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An internal model principle for open systems

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***Abstract*— The paper explores an extension of the classical internal model principle of Francis and Wonham to cases where the exogeneous signals (references or disturbances) are generated by an open, rather than closed (i.e., autonomous), exosystem. We study this extension both in the linear and nonlinear case, showing that the internal model principle is necessary for robust regulation of a contractive closed-loop system. While preliminary, our results motivate a generalization of nonlinear regulation theory to open exosystems.**

I. Introduction

The internal model principle is an important result of classical control theory and a pillar of regulation theory. Originally introduced by Francis and Wonham in the context of linear time-invariant systems [1], it states that every controller solving the problem of output regulation robustly with respect to uncertainties in the plant's dynamics necessarily embeds a suitable copy of the exosystem's model.

In its original formulation, the exosystem is a linear time-invariant autonomous system $\dot{x}_E = A_E x_E$. If A_E is marginally stable, such autonomous system can generate arbitrary linear combinations of sinusoids to model steady-state exogeneous references and disturbances. The internal model principle states that, to have the capacity of internally generating those sinusoids that span the exogenous signals, the controller state-space model must include a copy of the exosystem.

Regulation theory has received considerable attention over the last decades, with extensions of the LTI theory to nonlinear systems [2]–[11], time-varying systems [12], infinite-dimensional systems [13], and hybrid systems [14]. Remarkably, all those works have retained the assumption of an *autonomous* exosystem. Such assumption has been instrumental to ground regulation theory in (differential) geometric tools [2]–[6], [9], limit set characterizations of the steady-state behavior [7], [8], [10], and algebraic characterizations of the regulator [12], [13].

The present paper explores the possibility to include *open* exosystems in regulation theory. To the best of the authors' knowledge, this question has received little attention to date. Our motivation stems from the fact that open exosystems are natural in nonlinear control and that the assumption of an autonomous exosystem is restrictive even in quite simple physical examples of nonlinear regulation. One of such elementary examples is described in Section II. More generally, we expect that allowing for open exosystems might facilitate the connection to internal model principles in related questions such as network synchronization [15] and observer design [16].

The main result of this paper is to provide an extension of the original internal model principle of Francis and Wonham to open exosystems. For this preliminary version, we concentrate on the simple single-input single-output (SISO) framework. Our starting point is a reinterpretation of Francis and Wonham's internal model principle in the behavioral framework of Willems [17] and its reformulation in terms of a *robust factorization* property of the controller's steady-state behavior (Sections IV and V). This reformulation of the internal model principle is shown to extend with no difficulty to time-varying open linear systems (Section VI). Subsequently, it enables an extension of the internal model principle to nonlinear systems provided that the closed-loop system is required to be contractive (Section VII). This makes our approach akin to the methodology of the monograph [9], which also employs contraction for a nonlinear theory of regulation. While [9] only considers autonomous exosystems, the present paper aims at generalizing the approach to non-autonomous exosystems.

II. Motivating Example

Motivated by questions in neurophysiology [18], we consider a classical FitzHugh-Nagumo nonlinear circuit *perturbed* by a parallel port interconnection to a synaptic current source. The perturbed circuit has the state-space representation

$$\begin{aligned} \dot{x}_1 &= -\tfrac{1}{3}x_1^3 + x_1 - x_2 + u + d \\ \tau\dot{x}_2 &= -a x_2 + x_1 \end{aligned} \tag{1}$$

where x_1 denotes the output voltage, u the controlled current source, and d the synaptic current with state-space model

$$\begin{aligned} d &= g x_3 (x_1 - E_{syn}) \\ \dot{x}_3 &= -x_3 + h(v). \end{aligned} \tag{2}$$

The nonlinear conductance of the synaptic current depends on a measured presynaptic voltage v via the nonlinear function h. The constants τ, a, g, E_{syn} are the parameters of the circuit.

An elementary control problem is to design the current source u such that the synaptic current is asymptotically rejected from the FitzHugh-Nagumo circuit. We assume that the voltages x_1 and v are measured, but that the synaptic current d is not. The obvious solution to this problem is to

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model the synaptic current as

$$\begin{array}{rcl} \hat{d} & = & g x_c (x_1 - E_{syn}) \\ \dot{x}_c & = & -x_c + h(v) \end{array} \tag{3}$$

and to choose the control $u = -\hat{d}$.

We wish to interpret this design question in the classical framework of regulation theory: the model (2) is an *exosystem* that generates the unmeasured current d perturbing the nominal *plant* modeled by a FitzHugh-Nagumo circuit. System (3) is an observer of the disturbance model (2). Equivalently, it can be regarded as an internal model of the exosystem. In spite of the simplicity of the previous example, there are difficulties to conceive its solution as a straightforward application of the classical theory of nonlinear regulation. Most importantly, the exosystem (2) is non-autonomous, i.e., it is an *open* system with input v, whereas the exosystem of regulation theory is always an autonomous system. Also, the physical (i.e. port interconnection) coupling between the exosystem and the plant is bidirectional, that is, the exosystem depends on the variable x_1 of the plant.

In the rest of the paper, we concentrate on the first difficulty only, that is, considering an open exosystem. For the considered example, the nonlinear coupling between the exosystem and the plant is immaterial to the solution (3) since x_1 is a measured variable of the plant and it only affects the output map of (3). A more general dependence of the exosystem on measured variables of the plant is not considered in the rest of the paper, and is instead deferred to future research.

The internal model principe of linear regulation theory says that the feedback controller *must* contain an internal model of the exosystem if there are parametric uncertainties in the plant model. In the nonlinear example above, those correspond to small variations of the parameters τ and a. The main result of the present paper is to show under what conditions this principle remains valid for nonlinear open systems whenever the closed-loop system is required to be contractive. Contraction of the closed-loop system means that the effect of initial conditions decays exponentially along any solution [19].

III. Notations and Preliminary Notions

$\mathcal{C}^k(\mathbb{R}^n)$ denotes the set of k-times continuously differentiable functions $\mathbb{R} \to \mathbb{R}^n$, $\mathcal{C}^k_{\mathrm{b}}(\mathbb{R}^n)$ that of bounded functions in $\mathcal{C}^k(\mathbb{R}^n)$, and $\mathcal{C} := \mathcal{C}^0$. With $\mathcal{U}$, $\mathcal{V}$ linear spaces, $\mathcal{L}(\mathcal{U}, \mathcal{V})$ denotes the space of linear operators $\mathcal{U} \to \mathcal{V}$. Given $P \in \mathcal{L}(\mathcal{U}, \mathcal{V})$, we let P^{-1}, $\operatorname{dom} P$, $\ker P$ and $\operatorname{Im} P$ denote the preimage, domain, kernel and image of P, respectively. Let $\mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{Z}$ be linear spaces, $\mathcal{M}$ a topological space, $P \in \mathcal{L}(\mathcal{U}, \mathcal{V})$, and $\mu^\star \in \mathcal{M}$. We say that an operator $E \in \mathcal{L}(\mathcal{W}, \mathcal{Z})$ is a *robust factor of* P *at* $\mu^\star$ if there exist a family $R = \{R^\mu\}_{\mu \in \mathcal{M}}$ in $\mathcal{L}(\mathcal{Z}, \mathcal{V})$ and an $\mathcal{M}$-neighborhood M of $\mu^\star$ such that

$$\forall \mu \in M, \qquad \operatorname{Im} P \supseteq \operatorname{Im}(R^\mu E) \tag{4a}$$

and, for all $\mathcal{M}$-neighborhoods M' of $\mu^\star$ included in M,

$$\left(\bigcap_{\mu \in M'} \ker R^\mu \right) \cap \operatorname{Im} E = \{0\}. \tag{4b}$$

An important role in the paper is played by differential operators, defined as follows. With $\mathcal{M}$ a topological space, we let $\mathcal{P}(\mathcal{M})$ be the set of continuous functions $p : \mathcal{M} \times \mathbb{R}^2 \to \mathbb{R}$ of the form $p(\mu, t, x) = \sum_{k=0}^n \alpha_k(\mu, t) x^k$ for some $n \in \mathbb{N}$ independent on μ, t or x. We write $p^{\mu,t}(x)$ in place of $p(\mu, t, x)$ and omit μ and/or t when clear, unimportant, or when the coefficients $\alpha_k(\mu, t)$ are constant in μ and/or t. Given $k \in \mathbb{N}$, we denote by $s^k := \frac{d^k}{dt^k}$ the k-fold differentiation operator and we let $s := s^1$. Given a $p \in \mathcal{P}(\mathcal{M})$, we denote by $p^\mu(s)$ the operator $f \mapsto \sum_{k=0}^n \alpha_k(\mu, \cdot) s^k f(\cdot)$, and by $p(s)$ the map $(\mu, f) \mapsto p^\mu(s) f$.

IV. A Definition of Internal Model

This section formalizes the concept of *internal model* used throughout the paper. The proposed definition confers a formal meaning on statements of the kind "the controller must embed an internal model of the exosystem" stated in the various presented results.

Let $\mathcal{Z}$ and $\mathcal{W}$ be linear spaces and $\mathcal{M}$ a topological space. Fix $\mu^\star \in \mathcal{M}$, and consider two sets $\mathcal{B}_1 \subset \mathcal{Z} \times \mathcal{C}(\mathbb{R})$ and $\mathcal{B}_2 \subset \mathcal{W} \times \mathcal{C}(\mathbb{R})$ representing (subsets of) the behavior of two systems called Σ_1 and Σ_2, respectively. Suppose that all $(z, d) \in \mathcal{B}_1$ and $(w, u) \in \mathcal{B}_2$ satisfy equations of the form

$$d = A_1 z, \qquad\qquad u = A_2 w \tag{5}$$

in which $A_1 \in \mathcal{L}(\mathcal{Z}, \mathcal{C}(\mathbb{R}))$ and $A_2 \in \mathcal{L}(\mathcal{W}, \mathcal{C}(\mathbb{R}))$. The notion of "internal model" is formalized as follows.

Definition 1. Σ_2 *is said to* embed an internal model *of* Σ_1 *if* A_1 *is a robust factor of* A_2 *at* $\mu^\star$.

The meaning of Definition 1 is the following. If A_1 is a robust factor of A_2 at $\mu^\star$ then, by (4a), there exists a family $\{R^\mu\}_{\mu \in \mathcal{M}}$ in $\mathcal{L}(\mathcal{C}(\mathbb{R}), \mathcal{C}(\mathbb{R}))$ and an $\mathcal{M}$-neighborhood M of $\mu^\star$, such that

$$\forall \mu \in M, \ \forall z \in \operatorname{dom} A_1, \ \exists w \in \operatorname{dom} A_2, \ R^\mu A_1 z = A_2 w.$$

In turn, this implies that Σ_2 can generate all the outputs u produced by the behavior

$$\mathcal{B}_{21} := \{(z, u) \in \operatorname{dom} A_1 \times \mathcal{C}(\mathbb{R}) \mid u = R^\mu A_1 z, \ \mu \in M\}.$$

Throughout the paper, Σ_1 will represent the exosystem whereas Σ_2 will represent the steady-state behavior of the controller. Instead, the family $\{R^\mu\}_{\mu \in M}$ will model the effect of the plant's residual dynamics. Such dynamics is uncertain, and the uncertainty is lumped in the parameter μ, of which $\mu^\star$ is a nominal value. In these terms, the previous output-reproducing property can be read as the ability of the controller to generate all outputs of A_1 (representing the exosystem) filtered by R^μ (representing the uncertain plant). In particular, the operators R^μ generalize in our setting the blocking property of the zeros of the plant's residual dynamics (in particular, $d \in \ker R^\mu$ means that d is blocked by the plant). In turn, the second condition (4b) of robust factorization can be interpreted as a *nonresonance property* requiring that, for every output d of Σ_1 (the exosystem) and every neighborhood $M' \subseteq M$ of the nominal $\mu^\star$, there exists at least a perturbation $\mu \in M'$ such that d is not blocked by the plant. This implies that it takes the full behavior $\mathcal{B}_1$ of Σ_1 (the exosystem) to describe $\mathcal{B}_{21}$. Therefore, Definition 1 ultimately asks that system Σ_2 (the controller) is able to generate all the outputs of the behavior $\mathcal{B}_{21}$, and such behavior needs the full model A_1 of Σ_1 (the exosystem) to be defined.

V. Internal Model Principle for Linear Systems

A. The Output Regulation Framework

This section introduces the basic output regulation framework used throughout the paper to deal with linear systems. For the sake of illustration, we restrict our attention to the SISO case. We consider a "plant" described by the state-space model

$$\dot{x}_G(t) = A_G^\mu(t)x_G(t) + B_G^\mu(t)u(t) + P_G^\mu(t)d(t) \tag{6a}$$

$$e(t) = C_G^\mu(t)x_G(t) + Q_G^\mu(t)d(t), \tag{6b}$$

where $\mu \in \mathcal{M}$ is an uncertain parameter belonging to a finite-dimensional topological vector space $\mathcal{M}$, $x_G(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}$ is a control input, $e(t) \in \mathbb{R}$ is the regulation error to be controlled to zero, and $d : \mathbb{R} \to \mathbb{R}$ is a bounded exogenous input generated by a steady-state solution of the *open* exosystem

$$\dot{x}_E(t) = A_E(t)x_E(t) + F_E(t)v(t) \tag{7a}$$

$$d(t) = C_E(t)x_E(t) + L_E(t)v(t), \tag{7b}$$

in which $v \in \mathcal{V} \subseteq \mathcal{C}_{\rm b}(\mathbb{R})$ is a *measured* open input available for feedback. In particular, $P_G^\mu(t)d(t)$ represents a disturbance term acting on the plant's dynamics, and $-Q_G^\mu(t)d(t)$ a reference signal to be tracked by the plant's output $C_G^\mu(t)x_G(t)$. The regulation error $e(t)$ is indeed the difference between $C_G^\mu(t)x_G(t)$ and the reference $-Q_G^\mu(t)d(t)$. In the following, we shall only consider the steady-state behavior of (7), i.e., the set of all input-output pairs of (7) that are bounded backward and forward in time. Nonemptyness of such set can be seen as a "non-resonance" condition between v and the dynamics of x_E, and poses an additional implicit constraint on the class $\mathcal{V}$.

The controller is a state-space model described by

$$\dot{x}_C(t) = A_C(t)x_C(t) + B_C(t)e(t) + F_C(t)v(t) \tag{8a}$$

$$u(t) = C_C(t)x_C(t) + D_C(t)e(t) + L_C(t)v(t). \tag{8b}$$

The output regulation problem consists in designing the controller (8) to achieve *robust* asymptotic regulation of the error e for every reference and every disturbance generated by the exosystem. Robustness is meant with respect to small perturbations of the uncertain parameter μ. Formally, let $\mu^\star$ denote the nominal value used to tune the controller (8). In general, we say that a property $\mathbb{P}$ of the closed-loop system (6), (7), (8) holds *robustly* if $\mathbb{P}$ holds for every choice of the uncertain parameter μ in an $\mathcal{M}$-neighborhood of $\mu^\star$ with the same controller and exosystem [20]. Asymptotic regulation, for instance, corresponds to the property $\mathbb{P}$ defined by $\lim_{t\to\infty} e(t) = 0$.

B. Francis and Wonham's Internal Model Principle

We briefly recall the internal model principle of [1] within the framework presented in the previous section. In the setting of [1], $v = 0$ in (7) and (8). Namely, the exosystem is autonomous and the controller is only driven by the regulation error. Moreover, all matrices are time-independent, and $\mathcal{M} := \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{1\times n} \times \mathbb{R}$ denotes the space of the plant's matrices with the Euclidean topology; namely, the uncertain parameter $\mu = (A_G, B_G, P_G, C_G, Q_G)$ contains all the entries defining the plant's matrices. Finally, we assume A_E marginally stable. All these conditions will be assumed throughout this section with no further mention.

In this autonomous case, Francis and Wonham's internal model principle can be stated as follows.

Theorem 1 (adapted from [1, Thm. 2])**.** *Suppose that:*

R1. *The closed-loop system* (6), (8) *with* $d = 0$ *is asymptotically stable.*

R2. *The property* $\lim_{t\to 0} e(t) = 0$ *is robust for the closed-loop system* (6), (7), (8).

Then, the minimal polynomial of A_E divides at least one invariant factor of A_C.

The statement of the theorem, that the minimal polynomial of A_E divides at least one invariant factor of A_C, is the formal meaning that [1] gives to the statement "the controller embeds an internal model of the exosystem". Indeed, it implies that every eigenvalue of A_E is also an eigenvalue of A_C. We shall see in next Section V-C, that it is not clear how this notion can be extended to time-varying and open systems, whereas the notion proposed in previous Section IV can (showing this will be the aim of Sections VI and VII).

Moreover, in view of a generalization to nonlinear systems, we notice that the closed-loop stability requirement **R1** is equivalent to

P1. The system (6), (8) is contractive.

In the following, we call a trajectory that is defined and bounded on $\mathbb{R}$ a *steady-state trajectory*. Contraction of the closed-loop system implies that each steady-state trajectory of the exosystem defines a unique steady-state trajectory $(x_G^{\rm ss}, x_C^{\rm ss})$ of the system (6), (8) and, hence, a unique steady-state trajectory $e^{\rm ss}$ for the closed-loop regulation error. Under the assumption of contraction, the requirement **R2** is then equivalently reformulated as follows:

P2. The property $e^{\rm ss} = 0$ is robust for the closed-loop system (6), (7), (8).

The equivalent formulation **P1** and **P2** decouples the objective of closed-loop stability from the regulation objective of steady-state performance. The reader will note that for a marginally stable exosystem, any initial condition uniquely determines one steady-state trajectory.

C. Francis and Wonham's Internal Model Principle and Robust Factorization: A Behavioral Perspective

In the same closed and time-invariant setting of previous Section V-B, alternatively to the state-space representation (6), (7), (8), we can adopt an "input-output" representation of the exosystem, plant, and controller (recall that s denotes the differential operator, see Section III):

$$p_E(s)d = 0 \tag{9a}$$
$$p_G^\mu(s)e = q_G^\mu(s)u + n_G^\mu(s)d \tag{9b}$$
$$p_C(s)u = q_C(s)e \tag{9c}$$

In the behavioral language of Willems [17], these polynomial representations are *kernel representations* [17, Ch. 6] of the exosystem behavior $\mathcal{B}_E$, the plant behavior $\mathcal{B}_G$, and the controller behavior $\mathcal{B}_C$, respectively. Each behavior is a set of trajectories. The *steady-state* behavior is the subset of the behavior containing all and only steady-state trajectories.

We can obtain from (9) an equivalent result to Theorem 1 as follows. First, owing to **P2**, for all μ in a suitable $\mathcal{M}$-neighborhood M of $\mu^\star$, we can set $e = 0$ in (9), obtaining the existence of u such that

$$0 = q_G^\mu(s)u + n_G^\mu(s)d \tag{10a}$$
$$0 = p_C(s)u. \tag{10b}$$

Multiplying (10a) by $p_C(s)$, and using the fact that $p_C(s)$, $q_G^\mu(s)$, and $n_G^\mu(s)$ commute, from (10b) we obtain

$$\forall \mu \in M, \quad n_G^\mu(s)p_C(s)d = 0.$$

As M is open, we then obtain

$$p_C(s)d = 0. \tag{11}$$

Namely, every steady-state disturbance is a solution of the controller's equation when $e = 0$. This, in turn, is the behavioral equivalent of the statement "the minimal polynomial of A_E divides at least one invariant factor of A_C" of Theorem 1.

However, we notice that, to pass from (10) to (11), we need $p_C(s)$, $q_G^\mu(s)$, and $n_G^\mu(s)$ to commute, which is only true, in general, under the assumption of time invariance. Hence, for time-varying linear systems (11) cannot be obtained any more. In addition, in the open case where the steady-state exosystem's and controller's equations read

$$p_E(s)d = r_E(s)v, \qquad p_C(s)u = r_C(s)v,$$

for some polynomial operators $r_E(s)$ and $r_C(s)$, proving that every exosystem pair (v, d) is also a solution pair to the steady-state controller's with $e = 0$ means to prove

$$p_C(s)d = r_C(s)v. \tag{12}$$

However, it is unclear how (12) could be proved from (10) outside the simple case where d and u are matched in the plant's equation. Ultimately, this discussion reveals that Francis and Wonham's internal model principle, if it is meant as (11), admits no trivial extension to time-varying and open cases. Nevertheless, we may proceed differently.

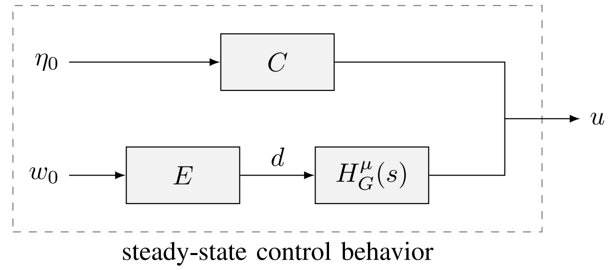

Fig. 1. The internal model principle as a robust factorization of the controller. The controller must include the mapping from w_0 to d, which is an image representation of the exosystem's steady-state behavior.

Instead of (9a), we can model the exosystem by means of the equivalent *image representation* [17, Ch. 6]

$$d = Ew_0 \tag{13}$$

in which $w_0 \in \mathbb{R}^{n_E}$ ($n_E \in \mathbb{N}$) represents the exosystem's initial conditions and $E \in \mathcal{L}(\mathbb{R}^{n_E}, \mathcal{C}_\mathrm{b}^\infty(\mathbb{R}))$. Then, for every $\mu \in M$, by letting $e = 0$ in (9), we obtain

$$u = H_G^\mu(s)Ew_0, \tag{14a}$$
$$0 = p_C(s)u, \tag{14b}$$

in which $H_G^\mu(s) := -q_G^\mu(s)^{-1}n_G^\mu(s)$ depends on the plant's data. Equation (14b) is a kernel representation of the controller's steady-state behavior, admitting the equivalent image representation

$$u = C\eta_0, \tag{15}$$

in which $\eta_0 \in \mathbb{R}^{n_C}$ ($n_C \in \mathbb{N}$) represents the controller's initial conditions and $C \in \mathcal{L}(\mathbb{R}^{n_C}, \mathcal{C}_\mathrm{b}^\infty(\mathbb{R}))$. From (14a) and (15), we then obtain that, for every $\mu \in M$, and every regulating control signal $u \in \mathrm{Im}(H_G^\mu(s)E)$, there exists a reproducible control $u' \in \mathrm{Im}(C)$ such that $u = u'$ (see Figure 1). In other terms, we obtain

$$\forall \mu \in M, \quad \mathrm{Im}\, C \supseteq \mathrm{Im}(H_G^\mu(s)E). \tag{16}$$

Moreover, by construction, for every open $M' \subseteq M$, we have

$$\left(\bigcap_{\mu \in M'} \ker H_G^\mu(s)\right) \cap \mathrm{Im}\, E = \{0\}. \tag{17}$$

From (16) and (17) we thus obtain that, in the terminology of Section III, E is a robust factor of C at $\mu^\star$. Therefore, we have proved the following version of the internal model principle, in which "embedding an internal model" is now formally expressed in terms of robust factorization as detailed in Section IV (see, in particular, Definition 1).

Theorem 2 (Internal Model Principle)**.** *Suppose that **P1** and **P2** hold. Then, the controller embeds an internal model of the exosystem, that is, E is a robust factor of C at $\mu^\star$.*

In the next sections, we show that this notion of internal model along with the internal model principle of Theorem 2 extends to open time-varying systems.

VI. Internal Model Principle for Time-Varying Open Systems

This section extends the internal model principle of Theorem 2 to time-varying open linear systems described by the equations (6), (7), (8) introduced in Section V-A. We refer to [21] for a behavioral theory of time-varying linear systems. We suppose that all matrices are bounded and continuously depend on the time t, and that the plant's matrices A_G, B_G, P_G, C_G, Q_G also continuously depend on μ.

In the Francis and Wonham's setting of Section V-B, the exosystem's behavior $\mathcal{B}_E$ was autonomous. Since A_E was assumed marginally stable, the exosystem's steady-state behavior $\mathcal{B}_E^{\rm ss}$ was the behavior itself, i.e., $\mathcal{B}_E = \mathcal{B}_E^{\rm ss}$. In this open case, instead, the exosystem's behavior is a set of input-output pairs (v, d) solving (7), and A_E needs not be marginally stable. In general, however, the exosystem's steady-state behavior will be a subset of the full behavior.

In this context, **P1** and **P2** can be restated as

O1. System (6), (8) is contractive.

O2. The property $e^{\rm ss} = 0$ is robust for the closed-loop system (6), (7), (8).

Let us fix arbitrarily an initial time t_0. In view of **O2**, there exists an $\mathcal{M}$-neighborhood M of $\mu^\star$ on which $e^{\rm ss} = 0$ holds. Along the lines of Section V-C, for each $\mu \in M$, we represent the steady-state closed-loop behavior, obtained by setting $e = 0$, through the image representation

$$d = E(w_0, v) \tag{18a}$$

$$0 = G_1^\mu u + G_2^\mu d, \tag{18b}$$

$$u = C(\eta_0, v) \tag{18c}$$

where now $E \in \mathcal{L}(\mathbb{R}^{n_E} \times \mathcal{C}_{\rm b}(\mathbb{R}), \mathcal{C}_{\rm b}(\mathbb{R}))$, $C \in \mathcal{L}(\mathbb{R}^{n_C} \times \mathcal{C}_{\rm b}(\mathbb{R}), \mathcal{C}_{\rm b}(\mathbb{R}))$, and $G_1^\mu, G_2^\mu \in \mathcal{L}(\mathcal{C}_{\rm b}(\mathbb{R}), \mathcal{C}_{\rm b}(\mathbb{R}))$. In (18), w_0 and η_0 represent initial conditions at time t_0. We observe that, while before the steady-state disturbances d and control inputs u were only parameterized by the exosystem's initial conditions, they are now parameterized by both initial conditions and the external signal v. We also observe that w_0, which represents the "past", is not measured, and that v, which is measured, needs only to be known from t_0 on.

From (18), we thus obtain the following two equations paralleling (14a) and (15) and describing the steady-state control behavior

$$u = H_G^\mu E(w_0, v), \qquad u = C(\eta_0, v),$$

where $H_G^\mu := -(G_1^\mu)^{-1} G_2^\mu$.

We assume that, as in the previous time-invariant case, the topology on $\mathcal{M}$ is such that

$$\left(\bigcap_{\mu \in M'} \ker H_G^\mu \right) \cap \operatorname{Im} E = \{0\} \tag{19}$$

for every $\mathcal{M}$-neighborhood $M' \subseteq M$ of $\mu^\star$. Then, by means of the same arguments of Section V-C, we obtain the following extension of the internal model principle of Theorem 2.

Theorem 3. *Suppose that **O1** and **O2** hold. Then, the controller embeds an internal model of the exosystem, that is, E is a robust factor of C at $\mu^\star$.*

VII. An Internal Model Principle for Nonlinear Contractive Systems

In this section, we extend Theorem 3 to a class of SISO nonlinear systems described by the following equations

$$\dot{x}_E = f_E(x_E, v), \qquad d = h_E(x_E, v), \tag{20}$$

$$\dot{x}_G = f_G^\mu(x_G, u, d), \qquad e = h_G^\mu(x_G, d), \tag{21}$$

in which $v \in \mathcal{V} \subseteq \mathcal{C}_{\rm b}(\mathbb{R})$, $\mu \in \mathcal{M}$, $f_E : \mathbb{R}^{n_E} \times \mathbb{R} \to \mathbb{R}^{n_E}$ and $h_E : \mathbb{R}^{n_E} \times \mathbb{R}$ are smooth functions, and $f_G^\mu : \mathbb{R}^{n_G} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n_G}$ and $h_G^\mu : \mathbb{R}^{n_G} \times \mathbb{R} \to \mathbb{R}$ are smooth for each $\mu \in \mathcal{M}$ and continuous in μ. We assume that to each steady-state input-output pair (v, d) of the exosystem there corresponds a steady-state trajectory $x_E^{\rm ss}$ satisfying (20).

We suppose that a controller of the form

$$\dot{x}_C = f_C(x_C, e, v), \qquad u = h_C(x_C, e, v) \tag{22}$$

has been designed for a nominal value $\mu^\star$ to ensure the following two properties:

N1. The system (21), (22) is contractive for every steady-state input-output pair (v, d) of the exosystem (20).

N2. The property $e^{\rm ss} = 0$ is robust for the closed-loop system (20), (21), (22).

Conditions **N1** and **N2** are the nonlinear counterparts of **O1** and **O2**. Let M be a neighborhood of $\mu^\star$ for which **N1** and $e^{\rm ss} = 0$ hold at each $\mu \in M$, and pick $\mu \in M$. For each steady-state input-output pair (v, d) of (20), contraction of (21), (22) implies convergence of its solutions to a unique steady-state trajectory $(x_G^{\rm ss}, x_C^{\rm ss})$. **N2** further implies regulation of the steady-state trajectory, that is $e^{\rm ss} = 0$.

Given steady-state input-output pair (v, d) of (20), let $x_E^{\rm ss}$ be the corresponding steady-state trajectory of the exosystem's state, and let

$$\begin{aligned} \dot{\delta x}_E(t) &= A_E(t)\delta x_E(t) + F_E(t)\delta v(t) \\ \delta d(t) &= C_E(t)\delta x_E(t) + L_E(t)\delta v(t) \end{aligned} \tag{23a}$$

$$\begin{aligned} \dot{\delta x}_G(t) &= A_G^\mu(t)\delta x_G(t) + B_G^\mu(t)\delta u(t) + P_G^\mu(t)\delta d(t) \\ \delta e(t) &= C_G^\mu(t)\delta x_G(t) + Q_G^\mu(t)\delta d(t) \end{aligned} \tag{23b}$$

$$\begin{aligned} \dot{\delta x}_C(t) &= A_C(t)\delta x_C(t) + B_C(t)\delta e(t) + F_C(t)\delta v(t) \\ \delta u(t) &= C_C(t)\delta x_C(t) + D_C(t)\delta e(t) + L_C(t)\delta v(t) \end{aligned} \tag{23c}$$

be the variational system obtained by linearizing (20), (21), (22) around the motion $(v, x_E^{\rm ss}, x_G^{\rm ss}, x_C^{\rm ss})$. We call $\delta\mathcal{B}_\mu(v, d)$ the behavior of (23), and we notice that requirement **N1** implies that (23b), (23c) is contractive for each steady-state input-output pair $(\delta v, \delta d)$ of (23a) [22].

We consider a subset $\widehat{\delta\mathcal{B}}_\mu^{\rm ss}(v, d) \subseteq \delta\mathcal{B}_\mu(v, d)$ of the steady-state behavior of the linearized system (23) defined by the constraint $\delta e = 0$, and we let $\mathcal{I}_\mu(v, d)$ denote the set of all steady-state input-output pairs $(\delta v, \delta d)$ associated with $\widehat{\delta\mathcal{B}}_\mu^{\rm ss}(v, d)$ (namely, leading to $\delta e = 0$ in (23)). Finally, we let $\mathcal{I}(v, d) := \cap_{\mu \in M} \mathcal{I}_\mu(v, d)$ and, for each $\mu \in M$, we let

$\widetilde{\delta\mathcal{B}}_{\mu}^{\mathrm{ss}}(v,d) \subseteq \widehat{\delta\mathcal{B}}_{\mu}^{\mathrm{ss}}(v,d)$ denote the steady-state behavior of the linearized system (23) restricted to $\mathcal{I}(v,d)$ (namely, we obtain $\widetilde{\delta\mathcal{B}}_{\mu}^{\mathrm{ss}}(v,d)$ from $\widehat{\delta\mathcal{B}}_{\mu}^{\mathrm{ss}}(v,d)$ by excluding all trajectories for which $(\delta v, \delta d) \notin \mathcal{I}(v,d)$). By construction, $\delta e = 0$ is then a robust property for the behavior $\widetilde{\delta\mathcal{B}}_{\mu}^{\mathrm{ss}}(v,d)$ in the sense that it holds for all μ in a neighborhood of $\mu^\star$. Then, from **N1** and **N2**, we obtain:

δ**N1.** The system (23b), (23c) is contractive.

δ**N2.** The property $\delta e = 0$ is robust for the behavior $\widetilde{\delta\mathcal{B}}_{\mu}^{\mathrm{ss}}(v,d)$.

As in Section VI, we fix an initial time t_0. Then, for every $\mu \in M$, we can describe the steady-state behavior $\widetilde{\delta\mathcal{B}}_{\mu}^{\mathrm{ss}}(v,d)$ through the image representation (cf. (18))

$$\delta d = E(\delta w_0, \delta v), \tag{24a}$$

$$\delta u = H_G^{\mu} \delta d, \tag{24b}$$

$$\delta u = C(\delta \eta_0, \delta v), \tag{24c}$$

in which E, H_G^{μ}, and C are linear operators defined as in Section VI but whose domain is suitably restricted so as (24) only describes $\widetilde{\delta\mathcal{B}}_{\mu}^{\mathrm{ss}}(v,d)$. As before, let us consider a topology on $\mathcal{M}$ such that

$$\left(\bigcap_{\mu \in M'} \ker H_G^{\mu}\right) \cap \operatorname{Im} E = \{0\} \tag{25}$$

for every $\mathcal{M}$-neighborhood $M' \subseteq M$ of $\mu^\star$. Then, by means of the same arguments of Section VI, we can use Theorem 3 to establish the following nonlinear version of the internal model principle.

Theorem 4. *Suppose that **N1** and **N2** hold. Then, the controller* (22) *embeds an internal model of the exosystem, that is, E is a robust factor of C at $\mu^\star$.*

Theorem 4 states that the linearization of the controller around each steady-state trajectory of the regulated system must include a copy of the linearized exosystem. This differential constraint on the controller along regulated trajectories provides a nonlinear version of the internal model principle including cases where the exosystem is open.

VIII. Discussion and Conclusions

Since Francis and Wonham's landmark paper, the internal model principle of control theory has always assumed an autonomous exosystem. This paper examined the extension to the case in which the exosystem is an open system with measured input. The reformulation of the internal model principle in the form of robust factorization was chosen because it extends without difficulties to open time-varying systems, and thereby also to contractive nonlinear systems.

We emphasize that robustness plays a crucial role in the necessity of the internal model principle. This is in line with the original formulation of Francis and Wonham's result, but departs from the subsequent nonlinear literature (e.g., [8], [10], [11]) that focuses on a weaker steady-state reproducing property, that is, the ability of the controller to generate the steady-state solution of the closed-loop system.

This paper is only a first step in the direction of a regulation theory for open systems. Future research will investigate the extension of Theorem 4 to the MIMO case and, starting from the above observations, constructive design techniques based on open internal models. A regulation theory for open systems should facilitate the interpretation of the internal model principle as a general design principle for feedback regulation, synchronization, and estimation.

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