

Reachable set-based dynamic quantization for the remote state estimation of linear systems

Yaodong Li, Michelle S. Chong

Abstract—We employ reachability analysis in designing dynamic quantization schemes for the remote state estimation of linear systems over a finite data rate communication channel. The quantization region is dynamically updated at each transmission instant, with an approximated reachable set of the linear system. We propose a set-based method using zonotopes and compare it to a norm-based method in dynamically updating the quantization region. For both methods, we guarantee that the quantization error is bounded and consequently, the remote state reconstruction error is also bounded. To the best of our knowledge, the set-based method using zonotopes has no precedent in the literature and admits a larger class of linear systems and communication channels, where the set-based method allows for a longer inter-transmission time and lower bit rate. Finally, we corroborate our theoretical guarantees with a numerical example.

I. INTRODUCTION

Cyber-physical systems integrate multiple agents and their sensing and actuation devices over a communication channel. Therefore, the remote estimator is responsible for reconstructing the state information based on the sensor's data transmitted over a communication channel [1]. For systems requiring signal communication via a bandwidth-limited network, the analog signal must be converted into discrete-valued digital symbols before being transmitted. This operation inevitably causes an error, called the quantization error. Furthermore, the total amount of information that may be transmitted per unit of time is often limited due to bandwidth constraints of the digital channels, which further degrades the precision of the information that is exchanged over the network. Therefore, novel approaches for remote state estimation over communication channels with bandwidth constraints are needed.

First, to minimize or even eliminate the effect of the quantization error, many studies of quantizer design have been done in the past decades. A quantizer is a mathematical mapping from a continuous region called the quantization region to a finite discrete set of indexes, which we call quantization levels. A quantizer with fixed parameters is called a static (or memory-less) quantizer. In this case, a low bit rate has a strong negative effect on the resolution of quantization. Another strategy for improving the resolution under a fixed bandwidth constraint is *dynamic quantization* [2], where the quantization parameters are dynamically adjusted based on the received data and knowledge of the plant

dynamics. In [3], [4], the robustness of dynamic quantization against external disturbances is studied based on the notion of input-to-state stability [5]. The surveys [6], [7] give recent developments of dynamic quantization for linear and nonlinear systems, respectively.

So far, stabilization problems with controller and quantizer designs are the main focus of the aforementioned papers, where the origin is often assumed to be a stable fixed point under the designed closed-loop controllers. On the other hand, the input and/or control laws are unknown to the remote state estimator. In this case, a conservative prediction for all the reachable states based on a constrained input is more reliable for adjusting the quantization parameters and resolution. In [8], [9], [10], a set-based observer without the consideration of quantization effects propagates the set of all possible states with bounded inputs using reachability analysis based on zonotopes [11]. We will use this result to propagate a conservative approximation of the quantization region for dynamic quantization. In this case, we choose the uniform quantizer [12] so that the resolution is evenly distributed across all components of the state.

In this paper, we design dynamic quantization schemes by performing reachability analysis of continuous-time linear time-invariant (LTI) dynamical systems with external disturbances, for remote state estimation. We over-approximate the terminal reachable set at each transmission instant to update the quantization region and further improve the resolution of the quantization. Polytope-based over-approximation of the reachable set can provide better results but the dramatic increase of vertices and surfaces makes it difficult to propagate. To that end, we were inspired by [11] which uses *zonotopes* to over-approximate the terminal reachable set of a LTI system which have desirable properties. The propagation of zonotopes relies on the *centroid* and *generators*, which are easy to compute and store. Although the number of generators increases linearly with time, in the dynamic quantization scheme, we only over-approximate the terminal set in one inter-transmission interval, by which the issue is avoided. We then compare our results to a norm-based method, which was inspired by [12] that uses the Lipschitz condition to upper-bound the terminal state. However, this only provides the norm bound instead of the more precise bound afforded by considering a component-wise bound of each individual state provided by the zonotopic method.

Similar to [13], we employ a pre-estimator before transmission to avoid remote estimation based on the outputs. We assume the input signal is known to the pre-estimator before transmission, but the remote state estimator has no

M. Chong is with the Department of Mechanical Engineering, Eindhoven University of Technology. This work was done when Y. Li was a student in the same department. Emails: lyd1106n1@gmail.com, m.s.t.chong@tue.nl

The authors thank K.J.A. Scheres for the initial discussions.

access to either the inputs or the control laws. Under the aforementioned setup and its corresponding properties, several objectives are achieved:

- 1) To the best of our knowledge, the first set-based dynamic quantization scheme using zonotopes.
- 2) Comparison between our set-based and a norm-based dynamic quantization scheme.
- 3) No overflow occurs for both dynamic quantization schemes.
- 4) Conditions for the transmission bit rate and inter-transmission interval are given to ensure the boundedness of the quantization error.
- 5) An upper bound is given for the state reconstruction error with respect to bounds on the quantization error, the input, and the disturbance.

All proofs can be found in the extended version of the paper [14].

II. PRELIMINARIES

A. Notations

Let $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_{\geq 0} := [0, \infty)$, $\mathbb{Z}_+ := \{0, 1, \dots\}$, $\mathbb{Z}_{>0} := \{1, 2, \dots\}$ and $\mathcal{I}_n := \{1, 2, \dots, n\}$. $\lambda_i(A)$ denotes the i -th eigenvalue of matrix A , $\lambda_{\max}(A)$, $\lambda_{\min}(A)$ denote the maximum and the minimum eigenvalues, respectively. I_n denotes the identity matrix of size $n \times n$ and $\mathbf{0}_n$ denotes the n -dimensional vector $[0, 0, \dots, 0]^T$, and $\mathbf{1}_{n \times m}$ denotes the $m \times n$ matrix with all elements as 1. $\sigma_{n,i}$ is the n -dimensional vector with 1 as its i -th element and 0 otherwise. We write σ_i when its dimension is clear from context. An n -dimensional hypercube with center $c \in \mathbb{R}^n$ and radius $l \in \mathbb{R}_{\geq 0}$ is denoted $\mathcal{B}(c, l)$. The infinity norm of a vector $x \in \mathbb{R}^n$ is denoted by $|x| := \max_{i \in \mathcal{I}_n} |x_i|$. For a matrix $A \in \mathbb{R}^{n \times n}$, $|A| := \max_{j \in \mathcal{I}_n} \sum_{i=1}^n |a_{ij}|$, where a_{ij} is the (i, j) -component of the matrix A . The Minkowski sum is denoted by \oplus . $\lfloor b \rfloor$ denotes the floor function that returns the greatest integer that is less than or equal to $b \in \mathbb{R}$. A continuous function $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a class- \mathcal{K} (\mathcal{L}) function, if it is strictly increasing (decreasing) and $\gamma(0) = 0$. A continuous function $\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a class- \mathcal{KL} function, if $\alpha(\cdot, s)$ is a class- \mathcal{K} for all $s \geq 0$, $\alpha(r, \cdot)$ is non-increasing and $\alpha(r, s) \rightarrow 0$ as $s \rightarrow \infty$ for all $r \geq 0$.

B. Set representation

A zonotope $\mathcal{Z} \subset \mathbb{R}^n$ is a set that satisfies

$$\mathcal{Z} := \left\{ x \in \mathbb{R}^n \mid x = c + \sum_{i=1}^p \epsilon_i g_i, \forall \epsilon_i \in [-1, 1] \right\}, \quad (1)$$

where $c \in \mathbb{R}^n$ is the geometric center of the zonotope and the line segments $g_i \in \mathbb{R}^n$ are called the generators of the zonotope. We denote $\mathcal{Z} = \langle c, G \rangle$, where $G := (g_1, \dots, g_p) \in \mathbb{R}^{n \times p}$.

C. Terminal reachable set and its over-approximation

The terminal reachable set of a dynamical system is defined as follows.

Definition 1: Consider $\dot{x} = f(x, u)$, for a finite time interval $[t_0, t_1]$, the terminal reachable set $\mathcal{R}_{[t_0, t_1]}(\mathcal{X}, \mathcal{U})$ is defined as the set of all states that are reachable at time t_1 with $x(t_0) \in \mathcal{X}$, with $u(t) \in \mathcal{U}$ for all $t \in [t_0, t_1]$, i.e.,

$$\begin{aligned} \mathcal{R}_{[t_0, t_1]}(\mathcal{X}, \mathcal{U}) \\ := \{x(t_1) : \forall x(t_0) \in \mathcal{X}, u(t) \in \mathcal{U}, \dot{x} = f(x, u), \forall t \in [t_0, t_1]\}. \end{aligned}$$

Next, we generalise Lemma 1 in [11] to arbitrary finite time intervals $[t_0, t_1]$ for $(t_0, t_1) \in \mathbb{R}_{\geq 0}^2$ which we state below and will form a crucial step in the reachable set based approach we will take for our dynamic quantization schemes in this paper.

Lemma 1: Consider a LTI system in the finite time interval $t \in [t_0, t_1]$, where $t_1 - t_0 \leq \tau$ and $(t_0, t_1) \in \mathbb{R}_{\geq 0}^2$, given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (2)$$

with $x(t_0) \in \mathcal{X}$, $Bu(t) \in \mathcal{U}$, $\forall t \in [t_0, t_1]$. The sets \mathcal{X} and \mathcal{U} are zonotopes. Its terminal reachable set $\mathcal{R}_{[t_0, t_1]}(\mathcal{X}, \mathcal{U})$ satisfies

$$\mathcal{R}_{[t_0, t_1]}(\mathcal{X}, \mathcal{U}) \subseteq e^{\tau A} \mathcal{X} \oplus \mathcal{B}(\mathbf{0}_n, \beta(\tau, \mu)), \quad (3)$$

where $\mu := \sup_{Bu \in \mathcal{U}} |Bu|$, and $\beta(\tau, \mu) := |A|^{-1} e^{\tau |A|} \mu$. Lemma 1 is a generalization of [11, Lemma 1] to an arbitrary finite time interval, which can be achieved through standard calculations based on the development in [11].

III. PROBLEM SETUP AND APPROACH

A. Setup and plant

We consider the problem of remote state estimation over a finite data rate channel in the setup depicted in Figure 1. The plant has dynamics

$$\dot{x} = Ax + Bu + Ed, \quad y = Hx, \quad (4)$$

with state $x \in \mathbb{R}^n$ input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^{n_y}$, and unknown disturbance $d \in \mathbb{R}^o$. The system matrices A, B, H, E are known, real matrices with appropriate dimensions. We assume that the pair (A, H) is observable.

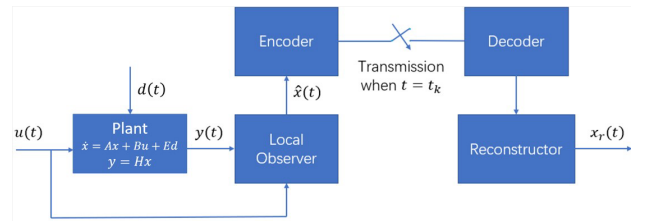


Fig. 1. Remote state estimation setup

Assumption 1: The initial state $x(0)$ and the input signals $Bu(t)$ and disturbance $Ed(t)$ for $t \in \mathbb{R}_{\geq 0}$ reside in known hypercubes $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{D} \subset \mathbb{R}^n$, respectively, defined as

$$\mathcal{X} := \mathcal{B}(x_c, x_b), \mathcal{U} := \mathcal{B}(\mathbf{0}_m, u_b), \mathcal{D} := \mathcal{B}(\mathbf{0}_o, d_b), \quad (5)$$

where $x_c \in \mathbb{R}^n$ is the center of the initial set \mathcal{X} , and $x_b, u_b, d_b \in \mathbb{R}_{\geq 0}$.

Assumption 1 requires the unknown initial state $x(0)$ of the system to reside in a known set \mathcal{X} and the input u to be bounded in \mathcal{U} , which is reasonable in most physical systems.

B. Local observer

The local observer estimates the state of the plant (4) which is designed as follows

$$\dot{\hat{x}} = A\hat{x} + Bu + K(H\hat{x} - y), \quad (6)$$

where $\hat{x} \in \mathbb{R}^n$ is the local state estimate. The local observer matrix $K \in \mathbb{R}^{n \times n_y}$ can be designed according to the following conditions.

Assumption 2: There exist a matrix $P = P^T \succ 0$, a matrix Q and two scalars $\nu_1 > 0, \nu_2 > 0$ such that

$$\begin{bmatrix} A^T P + PA + HQ^T + QH + \nu_1 I_n & P \\ P & -\nu_2 I_n \end{bmatrix} \preceq 0. \quad (7)$$

The inequality (7) is a linear matrix inequality (LMI) in P, Q, ν_1 and ν_2 which can be solved efficiently with computational tools such as the LMI toolbox in MATLAB. The local observer matrix K is then designed as $K = P^{-1}Q$.

We assume that the local observer (6) is initialized at x_c (recall that $x_c \in \mathbb{R}^n$ comes from Assumption 1), i.e.,

$$\hat{x}(0) = x_c. \quad (8)$$

Therefore, the initial local estimation error resides in the hypercube \mathcal{X} from Assumption 1, i.e., $\hat{e}(0) := x(0) - x_c \in \mathcal{X}$. This is essential as we use the bound on the local state estimation error \hat{e} in approximating the reachable set of the local observer (6) for the construction of the quantization region \mathcal{S}_Q^k at each $t_k, k \in \mathbb{N}$.

Lemma 2: Consider the plant (4) and local observer (6) under Assumptions 1 - 2. The local state estimation error $\hat{e}(t) := x(t) - \hat{x}(t)$ of the local observer (6) satisfies

$$|\hat{e}(t)| \leq \hat{\beta}(x_b, t) + \hat{\gamma}(d_b), \quad t \in \mathbb{R}_{\geq 0}, \quad (9)$$

where x_c, x_b come from (5), $\hat{\beta}(r, s) := \sqrt{\frac{n\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\frac{\lambda_e s}{2}} r$ and $\hat{\gamma}(r) := \sqrt{\frac{n\nu_2}{\lambda_{\min}(P)\lambda_e}} r$, with $\lambda_e := \frac{\nu_1}{n\lambda_{\max}(P)}$; ν_1 and ν_2 come from (7).

Corollary 1: The set \mathcal{E}_t of all the possible state estimation errors $\hat{e}(t)$ is given by

$$\hat{e}(t) \in \mathcal{E}_t := \mathcal{B}(\mathbf{0}_n, \beta_d(t)) \quad (10)$$

for all $t \in \mathbb{R}_{\geq 0}$ with $\beta_d(t) := \hat{\beta}(x_b, t) + \hat{\gamma}(d_b)$, $\hat{\beta}, \hat{\gamma}$ are defined in (9).

C. Transmission times and the dynamic quantization scheme

We assume that the inter-transmission times are known and is periodic with period $T \in \mathbb{R}_{> 0}$, i.e., the time at the k -th transmission satisfies $t_k = kT, k \in \mathbb{Z}_+$.

Given a quantization level* $N \in \mathbb{Z}_+$, we employ a dynamic quantization scheme by updating the quantization

*The quantization level N is related to the number of bits $B_r \in \mathbb{Z}_+$ available to the communication channel according to $N = 2^{B_r/n}$, where n is the dimension of the packet vector x .

region denoted by $\mathcal{S}_Q^k \subset \mathbb{R}^n$ at each transmission time $t_k, k \in \mathbb{Z}_+$. We perform reachability analysis to update the quantization region \mathcal{S}_Q^k . In Section IV, we propose using a set-based method of approximating the reachable set, which we achieve with zonotopes. We then compare it to a norm-based method of reachable set approximation in Section V. In both cases, we further regularize the reachable set as a hyperrectangle with $C^k \in \mathbb{R}^n$ being the center of the quantization region (the centroid), and $L^k \in \mathbb{R}^n$ being the segment length vector (the quantization range), i.e.,

$$\mathcal{S}_Q^k := \mathcal{H}(C^k, L^k). \quad (11)$$

We do so for ease of dividing \mathcal{S}_Q^k into hyperrectangular sub-regions denoted by $\mathcal{S}_{q,j}^k, j \in \mathcal{I}_{N^n}$. Although other quantizer designs such as logarithmic or the more general Voronoi quantizer exist [15], we consider the uniform quantizer in this paper for a fair comparison between the two reachable set approximation methods we will present later in Section V-A. To this end, with quantization level $N \in \mathbb{Z}_+$, the quantization region \mathcal{S}_Q^k is divided into $N^n \in \mathbb{Z}_+$ sub-regions at each $t_k, k \in \mathbb{Z}_+$. Each subregion is then numbered from 0 to $N - 1$ in the i -th dimension. Hence, the local state estimate $\hat{x}(t_k)$ is encoded into $\{0, 1, \dots, N - 1\}^n$, i.e., component $i \in \mathcal{I}_n$ of the encoded packet is defined by

$$P_{e,i}^k = Q_e(\hat{x}_i(t_k), C_i^k, L_i^k, N), \quad (12)$$

where the encoding map $Q_e : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{Z}_+ \rightarrow \{0, 1, \dots, N - 1\}$ is

$$Q_e(\hat{x}_i(t_k), C_i^k, L_i^k, N) := \left\lfloor \left(\hat{x}_i(t_k) + L_i^k - C_i^k \right) \frac{N}{2L_i^k} \right\rfloor. \quad (13)$$

The parameters L_i^k, C_i^k are the i -th element of the vector L^k and C^k , respectively.

Each subregion has a centroid $c_j^k \in \mathcal{S}_Q^k \subset \mathbb{R}^n$ with a quantization range $l^k := \frac{L^k}{N}$, which is defined as

$$\mathcal{S}_{q,j}^k := \mathcal{H}(c_j^k, l^k), \quad j \in \mathcal{I}_{N^n}. \quad (14)$$

In our communication scheme, the decoded packet $P_d^k = (P_{d,1}^k, P_{d,2}^k, \dots, P_{d,n}^k) \in \mathbb{R}^n$ is a member of the set of centroids $\{c_j^k\}_{j \in \mathcal{I}_{N^n}}$ of the subregions. Therefore, component $i \in \mathcal{I}_n$ of the decoded packet is given by

$$P_{d,i}^k = Q_d(P_{e,i}^k, C_i^k, L_i^k, N), \quad (15)$$

where the decoding map $Q_d : \mathbb{Z}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{Z}_+ \rightarrow \mathbb{R}$ is

$$Q_d(P_{e,i}^k, C_i^k, L_i^k, N) := C_i^k - \frac{L_i^k}{2} + \frac{L_i^k}{2N} (2P_{e,i}^k + 1). \quad (16)$$

Remark 1: We assume that the quantization parameters C^k and L^k are not transmitted over the communication channel. Instead, they are updated on both the encoder and the decoder sides simultaneously and therefore do not occupy any bandwidth. The robustness of a zoom-in and zoom-out dynamic quantization scheme in the context of stabilizing a linear system when the encoder and decoder asynchronously update their parameters was investigated in [16], but not in

the context of the reachable-set based dynamic quantization scheme considered in this paper.

To prevent overflow at the first quantization step ($k = 0$), the initial quantization parameters satisfy the following.

Assumption 3: The initial quantization region is chosen as $\mathcal{S}_Q^0 = \mathcal{H}(C^0, L^0)$ with $C^0 = x_c$, $L^0 = \mathbf{1}_n x_b$, where x_c and x_b come from Assumption 1.

Assumption 3 guarantees that the initial local state estimate $\hat{x}(0) \in \mathcal{S}_Q^0$ with arbitrary l_0 . Moreover, we will show in Sections IV and V that by updating C^k and L^k for $k \in \mathbb{Z}_+$ appropriately, we can always prevent overflow, i.e., $\hat{x}(t) \in \mathcal{S}_Q^{k+1}$, for $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}_+$ as stated below.

Assumption 4: The quantization parameters C^k and L^k for $k \in \mathbb{Z}_+$ are updated such that $\hat{x}(t) \in \mathcal{S}_Q^{k+1}$ for $t \in [t_k, t_{k+1})$.

D. Reconstructor

The reconstructor is responsible for producing a continuous prediction of the state x , based on the information received at discrete instances in time t_k . The dynamics of the reconstructed state $x_r \in \mathbb{R}^n$ satisfies, for all $k \in \mathbb{Z}_+$,

$$\dot{x}_r = Ax_r + K_r(Hx_r - HP_d^k), \quad \forall t \in [t_k, t_{k+1}), \quad (17)$$

and $x_r(t_k) = P_d^k$ with the reconstructor gain matrix $K_r \in \mathbb{R}^{n \times n_y}$ is chosen such that to satisfy $K_r = P^{-1}Q$, where matrices P and Q come from Assumption 2. The reconstructor (17) is initialized according to $x_r(0) = x_c$, where $x_c \in \mathbb{R}^n$ comes from Assumption 1. At every transmission instant t_k , the reconstructor is initialised at P_d^k , which we recall is the decoded state at t_k .

E. Objective and Approach

The overall objective is to design a dynamic quantization scheme for the encoder and decoder, as well as a local observer and a reconstructor that converts the discrete-time decoder data into continuous-time reconstructed state $x_r(t) \in \mathbb{R}^n$, such that the state reconstruction error $e_r(t) := x(t) - x_r(t)$ is upper-bounded with respect to the quantization error $e_q(t) := \hat{x}(t) - P_d^k$, input u , and the unknown disturbance d .

To this end, we provide the following guarantee for the state reconstruction error e_r of our remote state estimation setup depicted in Figure 1.

Theorem 1: Consider the plant (4), the local observer (6), encoder (12), decoder (15) and reconstructor (17) under Assumptions 1-4. The state reconstruction error $e_r(t) := x(t) - x_r(t)$ satisfies the following

$$|e_r(t)| \leq \beta_r (|e_r(t_k)|, t - t_k) + \gamma_r \left(\max\{|u|_{[t_k, t]}, |d|_{[t_k, t]}, |\hat{e}|_{[t_k, t]}, |e_q|_{[t_k, t]}\} \right), \quad (18)$$

for $t \in [t_k, t_{k+1})$ and for all $e_r(t_k) \in \mathbb{R}^n$, where $\beta_r \in \mathcal{KL}$, $\gamma_r \in \mathcal{K}$ and $|z|_{[t_k, t]}$ denotes $\sup_{s \in [t_k, t]} |z(s)|$.

From Theorem 1, we see that the reconstruction error e_r is ultimately bounded by the input u , disturbance d , local estimation error \hat{e} and the quantization error e_q . Since we

have established that u and d are bounded by Assumption 1 and \hat{e} by Lemma 2, we focus on establishing that the quantization error e_q is also bounded.

In this paper, we employ a reachable set-based approach in the dynamic quantization scheme. We propose over-approximating the reachable set with zonotopes in Section IV and compare it to a norm-based over-approximation of the reachable set in Section V. Both methods employ the idea of over-approximating the terminal reachable sets of the local state estimate $\hat{x}(t_k)$ and using it as the quantization region \mathcal{S}_Q^k . To the best of our knowledge, the set-based approach using zonotopes which we will present in Section IV, has not been done in the literature. We show that this provides relaxed conditions over the norm-based dynamic quantization scheme in Section V-A.

IV. SET-BASED DYNAMIC QUANTIZATION USING ZONOTOPES

First, the dynamics of the local observer (6) can be rewritten as

$$\dot{\hat{x}} = A\hat{x} + Bu - KH\hat{e}. \quad (19)$$

We can propagate the terminal reachable set of the local state estimate \hat{x} based on the dynamics (19) by treating the term $KH\hat{e}$ as an additional input. Since

$$KH\hat{e}(t) \in KH\mathcal{E}_t \subseteq KH\mathcal{E}_{t_k}, \quad \forall t \in [t_k, t_{k+1}),$$

where $\mathcal{E}_t = \mathcal{B}(\mathbf{0}_n, \beta_d(t))$ is defined in (10), and β_d satisfies $\beta_d(t') \geq \beta_d(t'')$, for all $t'' \geq t' \geq 0$. Then

$$Bu(t) - KH\hat{e}(t) \in \mathcal{U} \oplus KH\mathcal{E}_{t_k}, \quad \forall t \in [t_k, t_{k+1}).$$

Finally, given that $\hat{x}(t_k) \in \mathcal{S}_Q^k$, it implies that $\hat{x}(t_k) \in \mathcal{S}_{q,j}^k$ for some $j \in \mathcal{I}_{N^n}$. This is guaranteed for $k = 0$ by Assumption 3. Then, the terminal reachable set of the local state estimate \hat{x} at the next transmission time t_{k+1} can be deduced from Lemma 1, that for $t \in [t_k, t_{k+1})$

$$\hat{x}(t) \in \mathcal{R}_{[t_k, t_{k+1})}(\mathcal{S}_{q,j}^k, \mathcal{U} \oplus KH\mathcal{E}_{t_k}) \subseteq \Lambda \mathcal{S}_{q,j}^k \oplus \mathcal{B}(\mathbf{0}_n, \beta_{ue}^k), \quad (20)$$

with $\Lambda := e^{AT}$, where we recall that $T \in \mathbb{R}_{>0}$ is the inter-transmission interval, and β_{ue}^k is the upper bound of the input term $(Bu - KH\hat{e})$ of the local observer dynamics (19) for all $t \in [t_k, t_{k+1})$, which is

$$\beta_{ue}^k := |A|^{-1} e^{|A|T} (u_b + |KH|\beta_d(t_k)), \quad (21)$$

where u_b comes from (5) and β_d is defined in (9).

We then over-approximate the terminal reachable set of the state estimate with a hyperrectangle such that the quantization region \mathcal{S}_Q^{k+1} at the next transmission time t_{k+1} is

$$\mathcal{S}_Q^{k+1} := \mathcal{H}(C^{k+1}, L^{k+1}) \supseteq (\Lambda \mathcal{S}_{q,j}^k \oplus \mathcal{B}(\mathbf{0}_n, \beta_{ue}^k)). \quad (22)$$

Hence, the dynamic quantization update law can be formulated as

$$C^{k+1} = \Lambda P_d^k, \quad L_i^{k+1} = \beta_{ue}^k + \sum_{j=1}^n |\sigma_i^T \Lambda \sigma_j| \frac{L_j^k}{N}, \quad i \in \mathcal{I}_n, \quad (23)$$

with initialization $P_d^0 := x_c$, $L_i^0 := x_b$ for $i \in \mathcal{I}_n$ according to Assumption 3 and we recall that β_{ue}^k is defined in (21). We guarantee that by updating the dynamic quantization parameters C^k and L^k for $k \in \mathbb{Z}_+$ according to (23), the quantization region S_Q^{k+1} always contains the local state estimate $\hat{x}(t)$ for $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}_+$, which we state in the following lemma.

Lemma 3: Consider the plant (4), the local observer (6), encoder (12), decoder (15) under Assumptions 1-3. Suppose the quantization parameters C^k and L^k for $k \in \mathbb{Z}_+$ are updated according to the set-based scheme (23). Then Assumption 4 holds.

We denote the quantization error at $t = t_k$ as $e_q^k := \hat{x}(t_k) - P_d^k$. We show that its i -th component is always bounded by $\bar{e}_{q,i}^k$, i.e.,

$$|e_{q,i}^k| \leq \bar{e}_{q,i}^k, \quad k \in \mathbb{Z}_+. \quad (24)$$

We call $\bar{e}_q^k := (\bar{e}_{q,1}^k, \bar{e}_{q,2}^k, \dots, \bar{e}_{q,n}^k)^T$ the maximum quantization error which we will show is bounded. Therefore, in conjunction with Lemma 3, we guarantee that the quantization error $e_q(t)$ is also bounded in between transmission times, i.e. for $t \in [t_k, t_{k+1})$. By Lemma 3, we have that $\hat{x}(t) \in S_Q^{k+1}$ for $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}_+$ which implies that $\hat{x}(t_k) \in S_{q,j}^k$ for some $j \in \mathcal{I}_{N^n}$, where $S_{q,j}^k$ is defined in (14). Hence, the maximum quantization error satisfies

$$\bar{e}_q^k = l^k = \frac{L^k}{N}, \quad k \in \mathbb{Z}_+. \quad (25)$$

Therefore, by the dynamic quantization law in (23), the maximum quantization error satisfies

$$\bar{e}_q^{k+1} = \frac{\bar{\Lambda}}{N} \bar{e}_q^k + \frac{\beta_{ue}^k}{N} \mathbf{1}_n, \quad (26)$$

where

$$\bar{\Lambda} := (|\Lambda_{ij}|)_{i,j \in \mathcal{I}_n}, \quad (27)$$

is a matrix where each (i, j) -th component is the absolute value of Λ_{ij} and Λ_{ij} is the (i, j) -th component of matrix Λ (recall from (20) that $\Lambda = e^{AT}$).

We are now ready to show that by choosing the transmission time $T \in \mathbb{R}_{\geq 0}$ and total quantization level $N \in \mathbb{Z}_{>0}$ appropriately, the maximum quantization error \bar{e}_q^k is bounded for all $k \in \mathbb{Z}_+$.

Theorem 2: Consider the plant (4), the local observer (6), encoder (12), decoder (15) under Assumptions 1-3. Suppose the quantization parameters C^k and L^k for $k \in \mathbb{Z}_+$ are updated according to (23). If the transmission interval $T > 0$ and quantization level $N \in \mathbb{Z}_+$ are chosen such that the matrix $\frac{\bar{\Lambda}}{N}$ is Schur, i.e.,

$$\max_{i \in \mathcal{I}_n} \left| \lambda_i \left(\frac{\bar{\Lambda}}{N} \right) \right| < 1, \quad (28)$$

then the quantization error $e_q(t)$ is bounded for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}_+$.

V. NORM-BASED DYNAMIC QUANTIZATION

We now present a norm-based scheme in updating the quantization region S_Q^k , $k \in \mathbb{Z}_+$. We adapted the dynamic quantization scheme in [12] which was used in a stabilization setting to our remote state estimation problem. We choose the quantization region to be a hypercube, i.e., $S_Q^k := \mathcal{B}(C(t_k), L^k)$, where the centroid C has the following dynamics

$$\dot{C}(t) = AC(t), \quad t \in [t_k, t_{k+1}), \quad C(t_k) = P_d^k, \quad (29)$$

and we recall that P_d^k is the decoded packet defined in (15).

At the next transmission instant t_{k+1} , the quantization region S_Q^{k+1} is designed as

$$S_Q^{k+1} := \mathcal{B}(C(t_{k+1}), L^{k+1}), \quad (30)$$

where the quantization parameters are updated according to

$$C^{k+1} := C(t_{k+1}) = \Lambda P_d^k, \quad L^{k+1} = \frac{e^{|A|T}}{N} L^k + \beta_{ue}^k, \quad (31)$$

with initialization $P_d^0 := x_c$, $L^0 := x_b$, where x_c and x_b come from Assumption 1. Just as in the set-based method in Section IV, we show that the quantization region S_Q^{k+1} contains the local state estimate $\hat{x}(t)$ for $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}_+$.

Lemma 4: Consider the plant (4), the local observer (6), encoder (12), decoder (15) under Assumptions 1-3. Suppose the quantization parameters C^k and L^k for $k \in \mathbb{Z}_+$ are updated according to the norm-based scheme (30)-(31). Then Assumption 4 holds.

We can now provide conditions on the transmission time $T \in \mathbb{R}_{\geq 0}$ and the total quantization level $N \in \mathbb{Z}_{>0}$ such that quantization error $e_q(t) := \hat{x}(t) - P_d^k$ is bounded for $t \in [t_k, t_{k+1})$.

Theorem 3: Consider the plant (4), the local observer (6), encoder (12), decoder (15) under Assumptions 1-3. Suppose the quantization parameters C^k and L^k for $k \in \mathbb{Z}_+$ are updated according to the norm-based scheme (30)-(31). If the inter-transmission interval $T \in \mathbb{R}_{\geq 0}$ and total quantization level $N \in \mathbb{Z}_{>0}$ are chosen such that

$$\frac{e^{|A|T}}{N} < 1, \quad (32)$$

then the quantization error $e_q(t)$ is bounded for all $t \in \mathbb{R}_{\geq 0}$.

A. Comparison

The difference between the set-based and norm-based schemes lies in how they over-approximate the ‘size’ of the reachable set, which is captured by the quantization range L^k . The set-based method propagates the zonotope, and thereby provides upper bounds for each individual component in the state, while the norm-based method only considers the norm of e^{AT} . In fact, the set-based method provides a relaxed condition (28) in Theorem 2 compared to the norm-based method. We see this by first establishing the following.

Lemma 5: Given any matrix $A \in \mathbb{R}^{n \times n}$, scalars $T \in \mathbb{R}_{\geq 0}$ and $N \in \mathbb{Z}_{>0}$, the following holds

$$\max_{i \in \mathcal{I}_n} \left| \lambda_i \left(\frac{\bar{\Lambda}}{N} \right) \right| \leq \frac{e^{|A|T}}{N}, \quad (33)$$

where $\bar{\Lambda}$ is as defined in (27).

By Lemma 5, we see that condition (28) of the set-based scheme in Theorem 2 relaxes condition (32) of the norm-based scheme in Theorem 3. We state this result in the proposition below.

Proposition 1: Consider the plant (4), the transmission interval $T \in \mathbb{R}_{\geq 0}$ and quantization level $N \in \mathbb{Z}_{>0}$. If the parameters T and N satisfy condition (32) of the norm-based scheme, then condition (28) of the set-based scheme will also hold.

Consequently, when designing a dynamic quantization scheme, the designer first checks if the norm-based method (Theorem 3) is met. If so, then the designer has the option of both the set and norm-based methods presented so far in Sections IV and V, respectively. Future work would include deriving analytical bounds on the quantization error for both methods. In this paper, we compare them in simulation which we present in the next section.

VI. NUMERICAL SIMULATION

We consider a 2-dimensional LTI system in the form of (4) with

$$A = \begin{bmatrix} -1 & -4 \\ 4 & -1 \end{bmatrix}, \quad B = E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad H = [1 \quad 0], \quad (34)$$

with the initial state $x(0) \in \mathcal{X} = \mathcal{B}([10, -5]^T, 1) = [9, 11] \times [-6, -4]$, the input $u(t) = 0.5 \sin(t) \in \mathcal{B}(0, 0.5)$, $\forall t \geq 0$, and the disturbance $d(t)$ is a random noise that is upper-bounded by 0.05 for all $t \geq 0$. Hence, Assumption 1 is satisfied with $x_c = [10, 5]^T$, $x_b = 1$, $u_b = 0.5$ and $d_b = 0.05$.

We solve (7) using the LMI toolbox in MATLAB to obtain

$$P = \begin{bmatrix} 2.0648 & 0.9237 \\ 0.9237 & 1.9195 \end{bmatrix}, \quad Q = \begin{bmatrix} -7.7353 \\ -0.0248 \end{bmatrix}, \quad (35)$$

$$\nu_1 = 8.2561, \quad \nu_2 = 7.2571.$$

Hence, the local observer gain K of (6) and the reconstructor gain K_r of (17) are

$$K = K_r = P^{-1}Q = \begin{bmatrix} -4.7666 \\ 2.2808 \end{bmatrix}. \quad (36)$$

In our simulation study, the plant (4), the local observer (6) and reconstructor (17) are initialized at $x(0) = [10.5, -5.5]^T$, $\hat{x}(0) = x_c = [10, 5]^T$ and $x_r(0) = x_c = [10, 5]^T$, respectively.

Our communication setup depicted in Figure 1 has quantization level $N = 4$, bit rate $B_r = 4$, and the inter-transmission interval is $T = 0.1$. With these parameters, we can verify that $\frac{\bar{\Lambda}}{N}$ is Schur and $\frac{e^{|A|T}}{N} < 1$, where each satisfies the condition of the set-based (Theorem 2) and norm-based (Theorem 3) quantization schemes, respectively. Therefore, the set-based and norm-based dynamic

quantization schemes presented in Sections IV and V, respectively, are applicable. Moreover, the reconstructor x_r given by (17) has bounded reconstruction error according to Theorem 1. We simulated for the time-interval $[0, 20]s$, i.e., with transmission interval $T = 0.1$, we transmit 200 times. By defining the steady state quantization error as $e_q^\infty := \lim_{k \rightarrow 100} |e_q^k|$ and steady-state reconstruction error as $\bar{e}_r^\infty := \lim_{t \in [10, 20]} |e_r(t)|$, we compare them for each scheme, which we summarise in Table I. In both cases, the set-based scheme outperforms the norm-based scheme.

TABLE I

SET-BASED AND NORM-BASED DYNAMIC QUANTIZATION SCHEMES

	Dynamic quantization schemes	
	Set-based	Norm-based
Steady-state quantization error e_q^∞	0.0571	0.0684
Steady-state reconstruction error \bar{e}_r^∞	0.0921	0.1170

REFERENCES

- [1] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 138–162, 2007.
- [2] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Transactions on Automatic Control*, vol. 45, no. 7, pp. 1279–1289, 2000.
- [3] D. Liberzon and D. Nesić, "Input-to-state stabilization of linear systems with quantized state measurements," *IEEE Transactions on Automatic Control*, vol. 52, no. 5, pp. 767–781, 2007.
- [4] Y. Sharon and D. Liberzon, "Input to state stabilizing controller for systems with coarse quantization," *IEEE Transactions on Automatic Control*, vol. 57, no. 4, pp. 830–844, 2011.
- [5] E. D. Sontag, "Input to state stability: Basic concepts and results," in *Nonlinear and Optimal Control Theory*, pp. 163–220, Springer, 2008.
- [6] G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans, "Feedback control under data rate constraints: An overview," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 108–137, 2007.
- [7] Z. Jiang and T. Liu, "Quantized nonlinear control—a survey," *Acta Automatica Sinica*, vol. 39, no. 11, pp. 1820–1830, 2013.
- [8] C. Combastel, "A state bounding observer for uncertain non-linear continuous-time systems based on zonotopes," in *Proceedings of the 44th IEEE Conference on Decision and Control*, pp. 7228–7234, IEEE, 2005.
- [9] J. K. Scott, D. M. Raimondo, G. R. Marseglia, and R. D. Braatz, "Constrained zonotopes: A new tool for set-based estimation and fault detection," *Automatica*, vol. 69, pp. 126–136, 2016.
- [10] J. Su and W.-H. Chen, "Model-based fault diagnosis system verification using reachability analysis," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 49, no. 4, pp. 742–751, 2017.
- [11] A. Girard, "Reachability of uncertain linear systems using zonotopes," in *International Workshop on Hybrid Systems: Computation and Control*, pp. 291–305, Springer, 2005.
- [12] C. De Persis and A. Isidori, "Stabilizability by state feedback implies stabilizability by encoded state feedback," *Systems & Control Letters*, vol. 53, no. 3–4, pp. 249–258, 2004.
- [13] Y. Xu and J. P. Hespanha, "Estimation under uncontrolled and controlled communications in networked control systems," in *Proceedings of the 44th IEEE Conference on Decision and Control*, pp. 842–847, IEEE, 2005.
- [14] Y. Li and M. S. Chong, "Reachable set-based dynamic quantization for the remote state estimation of linear systems," 2023. Extended version on arXiv:2309.04006.
- [15] F. Bullo and D. Liberzon, "Quantized control via locational optimization," *IEEE Transactions on Automatic Control*, vol. 51, no. 1, pp. 2–13, 2006.
- [16] T. Kameneva and D. Nešić, "Robustness of quantized control systems with mismatch between coder/decoder initializations," *Automatica*, vol. 45, no. 3, pp. 817–822, 2009.