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Attitude Consensus Control with Disturbance Rejection for Incompletely Cooperative Multi-agent Systems on $SO(3)$

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Abstract— **The three-dimensional orthogonal matrix group $SO(3)$ offers global, unique, and nonsingular attitude representation of the rotational motion. This paper addresses the attitude consensus problem with disturbance rejection for multi-agent systems consisting of incompletely cooperative agents evolving on $SO(3)$. Firstly, we establish an individual cost functional that evaluates the agent consensus aim using the natural Riemannian metric on $SO(3)$, and formulate the considered consensus problem as a differential game. Secondly, a baseline consensus protocol is designed following the inverse optimal control procedure. In addition, a finite-time disturbance observer on $SO(3)$ is developed based on the non-singular terminal sliding mode technique, which is used for estimating and compensating for disturbances. The effectiveness of the proposed schemes is verified with simulations on a 4-vehicle formation setting.**

I. INTRODUCTION

Motivation and literature survey: The study of multi-agent systems (MASs) has a significant impact on several fields, such as power, transportation, finance, and healthcare [1], [2], [3]. Some vehicle applications of MASs are spacecraft formation, multi-unmanned aerial vehicles and underwater swarm systems. As one of the most important objectives of such systems, the problem of achieving attitude coordination through suitable consensus control schemes has attracted extensive attention in the literature [4], [5], [6].

Suitable approaches have been developed that enable improved attitude consensus in the Euclidean space [7], [8], [9], where the agent attitudes are parameterized by using Euler angles, quaternion and modified Rodrigues parameters methods. However, the existing parameterized representation methods have significant disadvantages such as the inability to achieve large-angle maneuvering and the unwinding phenomenon. On the contrary, the three-dimensional special orthogonal group $SO(3)$ can characterize attitude motion globally, uniquely, and non-singularly, which makes it an attractive option for controller design [10]. In the recent decade, the attitude control design on $SO(3)$ has attracted broad attention and led to valuable results [11], [12], [13].

For the MASs on $SO(3)$, several consensus control results can be found in the literature, utilizing switching topologies [14], optimal control methodologies [15], and enabling finite-time convergence [16]. It should be mentioned that, in the above papers, the agents are assumed to be fully cooperative, i.e. they have no individual objective or payoff to actively pursue by themselves. This may not appropriately characterize certain practical scenarios. To the authors' best knowledge, there are currently no results on the coordination control of incompletely cooperative MASs on $SO(3)$.

A problem of high importance associated with attitude control is that of disturbance/uncertainty rejection, since in practice the existence of external perturbations, measure noise, and un-modeled dynamics reduce the control accuracy and might even compromise the system stability [17]. The terminal sliding-mode based disturbance observer provides nonlinear disturbance compensation with strong robustness, high precision, and fast convergence rate, and is hence an effective solution approach for the attitude control system in non-$SO(3)$ models [18]. Therefore, extending this technique to MASs on $SO(3)$ is significant and highly relevant.

Contribution: In this paper, we address the attitude consensus control problem for incompletely cooperative MASs on $SO(3)$ with diverse individual objectives and disturbance rejection. The contributions of the paper are the following:

(i) it explores differential game-based coordination strategies for MASs with incompletely corporative agents on $SO(3)$ that aim to balance global control objectives and individual optimization goals;
(ii) it introduces the terminal sliding-mode technique into the finite-time and bounded error disturbance observation design of MASs on $SO(3)$ to achieve consensus in the presence of disturbances.

Paper structure: The rest of the paper is organized as follows. In Section II, we provide some math preliminaries, followed by a description of the attitude consensus control problem of the MASs on $SO(3)$. A game-based baseline control policy without disturbances and the disturbance rejection technique are presented in Sections III and IV, respectively. Section V verifies the effectiveness of the proposed control strategy via numerical simulations. Finally, Section VI provides some concluding remarks.

II. PROBLEM FORMULATION

A. Preliminaries

Consider a MAS consisting of N agents, described by the directed graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where $\mathscr{V} \triangleq \{1, 2, \cdots, N\}$ denotes the node set and $\mathscr{E} \triangleq \{(i, j) \mid j$ can obtain information from $i\}$ the edge set.

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A path p is a sequence of connected edges in the graph $\mathscr{G}$ that connects two nodes, i.e., $p \triangleq (v_1, v_2, \cdots, v_q)$, where $v_i \in \mathscr{V}$ and $(v_i, v_{i+1}) \in \mathscr{E}$, $i = 1, 2, \cdots, q-1$. The neighbor node-set of i is defined as $\mathscr{N}_i \triangleq \{j \in \mathscr{V} \mid (j,i) \in \mathscr{E}\}$. For a vector $x \in \mathbb{R}^n$, x_i denotes its ith element and x_{-i} the vector that follows by removing the ith element from x.

We let I_n denote the identity matrix of dimension n and $\det(\cdot)$ the determinant of $(\cdot)$. The state of each agent evolves in $SO(3)$, which consists of the matrices $R \in \mathbb{R}^{3\times 3}$ such that $R^\top R = I_3$ and $\det(R) = 1$. The identity element of $SO(3)$ is I_3, where the tangent space is defined as its Lie algebra $\mathfrak{so}(3) \triangleq \{X \in \mathbb{R}^{3\times 3} : X + X^\top = I\}$. For all $x, y \in \mathbb{R}^3$, the map $\hat{x} : \mathbb{R}^3 \to \mathfrak{so}(3)$ satisfies $\hat{x}y = x \times y$ and its inverse operator is $\text{vex}(\cdot) : \mathfrak{so}(3) \to \mathbb{R}^3$. For $\hat{x}, \hat{y} \in \mathfrak{so}(3)$, define the inner product:

$$\langle \hat{x}, \hat{y} \rangle \triangleq \text{tr}(\hat{x} \cdot \hat{y}),$$

where $\text{tr}(\cdot)$ is the trace of matrix $(\cdot)$. To possess the property $\langle \hat{x}', \hat{x} \rangle = x^\top x \in \mathbb{R}$, the dual space $\mathfrak{so}'(3)$ is defined as

$$\mathfrak{so}'(3) \triangleq \left\{ \hat{x}' \in \mathbb{R}^{3\times 3} \mid \hat{x}' = \frac{1}{2} I_3 \hat{x}^\top, \forall \hat{x} \in \mathfrak{so}(3) \right\}.$$

To formulate the coordinate independent distance function on $SO(3)$, the following logarithmic map is introduced.

Definition 1 ([19]): Let $R \in SO(3)$ be such that $\text{tr}(R) \neq -1$. Then

$$\log(R) \triangleq \frac{\phi}{2\sin\phi}\left(R - R^\top\right) \in \mathfrak{so}(3),$$

where ϕ satisfies $\cos\phi = \frac{1}{2}(\text{tr}(R) - 1)$ and $|\phi| < \pi$. □

Then, $\forall R \in SO(3)$, the metric $\|R\|_{SO(3)}$ satisfies

$$\|R\|^2_{SO(3)} \triangleq -\frac{1}{2}\text{tr}\left([\log(\text{R})]^2\right) = -\frac{1}{2}\langle \log(R), \log(R) \rangle.$$

The coordinate independent property of this metric is guaranteed since, $\forall S \in SO(3)$,

$$\langle \text{Ad}_S \log(R), \text{Ad}_S \log(R) \rangle = \langle \log(R), \log(R) \rangle,$$

where $\text{Ad}_S \log(R) \triangleq S \cdot \log(R) \cdot S^{-1}$ denotes the adjoint map that converts the state $\log(R)$ into coordinate S. The derivative of the above metric can be computed by using the following lemma.

Lemma 1 ([19]): For the compact Lie group $SO(3)$, let $R(t) \in SO(3)$ be a smooth trajectory that never passes through a singularity on $\text{tr}(R) = -1$. Then

$$\frac{d}{dt}\|R(t)\|^2_{SO(3)} = -\frac{1}{2}\langle \log(R(t)), \hat{\omega}(t) \rangle, \tag{1}$$

where $\hat{\omega}(t) \triangleq R^\top(t)\dot{R}(t)$. □

Similarly, the metric on $\mathfrak{so}(3)$ is defined as

$$\|\hat{\omega}\|^2_{\mathfrak{so}(3)} \triangleq -\frac{1}{2}\langle \hat{\omega}, \hat{\omega} \rangle.$$

A $n \times n$ matrix H is called Hurwitz if all eigenvalues of $\det(\lambda \mathbf{I}_n - H) = 0$ have negative real part. A Hermitian matrix is a complex square matrix that is equal to its complex transpose. We let $|\cdot|$ and $\|\cdot\|$ denote the absolute value of a scalar $x \in \mathbb{R}$ and the 2-norm of a vector $y \in \mathbb{R}^n$, respectively. The sign function is defined by $\text{sign}(\cdot) : \mathbb{R} \to \mathbb{R}$ satisfying, $\text{sign}(x) \triangleq \frac{x}{|x|}$, $x \in \mathbb{R} \setminus \{0\}$, and $\text{sign}(0) = 0$.

B. Incompletely cooperative consensus control problem with disturbance rejection

In a MAS, the attitude dynamics of each agent on $SO(3)$ are described by

$$\dot{R}_i(t) = R_i(t)\hat{\omega}_i(t) + \hat{d}_i, \ i \in \mathscr{V}, \tag{2}$$

where $R_i(t) \in SO(3)$ denotes the attitude, $\hat{\omega}_i(t) \in \mathfrak{so}(3)$ the velocity, and $\hat{d}_i \in \mathfrak{so}(3)$ the lumped disturbance of agent i. Here, we assume that $\hat{\omega}_i$ can be controlled directly.

The following assumptions are made to facilitate the controller design.

Assumption 1: The attitude $R_i(t)$ of MAS (2) is available to agent i, $i \in \mathscr{V}$. □

Assumption 2: There exist positive constants $\bar{d}_i$ and $\tilde{d}_i$ such that $\|\hat{d}_i\|_{\mathfrak{so}(3)} \leq \bar{d}_i$, $\left\|\dot{\hat{d}}_i\right\|_{\mathfrak{so}(3)} \leq \tilde{d}_i$, $i \in \mathscr{V}$. □

Assumption 3: The communication graph $\mathscr{G}$ of the MAS is assumed to be directed and at least connected. □

Denote the relative attitude between agents i and j by $R_{ij} \triangleq R_j^\top R_i$. As a result, the consensus task it to ensure that at the terminal time t_f:

$$R_{ij}(t_f) = I_3, \ i, j \in \mathscr{V}. \tag{3}$$

Considering the consensus aim (3) with disturbance rejection, the control input $\hat{\omega}_i$ can be designed as follows

$$\hat{\omega}_i = \hat{\omega}_{ic} + \hat{\omega}_{id} = \hat{\omega}_{ic} - R_i^\top \hat{d}_i, \tag{4}$$

where $\hat{\omega}_{ic}$ is responsible for the attitude maneuver of agent i so as to achieve the consensus, while $\hat{\omega}_{id}$ is used for compensating the lumped disturbance $\hat{d}_i$. In this case, the lumped disturbance $\hat{d}_i$ should be estimated by some means. Note that due to the randomness or uncertainty of disturbances, it is difficult that those are fully estimated and compensated. Therefore, what we call "disturbance rejection" in the following refers to restricting the effect of the disturbance to a sufficiently small neighborhood of the origin, see e.g. the definition in [18].

C. Optimal control problem

Consider an incompletely cooperative case where in addition to achieving consensus, each agent aims to minimize its energy consumption. From (4), the total energy consumption of agent i can be divided into two parts, described by $\hat{\omega}_{id}$ and $\hat{\omega}_{ic}$. For the former, the value of energy consumption $\|\hat{\omega}_{id}\|^2_{\mathfrak{so}(3)}$ is minimal if the disturbance is perfectly compensated. As for $\hat{\omega}_{ic}$, its energy consumption is represented by the term $\|\hat{\omega}_{ic}\|^2_{\mathfrak{so}(3)}$.

Combining the minimal energy consumption with the formation aim (3) yields the following cost functional

$$J_i = \sum_{j \in \mathscr{N}_i} \underbrace{\frac{1}{2} K_{f_i} \left\|R_{ij}(t_f)\right\|^2_{SO(3)}}_{\xi_i(R_{ij}(t_f))} + \frac{1}{2}\int_0^{t_f} K_{u_i} \|\hat{\omega}_{ic}\|^2_{\mathfrak{so}(3)} \text{d}t, \tag{5}$$

for each agent, where K_{f_i}, K_{u_i} are positive constants. In the above definition, the control for disturbance-rejection $\hat{\omega}_d$ is neglected, and is considered in the next section. Note that, in the presence of the dynamics (2) with $\hat{d}_i = 0$ and given initial conditions, then $R_{ij}(t_f)$ can be uniquely determined by the trajectories of $\hat{\omega}_c$. Hence, there exists some cost functional $\tilde{J}_i(\hat{\omega}_c)$ that satisfies $\tilde{J}_i(\hat{\omega}_c) = J_i(\hat{\omega}_c, R_{ij}(\hat{\omega}_c, t_f, R_0), t_f, R_0)$, where $R_0 \triangleq (R_1(0), R_2(0), \cdots, R_N(0))$, and $\hat{\omega}_c \in \mathscr{U} \triangleq \mathscr{U}_1 \times \mathscr{U}_2 \times \cdots \times \mathscr{U}_N$ with $\mathscr{U}_i \triangleq \{\hat{\omega}_{ic}(t, R_0) \in \mathfrak{so}(3) \mid t \in [0, t_f]\}$. The above motivates the following optimization problem for each agent $i \in \mathscr{V}$

$$\begin{aligned} &\min_{\hat{\omega}_{ic} \in \mathscr{U}_i,\ t \in [0,t_f]} \tilde{J}_i(\hat{\omega}_c) \\ &\text{s.t. (2) and } \hat{d}_i = 0. \end{aligned} \tag{6}$$

Note that for the MAS (2), the individual cost functional (5) is a trade-off between the overall consensus goal and the aim to minimize individual energy consumption. Specifically, the greater the value of K_{f_i}, the more expensive it will be for agent i to achieve consensus, while a larger value for K_{u_i} implies that agent i is less willing to maneuver.

The consensus control problem (6) can be formulated as a differential game. In particular, it can be shown that the solutions of (6) coincide with the Nash equilibrium definition provided in Definition 2 below.

Definition 2 ([20]): The control strategy $\hat{\omega}_c^* \triangleq (\hat{\omega}_{1c}^*, \cdots, \hat{\omega}_{Nc}^*)$ defines a Nash equilibrium for cost $\tilde{J}_i(\hat{\omega}_c), i \in \mathscr{V}$, if the following inequalities hold:

$$\tilde{J}_i(\hat{\omega}_c^*) \leq \tilde{J}_i(\hat{\omega}_{ic}, \hat{\omega}_{-ic}^*),\ i \in \mathscr{V},$$

for each admissible control strategy $\hat{\omega}_{ic} \in \mathscr{U}_i$. □

D. Problem statement

This section presents a statement of the problem considered in this paper, which is provided below.

Problem 1: Design control schemes for the MAS (2) which, when Assumptions 1-3 hold, satisfy

(i) Constitute a Nash equilibrium according to Definition 2 for individual costs described by (5);
(ii) Achieve disturbance rejection;
(iii) Use locally available information. □

Property (i) minimizes the energy consumption of each agent under the premise of taking into account the global coordination objective (3). In the sense of bounded estimate errors, (ii) requires to compensate as much as possible for the effects brought by disturbances. Finally, (iii) aims for a distributed controller scheme that has significant advantages such as scalability and fault-tolerance in the presence of a single point of failure.

III. Consensus control without disturbances

For the case when the disturbances $\hat{d}_1, \hat{d}_2, \cdots, \hat{d}_N$ of the MAS (2) are neglected or compensated, this section proposes the following consensus protocol.

Theorem 1: Consider a group of N agents with dynamics described by (2) with $\hat{d}_i = 0$, $i \in \mathscr{V}$, and cost functional given by (5) and let Assumptions 1 and 3 hold. In addition, let $\hat{\omega}_{ic}$ satisfy

$$\hat{\omega}_{ic}^* = -\frac{1}{K_{u_i}} \sum_{j \in \mathscr{N}_i} K_i(t) \log(R_{ij}), i \in \mathscr{V}, \tag{7}$$

where $K_i(t)$ is a time-varying positive scalar function that satisfies

$$K_i(t) = -\frac{3}{K_{u_i}(t - \frac{3}{K_{f_i} K_{u_i}} - t_f)},\ t \in [0, t_f]. \tag{8}$$

Then, the control strategies $\hat{\omega}_{1c}^*, \hat{\omega}_{2c}^*, \cdots, \hat{\omega}_{Nc}^*$ consist a Nash equilibrium in accordance with Definition 2. □

Proof: According to (2) and (5), we introduce the following Hamiltonian function:

$$H_i(R_i, \hat{\omega}_i, \hat{p}_{R_i}) = \frac{1}{2} K_{u_i} \|\hat{\omega}_{ic}\|_{\mathfrak{so}(3)}^2 + \langle \hat{p}_{R_i}, R_i \hat{\omega}_{ic} \rangle, \tag{9}$$

where $\hat{p}_{R_i} \in \mathfrak{so}'(3)$ denotes the costate. The optimal control $\hat{\omega}_{ic}^*$ satisfies

$$H_i(R_i, \hat{\omega}_{ic}^*, \hat{p}_{R_i}) = \min_{\hat{\omega}_{ic} \in \mathfrak{so}(3)} H_i(R_i, \hat{\omega}_{ic}, \hat{p}_{R_i}).$$

Then, we can obtain the optimal control strategy for each agent by implementing the inverse optimal control procedure [21]. From the definitions of $\|\cdot\|_{SO(3)}$ and $\|\cdot\|_{\mathfrak{so}(3)}$, it can be verified that

$$\xi_i(I_3) = -\frac{1}{4} K_{f_i} \text{tr}\left[\log(I_3)^2\right] = 0.$$

Also, $\forall R \in SO(3)$ such that $R \neq I_3$, $\xi_i(R) > 0$. Therefore, we can select the following candidate Lyapunov function for agent i as,

$$V_i(R_i) \triangleq \frac{1}{2} \sum_{j \in \mathscr{N}_i} K_i(t) \|R_{ij}\|_{SO(3)}^2, \tag{10}$$

where $K_i(t)$ is a positive time-dependent and first-order differentiable function to be determined. Therefore, by using (1), the derivative of V_i is

$$\dot{V}_i = \sum_{j \in \mathscr{N}_i} \left[\frac{1}{2} \dot{K}_i(t) \|R_{ij}\|^2 + K_i(t) \left\langle R_i \log(R_{ij}), R_i \hat{\omega}_{ic} \right\rangle\right].$$

These relations give the resulting gradient equation of V_i as

$$\hat{p}_{R_i} = \nabla_{R_i} V_i = \frac{1}{2} \sum_{j \in \mathscr{N}_i} K_i(t) R_i \log(R_{ij}), \tag{11}$$

where $\nabla_{R_i} V_i \triangleq \frac{\partial V_i}{\partial R_i}$ denotes the partial derivative of V_i with respect to R_i. Substituting (11) into (9) yields

$$H_i = \frac{1}{2} K_{u_i} \|\hat{\omega}_{ic}\|_{\mathfrak{so}(3)}^2 + \sum_{j \in \mathscr{N}_i} \left\langle K_i(t) R_i \log(R_{ij}), R_i \hat{\omega}_{ic} \right\rangle. \tag{12}$$

As a result, one can see that the Hamiltonian function H_i of agent i is also influenced by other agents' states $R_j, j \in \mathscr{V}_i$, which depend on the trajectories of $\hat{\omega}_j$. Specifically, to achieve Nash equilibrium, each agent needs to adopt a control strategy $\hat{\omega}_i^*$ with the goal of minimizing their own Hamiltonian function (12) so as to satisfy

$$H_i(\hat{\omega}_i^*, \hat{\omega}_{-i}^*) \leq H_i(\hat{\omega}_i, \hat{\omega}_{-i}^*).$$

In this case, no agent can improve their Hamiltonian functions by changing their control strategy alone. Therefore, the solution of the differential game is transferred into the individual optimal control problem under the remaining agents' influence.

From the stationarity condition $\frac{\partial H_i}{\partial \hat{\omega}_i} = 0$, the form of the optimal control $\hat{\omega}_{ic}^*$ is given by

$$\hat{\omega}_{ic}^* = -\frac{1}{K_{u_i}} \sum_{j\in\mathcal{N}_i} K_i(t)\log(R_{ij}).$$

Since the Hamilton-Jacobi-Bellman (HJB) function:

$$\begin{aligned}&\dot{V}_i(R_i) + H_i(R_i, \hat{\omega}_{ic}, \nabla V_i)\\ &\quad = \sum_{j\in\mathcal{N}_i} \frac{1}{2}\dot{K}_i(t)\|R_{ij}\|^2 - \frac{3}{2K_{u_i}} K_i^2(t)\|R_{ij}\|^2 = 0,\end{aligned}$$

and the terminal condition $\varphi_i(R_t(t_f))$, we have

$$\frac{1}{2}\dot{K}_i(t) - \frac{3}{2K_{u_i}} K_i^2(t) = 0,\ K_i(t_f) = K_{f_i}. \tag{13}$$

Solving (13) yields (8) and completes the proof. ■

Theorem 1 offers a comprehensive result that takes into account the specified terminal time t_f, and the cost parameters K_{u_i} and K_{f_i}. The amplitude of the control signal $\|\omega_{ic}\|_{\mathfrak{so}(3)}$ increases as K_{f_i} increases and K_{u_i}, t_f decrease. Note also that (7) is a feasible distributed solution of the consensus problem under Assumptions 1 and 3, since for each agent only its own information and that of its neighbors are used. One can see from (8) that only local parameters K_{u_i}, K_{f_i} are required for the parameter design of agent i, although the scheme is distributed in terms of its information exchange. This admits a decentralized design, which facilitates the simplicity of control algorithm.

IV. Consensus control with disturbance rejection

In this section, we combine the proposed consensus protocol described by (7)-(8), with a sliding-mode disturbance observer on $SO(3)$. The latter aims to compensate the impact of the lumped disturbances, resulting in a finite-time consensus controller with disturbance rejection.

The following sliding variable is defined for agent i

$$\sigma_i \triangleq \mathrm{vex}(R_i - R_i^\top).$$

Then, the derivative of σ_i is given by

$$\dot{\sigma}_i = \mathrm{vex}\left(R_i\hat{\omega}_i + \hat{d}_i + \hat{\omega}_i R_i^\top - \hat{d}_i^\top\right). \tag{14}$$

Define the estimations of $\sigma_i, \hat{d}_i$ by $\sigma_i^o, \hat{d}_i^o$, and the associated errors by $e_{\sigma_i} \triangleq \sigma_i^o - \sigma_i$, $e_{d_i} \triangleq \hat{d}_i^o - \hat{d}_i$. Then, a finite-time disturbance observer is established that satisfies

$$\begin{aligned}\dot{\sigma}_i^o &= \mathrm{vex}\left(R_i\hat{\omega}_i + \hat{\omega}_i R_i^\top\right) + \mathrm{vex}\left(\hat{d}_i^o - (\hat{d}_i^o)^\top\right)\\ &\quad - \alpha_1\left(\lceil e_{\sigma_i}\rfloor^{\gamma_i} + e_{\sigma_i}\right)\\ \dot{d}_i^o &= -\alpha_2\left(\lceil e_{\sigma_i}\rfloor^{\gamma_2} + 2\lceil e_{\sigma_i}\rfloor^{\gamma_1} + e_{\sigma_i}\right),\end{aligned} \tag{15}$$

where $\lceil e_{\sigma_i}\rfloor^{\gamma_j} \triangleq \left[\lceil e_{\sigma_{i1}}\rfloor^{\gamma_j}, \lceil e_{\sigma_{i2}}\rfloor^{\gamma_j}, \lceil e_{\sigma_{i3}}\rfloor^{\gamma_j}\right]^\top$ with elements $\lceil e_{\sigma_{ip}}\rfloor^{\gamma_j} \triangleq \mathrm{sign}(e_{\sigma_{ip}})|e_{\sigma_{ip}}|_j^\gamma$, $j = 1,2$, $p = 1,2,3$, and gain parameters $\alpha_1 > 0$, $\alpha_2 > 0$, $\frac{1}{2} < \gamma_1 < 1$, $\gamma_2 = 2\gamma_1 - 1$.

This enables a finite time estimate of the lumped disturbance $\hat{d}_i$ as shown in the following theorem.

Theorem 2: Consider the MAS (2) and the disturbance observer (15), and let Assumptions 1-2 hold. Then, the observation error $e_i^o \triangleq \left[e_{\sigma_i}^\top, e_{d_i}^\top\right]^\top$ converges to a bounded set that includes the origin in finite time. □

Proof: Introduce the auxiliary variable

$$\varepsilon \triangleq \left[\left(\lceil e_{\sigma_i}\rfloor^{\gamma_1} + e_{\sigma_i}\right)^\top, e_{d_i}^\top\right]^\top.$$

The convergence of ε implies that e_i^o converges to the origin. Therefore, we focus on investigating the convergence of ε by defining the Lyapunov function

$$V_o\left(e_i^o\right) = \varepsilon^\top P\varepsilon$$

where P is a symmetric and positive-definite matrix. Taking the derivative of V_o along the solutions of (14)-(15) yields

$$\dot{V}_o = \varepsilon^\top P\left(\mathrm{diag}\left(|e_{\sigma_i}|^{\gamma_1-1}, |e_{\sigma_i}|^{\gamma_1-1}\right)H_1\varepsilon + H_2\varepsilon + D\right) \tag{16}$$

where $H_1 = [-\alpha_1\gamma_1\mathbf{I}_3, \gamma_1\mathbf{I}_3; -\alpha_2\mathbf{I}_3, \mathbf{0}_{3\times3}]$, $H_2 = [-\alpha_1\mathbf{I}_3, \mathbf{I}_3; -\alpha_2\mathbf{I}_3, \mathbf{0}_{3\times3}]$, $D = \left[\mathbf{0}_3; \dot{d}_i\right]$. It can be verified that H_1 and H_2 are Hurwitz matrices. Therefore there exist symmetric Hermitian matrixes Q_1, Q_2 such that

$$H_1^\top P + PH_1 = -Q_1,\ H_2^\top P + PH_2 = -Q_2.$$

Substituting the above equalities into (16) yields

$$\begin{aligned}\dot{V}_o =& \varepsilon^\top\left[\mathrm{diag}\left(\left[|e_{\sigma_i}|^{\gamma_1-1}, |e_{\sigma_i}|^{\gamma_1-1}\right]\right)\left(H_1^\top P + PH_1\right)\right]\varepsilon\\ &+ \varepsilon^\top\left(H_2^\top P + PH_2\right)\varepsilon + D^\top P\varepsilon + \varepsilon^\top PD\\ \le& -\left(|e_{\sigma_i}|_{\max}\right)^{\gamma_1-1}\varepsilon^\top Q_1\varepsilon - \varepsilon^\top Q_2\varepsilon + 2\|\varepsilon\|\|P\|\|D\|,\end{aligned}$$

where $|e_{\sigma_i}|_{\max} \triangleq \max\{|e_{\sigma_i 1}|, |e_{\sigma_i 2}|, |e_{\sigma_i 3}|\}$. By Assumption 2, since $\|d_i\| \le \bar{d}_i$ and $\|\dot{d}_i\| \le \tilde{d}_i$, we have $2\|\varepsilon\|\|P\|\|D\| \le 2\sqrt{3}\tilde{d}_i\|\varepsilon\|\|P\| \le 2\sqrt{3}\tilde{d}_i\lambda_{\min}(P)V^{\frac{1}{2}}\|P\|$ with the minimum eigenvalue of $(\cdot)$ as $\lambda_{\min}(\cdot)$. Given $|e_{\sigma_i}|_{\max} \le \|e_{\sigma_i}\| \le \|\varepsilon\|^{\frac{1}{\gamma_1}}$, one further has

$$\dot{V}_o \le -\lambda_1 V_o^{\frac{3}{2}-\frac{1}{2\gamma_1}} - \lambda_2 V_o + \lambda_3 V_o^{\frac{1}{2}}$$

where $\lambda_1 = \lambda_{\min}(Q_1)\lambda_{\max}(P)^{\frac{1}{2\gamma_1}-\frac{3}{2}}$, $\lambda_2 = \lambda_{\min}(Q_2)\lambda_{\max}(P)^{-1}$, $\lambda_3 = 2\sqrt{3}\dot{\tilde{d}}_i\lambda_{\min}(P)^{-\frac{1}{2}}\|P\|$. According to Proposition 2 in [18], the trajectory of the proposed disturbance observer is finite-time uniformly ultimately bounded stable, which implies the error e_i^o will converge to the following small region near origin

$$\mathscr{C}_o \triangleq \left\{e_i^o \mid \beta_1 V_o\left(e_i^o\right)^{\frac{1}{2}-\frac{1}{2\gamma_1}} + \beta_2 V_o\left(e_i^o\right)^{\frac{1}{2}} < \lambda_3\right\} \tag{17}$$

with settling time

$$T_o \le \frac{\ln\left[1 + (\lambda_2 - \beta_2)V_o(e_i^o(0))^{\frac{1}{2\gamma_1}-\frac{1}{2}}/(\lambda_1 - \beta_1)\right]}{(\lambda_2 - \beta_2)\left(\frac{1}{2\gamma_1} - \frac{1}{2}\right)}, \tag{18}$$

where $\beta_i \in (0, \lambda_i), i = 1,2$. This completes the proof. ■

From the above result, one can see that the estimation error e_{d_i} can be adjusted by appropriately selecting the values of parameters α_i, $\gamma_i, i = 1,2$, as follows from (17). Also, the settling time can be specified through the system parameters, as shown in (18). Thus, the lumped disturbance term $\hat{d}_i$ can be estimated by $\hat{d}_i^o$, which combined with (7) yields the following controllers with disturbance rejection.

$$\hat{\omega}_i^* = -\frac{3}{K_{ui}^2\left(t - \frac{3}{K_{fi}K_{ui} - t_f}\right)} \sum_{j\in\mathcal{N}_i} \log(\mathrm{R_{ij}}) - R_i^\top \hat{d}_i^o,\ i \in \mathscr{V}. \quad (19)$$

Based on the separation principle and Theorems 1-2, we draw the following conclusion directly.

Theorem 3: Consider the MAS (2) satisfying Assumptions 1-3 and the cost functional (5). The control policy (19) achieves the attitude consensus with disturbance rejection. □

V. SIMULATIONS

This section validates our analytic results with numerical simulations on a MAS consisting of 4 agents with dynamics given by (2) and the communication network topology depicted in Fig. 1 below. It can be seen that the communication network satisfies Assumption 3 since there always exists a path from agent 4 to the other agents.

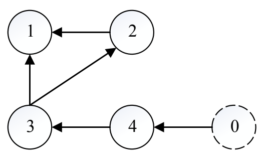

Fig. 1. The communication topology of the considered MAS with 4 agents.

For our simulation, we consider the initial conditions of the attitude represented by Euler angles as shown in Table I, and the parameters presented in Table II. In the simulation, we consider noise, denoted by d_{noise}, with values randomly selected from the uniform distribution $[-0.01, 0.01]$. The lumped disturbances of agents are given by

$$\begin{aligned} d_{11} &= d_{42} = 0.01 \text{ rad/s}, \\ d_{21} &= d_{41} = 0.01\sin\left(\frac{1}{2\pi}t\right) \text{ rad/s}, \\ d_{32} &= d_{43} = d_{noise}, \\ d_{12} &= d_{13} = d_{22} = d_{23} = d_{31} = d_{33} = 0. \end{aligned} \quad (20)$$

It can be seen that Assumption 2 is satisfied for the above disturbances with the forms of constant, sine-wave, and random noise. As shown in Table. I, agents 1-3 are subjected to a single type of disturbance, while agent 4 is disturbed by the hybrid disturbances consisting of constant, sine and random noise.

TABLE I

THE INITIAL CONDITIONS OF THE MAS.

No.	Pitch angle	Yaw angle	Roll angle	disturbance
1	50°	75°	30°	constant
2	75°	-25°	65°	sine
3	-50°	30°	-100°	random noise
4	120°	-45°	85°	hybrid

To make the control effect more intuitive, assume that agent 4 implements the attitude regulation strategy, which can be achieved by using a virtual neighbor agent 0 that stays at the origin of the attitude. The time response of the Euler angles, control variables and disturbance estimation errors are shown in Fig. 2, Fig. 3, and Fig. 4, respectively.

TABLE II

THE CONTROLLER PARAMETER VALUES.

No.	t_f	K_u	K_f	α_1	α_2	γ_1	γ_2
1	15	0.05	10	2	1	0.66	0.32
2	15	0.5	10	2	1	0.66	0.32
3	15	0.5	2	0.2	0.07	0.8	0.6
4	15	0.05	2	0.2	0.07	0.66	0.32

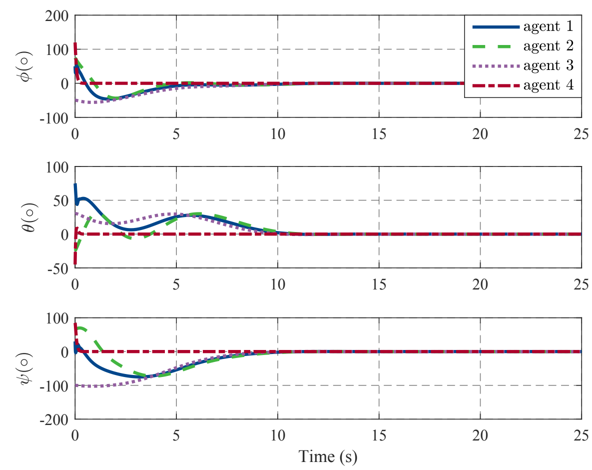

Fig. 2. The response of pitch angle ϕ_i, roll angles θ_i and yaw angles ψ_i, $i = 1,2,3,4$, to the disturbance characterized by (20).

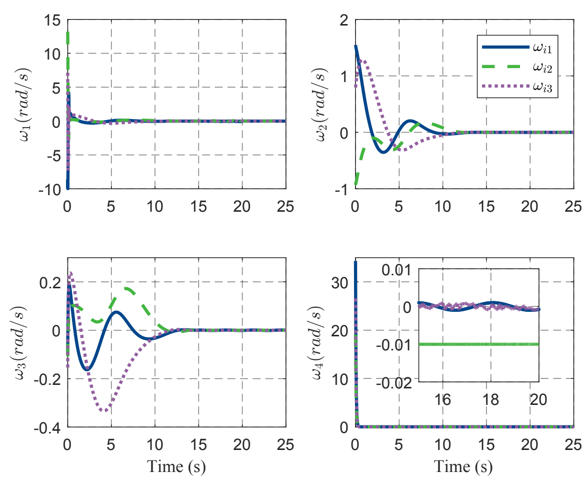

Fig. 3. The response of velocity $\omega_1, \omega_2, \omega_3, \omega_4$ to the disturbance characterized by (20).

As depicted in Fig. 2, controllers (19) achieve attitude consensus at $t_f = 15$ s, which demonstrates the validity of Theorem 3. Moreover, by comparing the responses between agents with different cost functionals, one can see that the control amplitude increases when K_{u_i} decreases. For example, by comparing the responses of agents 1 and 2, from Fig. 3, it can be seen that, although all parameters except for K_{u_i} are the same, the control amplitude of agent 1

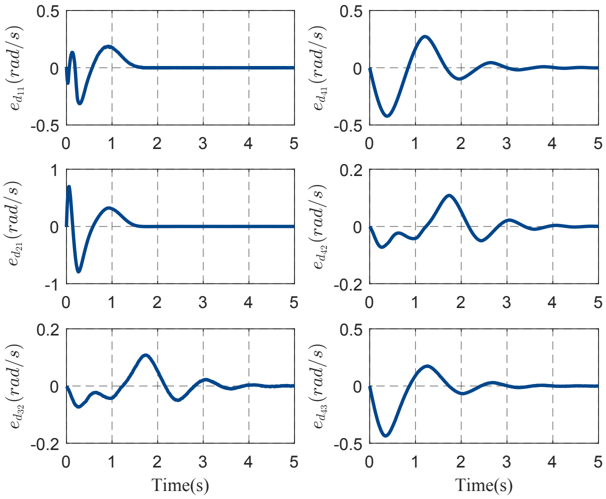

Fig. 4. The response of error lumped disturbance $e_{d_{11}}, e_{d_{21}}, e_{d_{32}}$, e_{d_4}.

is two orders of magnitude greater than that of agent 2. Correspondingly, the attitude angle response of agent 1 is much faster than that of agent 2, as shown in Fig. 2. Notably, according to the communication network depicted in Fig. 1 and the form of controller (19), the control effect of agent 1 is determined by the attitude error with both agent 2 and agent 3. Therefore, the attitude angles of agent 1 converge to the average of the attitude angles of agents 2 and 3 at a relatively fast rate, and the situation is maintained until convergence is reached.

The effectiveness of the proposed disturbance observer (15) can be verified in Fig. 4, where the errors of the estimation disturbances can be seen to converge in a small region around the origin. To be more specific, it can be seen that the settling time of $e_{d_{11}}$ and $e_{d_{21}}$ is shorter than that of $e_{d_{32}}$. This can be explained from (18), since as parameters α_1, α_2 increase, the settling time T_o decreases. However, as described by (17), the estimation error will increase with the increase of parameters α_1, α_2. This is verified by the zoomed-in figures in Fig 3. To compensate for the disturbances, after the attitude converges, the controllers need to generate signals of equal magnitude and opposite sign to the estimation signals from the disturbance observers. Disturbed by the same constant disturbance d_{11} and d_{43}, as shown by the responses of ω_{13} and ω_{43}, the disturbance observer of agent 4 generates more accurate estimates than the disturbance observer of agent 1.

In conclusion, this simulation results verify that under the control schemes proposed in this paper, the MAS (2) subject to disturbances and uncertainties achieves attitude consensus and disturbance rejection under diverse performance indexes.

VI. CONCLUSION

This paper considered the problem of consensus control of incompletely cooperative MAS with disturbance rejection in $SO(3)$. A suitable optimal control problem has been formulated, taking into account the conflicting objectives of reaching consensus and minimizing individual energy consumption. In addition, it is analytically shown that the considered problem is solved by means of a distributed control design, which results in Nash equilibrium trajectories of individual velocities. The developed framework is combined with finite-time disturbance observers, enabling a disturbance rejection attitude consensus control strategy. This provides a solution for the incompletely cooperative MASs with global, nonsingular, and large angle attitude consensus maneuver.

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