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Open-loop and feedback LQ potential differential games for Multi-Agent Systems

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***Abstract*— Open-loop and feedback potential differential games for multi-agent systems are considered in this paper. Constructive sufficient conditions under which a linear quadratic differential game constitutes a potential differential game are provided. The conditions enable the construction of associated optimal control problems that yield (at significantly reduced computational complexity) solutions (in terms of open-loop and feedback Nash equilibrium strategies) of the original differential game. The results are demonstrated on a practically-motivated example that concerns spacecraft formation control.**

I. Introduction

Differential games provide a framework to model and analyse how multiple players interact within a dynamic environment, where each player is associated with an individual objective to be optimised while accounting for the actions of other players. Such problems play an important role in control engineering (*e.g.* in the context of robust control [1], [2], [3], power systems [4] and robotics [5], [6]). Notably, game theory has gained significant interest in the context of Multi-Agent Systems (MAS), *i.e.* systems consisting of several *agents* that together are able to perform *complex* tasks (see *e.g.* [7], [8], [9], [10], [11] and references therein).

Considering differential games, Nash equilibrium solutions are among the most commonly used solution concepts and can be classified into two types: open-loop Nash equilibrium solutions and feedback Nash equilibrium solutions (see *e.g.* [12]). However, obtaining such solutions is, in general, a challenging task. Thus, inspired by potential games arising in the context of static games, the notion of *potential differential games* (see [13], [14], [15] and references therein) has gained interest recently. Loosely speaking, potential differential games are differential games that can be associated with an optimal control problem whose solution yields a Nash equilibrium of the original differential game. This notion is an extension, to the dynamic setting, of the notion of *static potential games* which has been derived, as the name suggests, for static games (see [16], [17], [18]). The main motivation behind characterising potential differential games is that an associated optimal control problem is generally easier to solve than the original N-player differential game. This becomes immediately apparent even in the linear quadratic (LQ) case where differential games involve solving N coupled asymmetric/symmetric algebraic Riccati equations (AREs), whereas the solution of an LQ optimal control counterpart, *i.e.* the linear quadratic regulator (LQR) problem, involves the solution of a single ARE, for which many efficient algorithms are available. However, the available literature on the topic of potential *differential* games is somewhat scarce, with just a handful of results available (compared to their static counterpart). While some definitions and results are in place in the context of *open-loop* Nash equilibrium solutions (see *e.g.* [13], [14], [19], [20]), very few results are available for feedback Nash equilibrium solutions (such as [21] that considers discrete-time dynamic games).In this paper, we focus on MAS described by linear systems where each agent is associated with a quadratic cost functional. We provide a definition of potential differential game both in the context of open-loop and feedback Nash equilibrium solution, and we provide sufficient conditions under which a differential game constitutes a potential differential game. The provided conditions are constructive in the sense that they outline how to design an LQR problem whose solution yields a solution of the original differential game in the sense of open-loop/feedback Nash equilibrium solution. Thus, for a given LQ differential game, such LQR problem is referred to as an *associated* LQR problem. The remainder of this paper is organised as follows. Some preliminaries on differential games, specific to the setting of MAS are provided in Section II. The main result of this paper, namely *constructive* sufficient conditions under which the original differential game constitutes a potential (open-loop/feedback) differential game are presented in Section III for open-loop Nash equilibrium (OL-NE) strategies, and in Section IV for feedback Nash equilibrium (F-NE) strategies. A practically motivated example is presented in Section V, demonstrating the efficacy of the proposed results, before some concluding remarks are provided in Section VI.

Notation. Given a matrix $A \in \mathbb{R}^{n\times n}$, $A > 0$ ($A \geq 0$) indicates that the matrix is positive definite (positive semi-definite). Given a vector $v \in \mathbb{R}^n$, $||v||$ denotes its Euclidean norm. blkdiag$\{A_1, \dots, A_N\}$ denotes a block diagonal matrix with diagonal blocks $A_1, \dots, A_N$. A block matrix M is denoted by $M = [M^{jk}]$, where M^{jk} is the $j-$th row, k-th column block of M. Similarly, given a matrix $M = [M^{j\cdot}]$, $M^{j\cdot}$ denotes the j-th row block of M.

II. Preliminaries on the differential game representation of a multi-agent system

In this section we recall some preliminaries on differential games in the context of MAS. Namely consider a MAS,

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described by the dynamics

$$\dot{x}_i = A_i x_i + B_i u_i \,, \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$ correspond, respectively, to the state and control input of agent i and where $A_i \in \mathbb{R}^{n_i \times n_i}$ and $B_i \in \mathbb{R}^{n_i \times m_i}$, for $i = 1, \ldots, N$. We refer to the overall system of N agents (1), $i = 1, \ldots, N$, as the the *global system*, which is described by the dynamics

$$\dot{x} = Ax + \sum_{i=1}^{N} \bar{B}_i u_i \,, \tag{2}$$

where $x = [x_1^\top, \ldots, x_N^\top]^\top \in \mathbb{R}^n$ is the state of the global system, $n = \sum_{i=1}^{N} n_i$, $A = \text{blkdiag}\{A_1, \ldots, A_N\}$ and

$$\bar{B}_1 = [B_1^\top, \ldots, 0]^\top \,, \quad \ldots \,, \quad \bar{B}_N = [0, \ldots, B_N^\top]^\top \,.$$

Let x_0 denote the initial condition of the global system, *i.e.* $x(0) = x_0$. In what follows, we refer to u_i as the *strategy* of agent i and we let u_{-i} indicate the set of strategies of all agents *excluding* agent i (*i.e.* $u_{-i} = \{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N\}$), $i = 1, \ldots, N$.

Consider the case in which each agent i is associated with a quadratic cost functional given by

$$J_i(u_i, u_{-i}) = \frac{1}{2} \int_0^\infty x^\top Q_i x + u_i^\top R_i u_i dt \,, \tag{3}$$

where $R_i = R_i^\top > 0$, $Q_i = Q_i^\top$. Moreover, consider the matrix partition $Q_i = [Q_i^{jk}]$, where $Q_i^{jk} \in \mathbb{R}^{n_j \times n_k}$, for $j = 1, \ldots, N$ and $k = 1, \ldots, N$, and let $Q_i^{j\cdot} \in \mathbb{R}^{n_j \times n}$ denote the j-th row block of Q_i, for $j = 1, \ldots, N$. Loosely speaking, the objective of each agent (also referred to as *player* in the following) i consists in finding a control policy $u_i(\cdot)$ which minimises (3), subject to the actions of all other players. That is, the dynamics (2), together with cost functionals (3) for $i = 1, \ldots, N$, constitutes a N-player LQ, infinite horizon, non-cooperative differential game.

While different solution concepts exist for such differential games, we focus on Nash equilibrium solutions. These can, depending on the information available to each player, be further divided into *open-loop* strategies, and *feedback* strategies. In the former, each player seeks the "best" strategy amongst all feasible strategies that are functions of time and the initial state (*i.e.*, $u_i(\cdot) = u_i(t, x_0)$, $i = 1, \ldots, N$). In the latter, each player seeks the "best" strategy amongst all feasible feedback policies (*i.e.* $u_i(\cdot) = u_i(x)$). The distinction between OL-NE and F-NE strategies has been widely analysed in the literature (see *e.g.* [12], [22]). In what follows, to distinguish between the two solution concepts, we use asterisks (*i.e.* u_i^*, $i = 1, \ldots, N$) to indicate OL-NE strategies, and stars (*i.e.* $u_i^\star$, $i = 1, \ldots, N$) to indicate F-NE strategies. We recall the notion of OL-NE first.

Definition 1. *Consider the differential game defined by the dynamics* (2) *and the cost functionals* (3)*, for* $i = 1, \ldots, N$*. A set of strategies* $\{u_1^*(t, x_0), \ldots, u_N^*(t, x_0)\}$ *constitutes an OL-NE solution if*

$$J_i(u_i^*(\cdot), u_{-i}^*(\cdot)) \leq J_i(u_i(\cdot), u_{-i}^*(\cdot)) \,, \tag{4}$$

is satisfied for all $\{u_i(\cdot), u_{-i}^*(\cdot)\}$ *such that* $\lim_{t \to \infty} x(t) = 0$ *and* $J_i(u_i, u_{-i}^*) < \infty$*, for* $i = 1, \ldots, N$.

OL-NE solutions can be obtained using Pontryagin's Minimum Principle, considering the associated state-costate dynamics. In the context of LQ games, it is well-known (see *e.g.* [22, Chapter 7] and [12, Chapter 6]) that under certain conditions OL-NE solutions admit a feedback synthesis, *i.e.* the open-loop strategies $u_i^*(t, x_0)$ admit an equivalent representation in terms of a feedback control law. Namely, given a solution P_i, $i = 1, \ldots, N$, of the *asymmetric* AREs

$$P_i A + A^\top P_i + Q_i - P_i \left(\sum_{j=1}^{N} \bar{B}_j R_j^{-1} \bar{B}_j^\top P_j \right) = 0 \tag{5}$$

for $i = 1, \ldots, N$, which is such that $A_{\text{cl}}^* = (A - \sum_{i=1}^{N} \bar{B}_i R_i^{-1} \bar{B}_i^\top P_i) \in \mathbb{C}^-$, the set of strategies

$$u_i^*(x) = -R_i^{-1} \bar{B}_i^\top P_i x \,, \tag{6}$$

$i = 1, \ldots, N$, constitutes an OL-NE solution of the differential game. Moreover, the strategy (6), is referred to as the *feedback synthesis* of the corresponding OL-NE strategies $u_i^*(t, x_0)$, for $i = 1, \ldots, N$. In the remainder of this paper, we consider solely OL-NE strategies that admit a feedback synthesis.

The notion of F-NE solutions is recalled next.

Definition 2. *Consider the differential game defined by the dynamics* (2) *and the cost functionals* (3)*, for* $i = 1, \ldots, N$*. A set of strategies* $\{u_1^\star(x), \ldots, u_N^\star(x)\}$ *constitutes a F-NE solution if*

$$J_i(x_0, u_i^\star(x), u_{-i}^\star(x)) \leq J_i(x_0, u_i(x), u_{-i}^\star(x)) \tag{7}$$

is satisfied for all $\{u_i(x), u_{-i}^\star(x)\}$ *such that the origin of the system* (2) *in closed loop with* $\{u_i(x), u_{-i}^\star(x)\}$ *is (locally) asymptotically stable, for* $i = 1, \ldots, N$.

Considering linear feedback strategies only, it is well-known (see, *e.g.* [12], [22], [23]) that if one can obtain a solution $P_i = P_i^\top$ to the set of *coupled (symmetric) AREs*

$$\begin{aligned} &P_i A + A^\top P_i + Q_i + P_i \bar{B}_i R_i^{-1} \bar{B}_i^\top P_i \\ &- P_i \left(\sum_{j=1}^{N} \bar{B}_j R_j^{-1} \bar{B}_j^\top P_j \right) - \left(\sum_{j=1}^{N} P_j \bar{B}_j R_j^{-1} \bar{B}_j^\top \right) P_i = 0 \end{aligned} \tag{8}$$

for $i = 1, \ldots, N$, such that $A_{\text{cl}}^\star = (A - \sum_{i=1}^{N} \bar{B}_i R_i^{-1} \bar{B}_i^\top P_i) \in \mathbb{C}^-$, then the set of strategies

$$u_i^\star(x) = -R_i^{-1} \bar{B}_i^\top P_i x \,, \tag{9}$$

$i = 1, \ldots, N$, constitutes a F-NE solution of the game. Note that (5) and (8), for $i = 1, \ldots, N$, are not equivalent (even in terms of symmetry properties of their solutions). In the context of F-NE, it has been demonstrated (see [24], [25] for further details) that replacing (8) with the matrix inequalities

$$\begin{aligned} &P_i A + A^\top P_i + Q_i + P_i \bar{B}_i R_i^{-1} \bar{B}_i^\top P_i \\ &- P_i \left(\sum_{j=1}^{N} \bar{B}_j R_j^{-1} \bar{B}_j^\top P_j \right) - \left(\sum_{j=1}^{N} P_j \bar{B}_j R_j^{-1} \bar{B}_j^\top \right) P_i \leq 0 \end{aligned} \tag{10}$$

for $i = 1, \ldots, N$, results in a set of strategies (9), $i = 1, \ldots, N$, that consitutes a so-called ϵ_α-F-NE solution of the game. Namely, the set of strategies is such that

$$J_i(x_0, u_i^\star(x), u_{-i}^\star(x)) \leq J_i(x_0, u_i(x), u_{-i}^\star(x)) + \epsilon_\alpha \quad (11)$$

is satisfied for all $\{u_i(x), u_{-i}^\star(x)\}$ such that $(A_{\mathrm{cl}}^\star + I\alpha) \in \mathbb{C}^-$, for $i = 1, \ldots, N$, where ϵ_α is a non negative constant parametrised with respect to $\alpha > 0$, x_0.

Obtaining a solution of (5), (8) or (10), for $i = 1, \ldots, N$, for OL-NE, F-NE and ϵ_α-F-NE solutions, respectively, is, in general, computationally demanding, especially when the number of players N is large. Thus, the main objective of this paper is to provide a constructive approach to formulate optimal control problems, more precisely LQR problems, whose solutions are equivalent to an OL-NE or F-NE solution of the LQ differential game. If a differential game can be related to an optimal control problem (whose solution yields a Nash equilibrium of the original game) we refer to it as a *potential differential game*. Such problems have been considered in the context of OL-NE solutions in [13], [14], [15] and in [21] in the context of OL-NE and F-NE solutions of discrete-time dynamic games. We limit our attention to the class of differential games described by (2)-(3), $i = 1, \ldots, N$.

To streamline the presentation, consider the LQR problem defined by the dynamics

$$\dot{x}_{\mathrm{ocp}} = A x_{\mathrm{ocp}} + B_{\mathrm{ocp}} u_{\mathrm{ocp}} \,, \quad (12)$$

subject to the initial condition $x_{\mathrm{ocp}}(0) = x_0$, where $B_{\mathrm{ocp}} = [\bar{B}_1, \ldots, \bar{B}_N] \in \mathbb{R}^{n \times m}$, with $m = \sum_{i=1}^N m_i$, and $x_{\mathrm{ocp}} \in \mathbb{R}^n$ and $u_{\mathrm{ocp}} \in \mathbb{R}^m$ correspond to the state and control input, respectively, and by the cost functional

$$J_{\mathrm{ocp}} = \frac{1}{2} \int_0^\infty x_{\mathrm{ocp}}^\top Q_{\mathrm{ocp}} x_{\mathrm{ocp}} + u_{\mathrm{ocp}}^\top R_{\mathrm{ocp}} u_{\mathrm{ocp}} dt \quad (13)$$

with $Q_{\mathrm{ocp}} = Q_{\mathrm{ocp}}^\top \geq 0$ and $R_{\mathrm{ocp}} = R_{\mathrm{ocp}}^\top > 0$. Given a solution $P = P^\top > 0$ of the ARE

$$PA + A^\top P + Q_{\mathrm{ocp}} - P B_{\mathrm{ocp}} R_{\mathrm{ocp}}^{-1} B_{\mathrm{ocp}}^\top P = 0 \,, \quad (14)$$

the solution of the LQR problem (12)-(13) is given by

$$u_{\mathrm{ocp}}^\star = -R_{\mathrm{ocp}}^{-1} B_{\mathrm{ocp}}^\top P x_{\mathrm{ocp}} \,. \quad (15)$$

In what follows, let[1] $A_{\mathrm{cl}} = A - B_{\mathrm{ocp}} R_{\mathrm{ocp}}^{-1} B_{\mathrm{ocp}}^\top P$ denote the matrix describing the optimal closed-loop system. Finally, note that obtaining a solution of (14) is considerably easier than finding a solution of (5), (8) or (10), $i = 1, \ldots, N$. Motivated by this, in the following sections we provide constructive sufficient conditions under which the considered differential game is a potential differential game. OL-NE solutions and (exact/approximate) F-NE solutions are considered separately in the following sections. Loosely speaking, a differential game constitutes a potential (open-loop/feedback) differential game if it admits a (OL-NE/F-NE) solution that can be generated by an optimal control problem. From hereon, we refer to such optimal control problem as an *associated optimal control problem* (or an associated LQR problem, since we focus on the LQ setting).

[1] Note that A_{cl} denotes the matrix describing the system (2) in closed-loop with $[u_1^\top, u_2^\top, \ldots, u_N^\top]^\top = u_{\mathrm{ocp}}$, with the latter given in (15), whereas A_{cl}^* denotes the matrix describing the system in closed loop with the OL-NE strategies (6), $i = 1, \ldots, N$. Similarly, $A_{\mathrm{cl}}^\star$ denotes the matrix describing the system in closed loop with the F-NE strategies (9), $i = 1, \ldots, N$ (or, the ϵ_α-F-NE strategies characterised by the matrix inequalities (10) in place of (8), $i = 1, \ldots, N$).

III. Open-loop potential differential games

Consider the game (2)-(3) and its solutions in terms of OL-NE strategies. The notion of *open-loop potential differential games* has been previously introduced and considered in the literature (see *e.g.* [13], [14], [15]). In the following statement, we recall the definition of open-loop potential differential games as given in [13] (tailored to the MAS setting considered herein).

Definition 3. *The LQ differential game* (2)-(3)*, $i = 1, \ldots, N$, is an LQ open-loop potential differential game if there exists an optimal control problem* (12)-(13)*, such that its solution corresponds to an OL-NE solution of the original game,* i.e.

$$u_{\mathrm{ocp}}(x) = [u_1^*(t, x_0)^\top, \ldots, u_N^*(t, x_0)^\top]^\top \quad (16)$$

for all $t \geq 0$, where $\{u_1^*, \ldots, u_N^*\}$, denotes an OL-NE solution of the differential game.

In the following statement we provide conditions under which we can design Q_{ocp} and R_{ocp} in (13) such that the LQR problem yields (as detailed in Definition 3) a solution of the original game.

Theorem 1. *Consider the differential game defined by* (2), (3)*, for $i = 1, \ldots, N$. Suppose Q_i is such that*

$$\alpha_i Q_i^{ij} = \alpha_j Q_j^{ij} \triangleq \bar{Q}^{ij} \,, \quad (17)$$

for some $\alpha_i > 0$ and $\alpha_j > 0$, for all $j = 1, \ldots, N$ and $i = 1, \ldots, N$. Let

$$Q_{\mathrm{ocp}} = \left[\bar{Q}^{kj}\right] \,, \; R_{\mathrm{ocp}} = \mathrm{blkdiag}\{\alpha_1 R_1 \,, \ldots \,, \alpha_N R_N\} \,. \quad (18)$$

Suppose the ARE (14) *admits a solution $P = P^\top > 0$ and that the Sylvester equation*

$$A_j^\top X_{i,j} + X_{i,j} A_{\mathrm{cl}} + Q_i^j = 0 \,, \quad (19)$$

admits a solution, for $j = 1, \ldots, N$, $j \neq i$ and $i = 1, \ldots, N$. Then, the differential game (2)-(3) *is an open-loop potential differential game with an associated LQR problem defined by* (12) *and* (13)*.*

Remark 1. *Note that the condition* (19)*, for $j = 1, \ldots, N$ and $j \neq i$, for $i = 1, \ldots, N$, involves only the solution of the ARE* (14) *(associated with the LQR problem), via the matrix A_{cl}. Furthermore, a consequence of the result in Theorem 1 is that under this condition, which represents a set of (readily-solved) linear matrix equalities, an OL-NE solution of the LQ differential game can be obtained by solving an LQR problem (as detailed in the statement) via the relation* (16)*. Thus, the result provides a means to bypass the need to directly solve* (5)*, $i = 1, \ldots, N$, at any stage, which is computationally appealing. However, it* *is worth noting that the result also enables to construct a solution of* (5)*,* $i = 1, \dots, N$*, based solely on the solutions of* (19)*, for* $j = 1, \dots, N$ *and* $j \neq i$*, for* $i = 1, \dots, N$*, and* (14)*, for which several efficient algorithms are available. Namely, let* $P_i \in \mathbb{R}^{n \times n}$ *and consider a partition of* P *and* P_i*,* $i = 1, \dots, N$*, similar to that of* Q_i*. That is, let* $P = [P^{kj}]$ *and* $P_i = [P_i^{kj}]$*, for* $j = 1, \dots, N$ *and* $k = 1, \dots, N$*, and let* $P^{j\cdot} \in \mathbb{R}^{n_j \times n}$ *and* $P_i^{j\cdot} \in \mathbb{R}^{n_j \times n}$ *denote the* j*-th row block of matrices* P *and* P_i*, for* $i = 1, \dots, N$*. Then, the* i*-th row block of* P_i *is given by*

$$P_i^{i\cdot} = \frac{1}{\alpha_i} P^{i\cdot} \tag{20}$$

and the j*-th row block of* P_i *is given by*

$$P_i^{j\cdot} = X_{i,j} \, , \tag{21}$$

for $j = 1, \dots, N$ *and* $j \neq i$*, for* $i = 1, \dots, N$*.*

Remark 2. *The results of Theorem 1 can be related to the results of Theorem 1 in [13]. More precisely, while a wider range of differential games (with potentially more complex dynamics and cost functionals) are addressed in [13], in the context of linear LQ MAS, the results of Theorem 1 can be seen as a generalisation of the condition c) of Theorem 1 in [13]. Namely, herein, via the introduction of the parameters* α_i*, for* $i = 1, \dots, N$*, less strict conditions on the structure of the matrices defining the LQ differential game are imposed (compared to the one considered in [13]).*

Remark 3. *Since we consider OL-NE strategies that admit a feedback synthesis, it is straightforward to see that the original game inherits "stability properties" of its associated optimal control problem. This addresses, to some extent, one of the open problems mentioned in [13].*

IV. Feedback potential differential games

The aim of this section is to introduce *feedback* potential differential games, which to the best of our knowledge have not been extensively addressed in the literature to date. Consider first the following definition of exact (approximate) feedback potential games (specific to the MAS setting considered herein), which can be seen as a natural extension of Definition 3 to the context of F-NE solutions.

Definition 4. *The LQ differential game* (2)-(3) *is an LQ exact (approximate) feedback potential differential game if there exists an optimal control problem* (12)-(13)*, such that its solution corresponds to a F-NE (*ϵ_α*-F-NE) solution of the original game,* i.e.

$$u_{\rm ocp}(x) = [u_1^\star(x)^\top, \dots, u_N^\star(x)^\top]^\top \tag{22}$$

for all $t \geq 0$*, where* $\{u_1^\star, \dots, u_N^\star\}$*, denotes a F-NE solution (*ϵ_α*-F-NE solution) of the differential game.*

In the following, we provide constructive sufficient conditions under which the differential game described by equations (2) and (3) can be classified as either an *exact* or *approximate* feedback potential game. Note that since Definition 4 concerns F-NE (ϵ_α-F-NE) solutions, as opposed to Definition 3, the relevant matrix equations (inequalities) characterising such solutions are (8) (or alternatively (10)), for $i = 1, \dots, N$, in place of (5), $i = 1, \dots, N$, that were relevant in the context of *OL-NE* solutions (Definition 3).

A. Exact feedback potential differential games

In the following statement, we consider exact feedback potential differential games and provide conditions under which we can design $Q_{\rm ocp}$ and $R_{\rm ocp}$ in (13) such that the LQR problem yields a solution of the original game.

Theorem 2. *Consider the differential game defined by* (2)-(3)*, for* $i = 1, \dots, N$*. Suppose* $Q_i = [Q_i^{jk}]$ *is such that*

$$Q_i^{jk} = 0 \, , \tag{23}$$

for all $j = 1, \dots, N$ *and* $k = 1, \dots, N$ *with* $j \neq k$*. Let*

$$Q_{\rm ocp} = {\rm blkdiag}\{\alpha_1 Q_1^{11}, \dots, \alpha_N Q_N^{NN}\} \, , \tag{24}$$

$$R_{\rm ocp} = {\rm blkdiag}\{\alpha_1 R_1 \, , \dots \, , \alpha_N R_N\} \, , \tag{25}$$

for any $\alpha_i > 0$*, for* $i = 1, \dots, N$*. Suppose the ARE* (14) *admits a solution* $P = P^\top > 0$*. Then, the differential game* (2)-(3) *is an exact feedback potential differential game with an associated LQR problem defined by* (12) *and* (13)*.*

Remark 4. *Similar to the observations made in the context of OL-NE strategies (see Remark 1), the result in Theorem 2 entails that a F-NE solution of the LQ differential game can be obtained by solving a certain LQR problem. This bypasses the requirement of obtaining a solution of* (8)*,* $i = 1, \dots, N$*. However, if required, a solution of* (8)*, for* $i = 1, \dots, N$*, can be constructed from the solution of* (14)*. Namely,* P_i*, for* $i = 1, \dots, N$*, is given by*

$$P_i = blkdiag\{X_{i,1}, \dots, X_{i,i-1}, \frac{P^{ii}}{\alpha_i}, X_{i,i+1}, \dots, X_{i,N}\} \, , \tag{26}$$

where $X_{i,j}$ *is the solution of the Lyapunov equation*

$$(A_{\rm cl}^{jj})^\top X_{i,j} + X_{i,j}(A_{\rm cl}^{jj}) + Q_i^{jj} = 0 \, , \tag{27}$$

for $j = 1, \dots, N$*,* $j \neq i$*. Note that the solutions* $X_{i,j}$ *are guaranteed to exist, for* $j = 1, \dots, N$ *and* $j \neq i$*, and for* $i = 1, \dots, N$*, by the fact that* $A_{\rm cl}^{ii} \in \mathbb{C}^-$*, for* $i = 1, \dots, N$*.*

B. Approximate potential differential game

Theorem 2 is particularly interesting because, aside from being the first effort in defining and characterising (exact) *feedback* potential differential games, it demonstrates how a solution of the differential game (2)-(3) can be obtained from the solution of an associated LQR problem (12), (13) (as discussed in Remark 4). However, the result holds only when the agents' cost functionals satisfy (23), for $j = 1, \dots, N$ and $j \neq i$, for $i = 1, \dots, N$. This condition is rather limiting and imposes a strict structure on the matrices defining the LQ differential game. To circumvent this limitation, we turn our attention to *approximate* feedback potential differential games. Namely, considering ϵ_α-F-NE solutions we provide conditions under which the LQ differential game constitutes an approximate feedback potential differential game and in doing so, the stringent condition on the structure of Q_i, $i = 1, \ldots, N$, can be relaxed. More specifically, we consider approximate feedback potential differential games and provide conditions under which we can design $Q_{\rm ocp}$ and $R_{\rm ocp}$ in (13) such that the LQR problem yields (as detailed in Definition 4) an *approximate* solution, in the sense of ϵ_α-F-NE solutions, of the original game. To streamline the presentation, let

$$S_{-i} = \mathrm{blkdiag}\{S_1, \ldots, S_{i-1}, 0, S_{i+1}, \ldots, S_N\},$$

where $S_i = B_i(R_{\rm ocp}^{ii})^{-1}B_i^\top$. Furthermore, let the left hand side of (10), be denoted by

$$L_i = P_iA + A^\top P_i + Q_i + P_i\bar{B}_iR_i^{-1}\bar{B}_i^\top P_i \qquad (28)$$
$$-P_i\left(\sum_{j=1}^N \bar{B}_jR_j^{-1}\bar{B}_j^\top P_j\right) - \left(\sum_{j=1}^N P_j\bar{B}_jR_j^{-1}\bar{B}_j^\top\right)P_i\,.$$

Theorem 3. *Consider the differential game defined by* (2)-(3)*, for* $i = 1, \ldots, N$. *Suppose the ARE* (14) *admits a solution* $P = P^\top > 0$. *Let* $Q_{\rm ocp}$ *be a matrix satisfying*

$$Q_{\rm ocp} - \alpha_iQ_i + PS_{-i}P \geq 0\,, \qquad (29)$$

for $i = 1, \ldots, N$, *and let* $R_{\rm ocp}$, *be given by* (25) *for any* $\alpha_i > 0$, *for* $i = 1, \ldots, N$. *Then, the differential game* (2), (3) *is an approximate feedback potential differential game with an associated LQR problem defined by* (12) *and* (13).

Remark 5. *A few observations regarding the result in Theorem 3 are in order. First, note that if* L_i *(given by* (28)*) is zero, the LQ differential game is an* exact *(as opposed to* approximate*) feedback potential differential game. Second, letting* $L_i = -\Upsilon_i$, *the resulting* ϵ_α*-F-NE solution constitutes an (exact) F-NE solution of a* modified *LQ differential game (see,* e.g. *[24] for further details), with dynamics* (2) *and* modified cost functionals

$$\tilde{J}_i(u_i, u_{-i}) = \frac{1}{2}\int_0^\infty x^\top(Q_i + \Upsilon_i)x + u_i^\top R_iu_i dt\,,$$

for $i = 1, \ldots, N$. *Thus, since the result of Theorem 3 holds for* any α_i *satisfying* (29) *and* (25)*,* $i = 1, \ldots, N$, *one could select the parameters* α_i*,* $i = 1, \ldots, N$, *to minimise* $x^\top\Upsilon_ix$, $i = 1, \ldots, N$, *so that the modified cost functionals are as close as possible to the original ones* (3)*,* $i = 1, \ldots, N$.

V. Example: Spacecraft formation control

The results presented in Section IV are demonstrated on a practically motivated example involving formation control of spacecraft. The advantages of employing potential game theory are especially pronounced in systems with a large number of agents, as is frequently encountered in formation control problems. This is due to the difference in computational cost between solving a single optimal control problem and a system of coupled AREs. The specific problem considered herein is motivated by the example presented in [26]. Namely, consider the problem of ensuring that N spacecraft achieve a desired formation about a circular reference orbit of radius $\bar{R}_0$. Then, the relative motion of the i-th spacecraft, for $i = 1, \ldots, N$, with respect to the (common) reference orbit, is given by the Hill-Clohessy-Wiltshire equations

$$\dot{\bar{R}}_i = \bar{A}\bar{R}_i + \bar{B}u_i\,,$$

where

$$\bar{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n_r^2 & 0 & 0 & 0 & 2n_r & 0 \\ 0 & 0 & 0 & -2n_r & 0 & 0 \\ 0 & 0 & -n_r^2 & 0 & 0 & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\bar{R}_i = [r_i^\top, \dot{r}_i^\top]^\top \in \mathbb{R}^6$, with $r_i \in \mathbb{R}^3$ and $\dot{r}_i \in \mathbb{R}^3$ denoting the vectors of relative positions and velocities of the i-th spacecraft with respect to the reference orbit, for $i = 1, \ldots, N$, and where $n_r = \sqrt{\mu/\bar{R}_0^3}$ with μ the gravitation constant of Earth. We consider the problem of designing the control inputs u_i of each spacecraft so that $\lim_{t\to\infty}\dot{r}_i(t) = c$, for some predefined $c \in \mathbb{R}^3$, the first spacecraft (referred to as the leader) converges to the reference orbit, *i.e.* $\lim_{t\to\infty} r_1(t) = 0$, and so that the remaining spacecraft maintain a certain position about the leader *i.e* $\lim_{t\to\infty} r_i(t) - r_1(t) = d_i$, for some predefined $d_i \in \mathbb{R}^3$, $i = 2, \ldots, N$. These objectives can be captured via an LQ differential game. Towards this end, consider the error coordinates $x_1 = [r_1^\top, (\dot{r}_1 - c)^\top]^\top$ and $x_i = [(r_i - r_1 - d_i)^\top, (\dot{r}_i - c)^\top]^\top$, for $i = 2, \ldots, N$. It follows that the error coordinates evolve according to the dynamics (1), with $A_i = \bar{A}$ and $B_i = \bar{B}$, for $i = 1, \ldots, N$.

Consider three spacecraft (*i.e.* $N = 3$) and consider the circular reference orbit given by $\bar{R} = \bar{R}_0[\cos n_rt, \sin n_rt, 0]^\top$ with $\bar{R}_0 = 4.224 \times 10^5$ m. Associate to each spacecraft a cost functional (3), with $Q_1^{11} = \mathrm{blkdiag}\{4I_3, 2I_3\}$, $Q_1^{22} = \mathrm{blkdiag}\{2I_3, 16I_3\}$, $Q_1^{33} = \mathrm{blkdiag}\{0.4I_3, 2I_3\}$, $Q_2^{11} = \mathrm{blkdiag}\{4I_3, 2I_3\}$, $Q_2^{22} = \mathrm{blkdiag}\{3I_3, 3I_3\}$, $Q_2^{33} = \mathrm{blkdiag}\{0.1I_3, 0.2I_3\}$, $Q_3^{11} = \mathrm{blkdiag}\{0.1I_3, 0.1I_3\}$, $Q_3^{22} = \mathrm{blkdiag}\{0.1I_3, 0.1I_3\}$, $Q_3^{33} = \mathrm{blkdiag}\{5I_3, 7I_3\}$ and $Q_i^{jk} = 0$ for $i, j, k = 1, 2, 3$ and $j \neq k$, and let $R_i = 100I_3$, for $i = 1, 2, 3$. It can be easily verified that condition (23) of Theorem 2 holds. Let $Q_{\rm ocp}$, $R_{\rm ocp}$ be given by (24) and (25), with $\alpha_i = 1$, for $i = 1, 2, 3$, *i.e.* $Q_{\rm ocp} = \mathrm{blkdiag}\{Q_1^{11}, Q_2^{22}, Q_3^{33}\}$ and $R_{\rm ocp} = \mathrm{blkdiag}\{R_1, R_2, R_3\}$. The corresponding ARE (14) admits a solution P and, thus, it follows from Theorem 2 that the LQ differential game is an exact feedback potential differential game and the control laws (9) given by $[u_1^\star(t)^\top, u_2^\star(t)^\top, u_3^\star(t)^\top]^\top = -R_{\rm ocp}^{-1}B_{\rm ocp}^\top Px$ constitute a F-NE solution.

Consider now the system (2) in closed-loop with the above F-NE strategies $u_i^\star$, $i = 1, 2, 3$. Let $d_2 = [50 \times 10^3, 50 \times 10^3, 0]^\top$ m and $d_3 = [-50 \times 10^3, -50 \times 10^3, 0]^\top$ m and let $x_1(0) = [10 \times 10^3, 70 \times 10^3, 86 \times 10^3, 0, 0, 0]^\top$, $x_2(0) = [10 \times 10^3, -10 \times 10^3, 30 \times 10^3, 0, 0, 0]^\top$ and $x_3(0) = [-20 \times 10^3, 40 \times 10^3, -50 \times 10^3, 0, 0, 0]^\top$. The trajectories of the three spacecraft, in the inertial frame, are shown in Figure 1, where the asterisks denote the initial conditions. The time histories of the norm of the error coordinates, namely $||x_i||$, for $i = 1, 2, 3$, are shown in Figure 2.

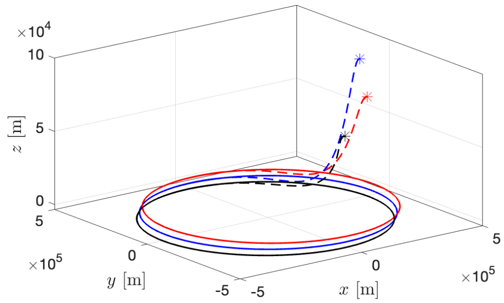

Fig. 1. Trajectories of first (dashed, blue line), second (dashed, red line) and third (dashed, black line) spacecraft. The asterisks denote the spacecraft's initial positions, whereas the solid lines indicate the final formation configuration.

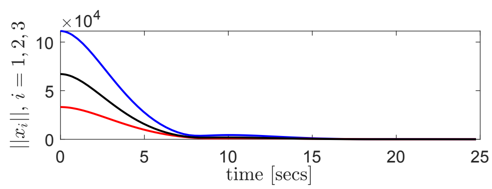

Fig. 2. Time histories of the norm of the error coordinates for first (blue line), second (red line) and third (black line) spacecraft.

VI. Conclusion

Open-loop and feedback potential differential games have been considered in the context of MAS. The problem of when and how it is possible to relate the original differential game to an LQR problem has been studied. We provide sufficient conditions under which an LQR problem can be constructed such that its solution yields a solution of the original game. A significant implication of this is that (exact/approximate) OL-NE/F-NE solutions can be obtained via the formulation of such LQR problems that are much more readily solved than the original LQ differential game. The results are then demonstrated on an illustrative example involving spacecraft formation control. Future work includes extending the previous results to more general, nonlinear differential games, which are not necessarily limited to the MAS setting.

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