $0 \leqslant s \leqslant \varepsilon$, and $s \mapsto \frac{\partial}{\partial s} u_s$ is integrable on $[0,\varepsilon]$, and thus locally integrable on $[0,1)$.

Denote by $\lambda_{\min}^{s,\delta}$ the minimal eigenvalue of the Gramian at $u_{s,\delta}$. Since $\lambda_{\min}^{s,\delta} = \|G_{u_{s,\delta}}^{-1}\|$ is continuous with respect to $G_{u_{s,\delta}}$ and $G_{u_{s,\delta}}$ is continuous with respect to $u_{s,\delta}$, it follows that $\lambda_{\min}^{s,\delta}$ is continuous with respect to s.

By Theorem 3.4, there exists a constant $\mu_0 > 0$, such that $u_{s,\delta} \notin \tilde{S}$, $\lambda_{\min}^{s,\delta} > 0$, $\forall s \in [0,1)$, $\forall \delta \in [0,\mu_0]$. Therefore $\frac{\delta}{\lambda_{\min}^{s,\delta}+\delta} \leqslant 1$ uniformly with respect to $(s,\delta) \in [0,\varepsilon]\times[0,\mu_0]$. By the dominated convergence theorem, we get

$$\begin{aligned}
&\lim_{\delta\to 0} \|\gamma(1) - x_1^\delta\| \\
&= \lim_{\delta\to 0}\lim_{\varepsilon\to 1} \| \int_0^\varepsilon \dot{\gamma}(s) - \frac{\partial}{\partial s}\mathcal{E}(u_{s,\delta})\mathrm{d}s\| \\
&\leqslant \lim_{\delta\to 0}\lim_{\varepsilon\to 1} \int_0^\varepsilon \|I_n - \mathrm{d}\mathcal{E}(u_{s,\delta})P_\delta(u_{s,\delta})\|\|\dot{\gamma}(s)\|\mathrm{d}s \\
&\leqslant \lim_{\varepsilon\to 1} \int_0^\varepsilon \lim_{\delta\to 0} \frac{n\delta}{\lambda_{\min}^{s,\delta}+\delta}\|\dot{\gamma}(s)\|\mathrm{d}s = 0
\end{aligned}$$

which proves the theorem. ■

Remark 3.6: Theorem 3.5 shows that if we plan to find a control driving the system from x to x_1 while $x_1 \in S$ lies in the critical values corresponding to singular controls, then as long as the solution of the PLE (4) is well posed before the curve γ reaches x_1, the endpoint corresponding to regularized solutions will reach x_1 as we take the regularization parameter δ to zero. This confirms that the regularization improves the applicability of the HCM, since the un-regularized PLE (4) is not well defined when $\gamma(s)$ intersects with the set S, which forbids the lifting of paths.

IV. Numerical Implementation

We adopt the model of rolling surfaces in [2] to illustrate the applicability of the regularized continuation method.

The system of a two-dimensional strictly convex manifold embedded in $\mathbb{R}^3$ rolling against the two-dimensional Euclidean surface $\mathbb{R}^2$ is described by the following control-affine system

$$\begin{aligned}
&\dot{v}_1 = u_1\cos\psi - u_2\sin\psi; && \dot{v}_2 = u_1; \\
&\dot{w}_1 = -\tfrac{1}{B}u_1\sin\psi - \tfrac{1}{B}u_2\cos\psi; && \dot{w}_2 = u_2; \\
&\dot{\psi} = -\tfrac{B_{v_1}}{B}u_1\sin\psi - \tfrac{B_{v_2}}{B}u_2\cos\psi
\end{aligned} \qquad (17)$$

where the state space, a 5-dimensional manifold M, is locally characterized in the geodesic coordinates as $((v_1,w_1),(v_2,w_2),\psi) \in S^2 \times \mathbb{R}^2 \times \mathrm{SO}(2)$, where B is a function on M, and B_{v_1}, B_{v_2} are its partial derivative with respect to the components v_1, v_2 respectively.

The singular controls corresponding to the rolling system (17) is the set

$$S := \{(v(\cdot)\cos\theta, v(\cdot)\sin\theta) | v \in L^2([0,T],\mathbb{R}), \theta \in [0,2\pi]\},$$

and the corresponding critical values are all the geodesics starting from $x \in M$.

We first show that the regularized solution converges to the original one when the latter is well posed. The regularized continuation method is stated as Algorithm 1.

Algorithm 1 Regularized continuation method

(i) Discretize the control space by piecewise linear functions; calculate the endpoint map and its differential by the basis functions;
(ii) Choose a nonsingular initial control u_0 and calculate the endpoint $\mathcal{E}(u_0)$;
(iii) Draw a smooth curve γ on M such that $\gamma(0) = \mathcal{E}(u_0)$, $\gamma(1) = x_1$;
(iv) Choose a regularisation parameter δ, solve the RPLE (15) along γ with boundary condition u_0;
(v) Set $u_{1,\delta}$ as the terminal value of control.

To be precise, suppose that the planning aims at steering the system (17) from (v_1,w_1,v_2,w_2,ψ) to $(\tilde{v}_1,\tilde{w}_1,\tilde{v}_2,\tilde{w}_2,\tilde{\psi})$. Algorithm 1 is executed as follows:

In step (i), we first partition the interval $[0,T]$ into N pieces $[\frac{i}{N}T, \frac{i+1}{N}T]$, $i = 0,\cdots,N-1$, and discretize the control space with piecewise linear functions, which is a $2N$-dimensional linear space. Then we use the 4th-order Runge-Kutta method to integrate the system dynamics and compute the endpoint map as well as its differential.

In steps (ii) and (iii), we set the initial control u_0 as a nonlinear function on $\mathbb{R}^2$, so that it does not belong to the singular set $\tilde{S}$; suppose the coordinate of $\mathcal{E}(u_0)$ is $\bar{v}_1,\bar{w}_1,\bar{v}_2,\bar{w}_2,\bar{\psi}$, then we draw an arbitrary smooth curve $\gamma_0(s)$ on $\mathbb{R}^2$ starting from $(\bar{v}_1,\bar{w}_1)$ and ending at $(\tilde{v}_1,\tilde{w}_1)$, and project $\gamma_0(s)$ to S^2 to derive the curve $\gamma(s)$ on M to be lifted by continuation (for details of projecting a curve from $\mathbb{R}^2$ to S^2, one may refer to [2]).

In steps (iv) and (v), we use again the 4th-order Runge-Kutta method to solve the RPLE (15) along γ in the discretized control space, as well as the endpoint corresponding to the solution of the RPLE at $s = 1$, and compare it with the desired terminal state $(\tilde{v}_1,\tilde{w}_1,\tilde{v}_2,\tilde{w}_2,\tilde{\psi})$.

In our experiments, the convex manifold is S^2, the initial position and the desired terminal of rolling are chosen as $(0,0,-1,0,0)^\top$ and $(0,0,-1,2,0)^\top$ respectively, and the initial control is $u_0(t) = (-\pi\sin(\frac{\pi}{T}t), \pi\cos(\frac{\pi}{T}t))^\top$.

For any $\delta > 0$, denote by $d(\delta) := \|\mathcal{E}(u_1) - \mathcal{E}(u_{1,\delta})\|$, $e(\delta) := \|u_s - u_{s,\delta}\|$ the discrepancies of the controls and of the endpoints respectively. Table I shows their values calculated following Algorithm 1, corresponding to the PLE and the RPLE, with respect to different choices of the

δ	0.04	0.02	0.006	0.002
$d(\delta)$	0.8582	0.1379	0.0007	0.0001
$e(\delta)$	19.2728	6.6486	2.0186	0.6409

TABLE I: Discrepancies between the controls and endpoints corresponding to the PLE and RPLE.

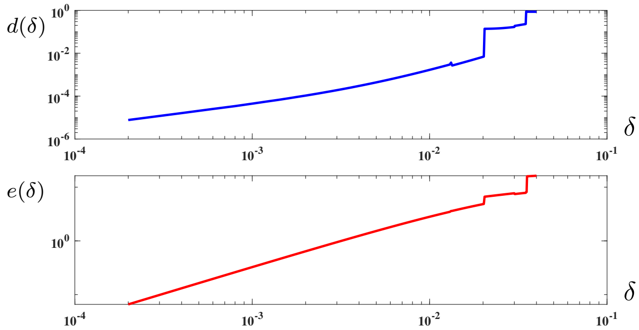

Fig. 2: log-log plot of discrepancies between the controls and endpoints corresponding to the PLE and RPLE.

parameter δ. From Table I we see that as the regularization parameter approaches zero, both the regularized control and the endpoint converge to the ones corresponding to the un-regularized PLE. Figure 2 is a log-log plot of the discrepancies, showing the convergence of the solution when δ is close to zero.

V. Conclusion

In this paper, we presented the regularized continuation method by deriving theoretical results on its effectiveness in motion planning, and showed how it overcomes the difficulty caused by singular controls, We illustrated our findings with some numerical experiments.

Note that Theorem 3.5 can be applied to motion planning when the whole trajectory to be lifted lies in the critical value set of the endpoint map. As long as $\tilde{u}_s := \lim_{\delta \to 0} u_{s,\delta}$ exists and the curve γ satisfies $\dot{\gamma}(s) \in \operatorname{Im}(\mathrm{d}\mathcal{E}(\tilde{u}_s))$, the endpoint derived from the solution of the RPLE converges to the desired target as δ tends to zero. For future investigations, we will focus on finding conditions ensuring well-posedness of the PLE.

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