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Polytopic Lyapunov Functions are not Straightforward for Minimum Dwell-Time Switched Affine Systems

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Abstract— **This paper addresses stabilization of switched affine systems relying on polytopic Lyapunov functions. Several methods have been developed for this family of systems, most of them based on quadratic Lyapunov functions for an average system. An immediate conjecture is that replacing such a function with one of polytopic type should also achieve stable behavior. We disprove this conjecture on a strategy that employs the hybrid systems formalism, by means of examples. Specifically, we show that for a given switching strategy based on a quadratic Lyapunov function, naïvely replacing this function by its polytopic counterpart does not guarantee the same stability properties, and that this happens even in the case of dwell-time switching. We deeply investigate such a problem, identify the conditions that prevent asymptotic stability, and give some directions on how to avoid them to ensure stability. These results are finally illustrated in simulation.**

Index Terms— **Switched affine systems, polytopic Lyapunov functions, hybrid dynamical systems.**

I. Introduction

An effective way to capture the behavior of processes including continuous and discontinuous dynamics is represented by switched systems [1]. Such systems are indeed characterized by a certain number of subsystems selected at each time instant by a properly designed switching signal, therefore capable of capturing the different operating modes exhibited by the plants. Concerning the design of the switching rule, two main design choices can be found in the literature, i.e., *arbitrarily fast switching* [2], which allows the selection of the next active subsystem regardless of when the last switching occurred, and *dwell-time switching* [3], which requires the existence of a minimum amount of time between any two consecutive switching instants.

In this context, stabilization of switched linear systems (SLS), where the objective is to stabilize an equilibrium point common to all subsystems (usually the origin), has been deeply studied [4]–[7]. A system class more general than SLS is that of switched affine systems (SAS), for which different equilibrium points for each subsystem may exist. While the stabilization problem of the origin for SLS has been successfully addressed in many works exploiting different methodologies, the problem of stabilizing a generic (not common) equilibrium point for SAS is still an open and more challenging objective. In fact, in this case, asymptotic stability of a desired equilibrium point can be achieved, under appropriate assumptions, only in the case of arbitrarily fast switching. In the case of dwell-time switching, instead, the existence of different equilibrium points for each subsystem prevents achieving asymptotic stability of the desired point unless it corresponds to the equilibrium of one of the subsystems. Therefore, designing a dwell-time constrained switching law that practically stabilizes the system requires the investigation of advanced control solutions.

Among many significant contributions in the literature, in [8] SASs are represented as nonlinear systems with input constraints in order to design switching laws derived by emulating locally classical controllers. An extension to sampled-data robust switching law is proposed in [9] by taking into account sampling variations or uncertainty. In [10] a state-dependent switching law satisfying dwell-time constraints is proposed. This strategy relies on Lyapunov-Metzler inequalities and takes into account the forward evolution during the dwell time of the value of quadratic Lyapunov functions for each subsystem. Periodic time- and event-triggered control laws are proposed in [11], where the class of SAS is represented via a hybrid dynamic system. By contrast, [12] introduces optimal control strategies to minimize a specific performance index. With reference to discrete-time SAS, instead, a solution based on the minimization of the size of the ultimate invariant set of attraction is proposed in [13]. A hybrid system framework is adopted also in [14]. The latter proposes a space regularization method and one based on time, ensuring a minimum dwell time and admissible chattering around the operating point.

A common feature of the aforementioned works is that they rely on the existence of a quadratic Lyapunov function. The current paper diverges from that trend, aiming to explore the circumstances under which a family of real-valued polytopic functions [15], rather than complex-valued ones (e.g, as in [16]–[18]), can serve as suitable Lyapunov functions for SAS. By employing the hybrid system formalism to model such systems, we introduce an extension of the approach presented in [14], wherein the switching conditions relying on a quadratic Lyapunov function are replaced by conditions based on a polytopic Lyapunov function. We show that such a naïve extension does not straightforwardly lead to the same stability guarantees presented in [14]. The latter reference shows that, in the case of the quadratic

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Lyapunov function framework, a switching law guaranteeing the asymptotic stability of the SAS can be constructed from a Lyapunov function for the average or convex combination system. By contrast, we here provide examples that show that constructing the switching law based solely on the decrease of the polytopic Lyapunov function for the average or convex combination system may not produce the desired stable behavior, and that this happens for both arbitrarily fast and dwell-time switching. Specifically, our examples show that unstable or eventually discrete solutions may appear (the latter being solutions of the hybrid system for which continuous time does not advance). Hence, in this work we identify the conditions that prevent eventually discrete solutions and divergence.

The reasons underlying the choice of a switching law based on polytopic Lyapunov function lies in the dwell-time switching design. In fact, keeping in mind that stability can be only practical when a dwell-time constraint is required, the adoption of polytopic functions looks appropriate to define the set of attraction, whose features could be related to the system performance on the application side. Differently from quadratic Lyapunov functions, such an attraction set can be indeed simply described by linear inequalities when polytopic functions are adopted, which enhance the interpretability of the region in terms of bounds on every separate state variable.

Notation. $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$, $\mathbb{N}$ denote the real, non-negative real, positive and natural numbers, respectively. For $N \in \mathbb{N}$, we define the set $\mathbb{N}_N := \{n \in \mathbb{N} : n \leq N\}$. $\bar{\mathbf{1}}$ denotes a vector all of whose components are equal to 1. The (i,j)-th entry of $R \in \mathbb{R}^{r\times n}$ is denoted by $R_{i,j}$ and its i-th row by $R_{i,:}$. If $X, Y \in \mathbb{R}^{r\times n}$, the expression '$X \preceq Y$' denotes the set of componentwise inequalities $X_{i,j} \leq Y_{i,j}$, $i = 1, ..., r$, $j = 1, ..., n$. The absolute value, euclidian norm and infinity norm are denoted as $|\cdot|$, $\|\cdot\|$, and $\|\cdot\|_\infty$, respectively. For $x \in \mathbb{R}^n$, x' denotes its transpose.

II. Preliminaries

A. Problem statement

Consider switched affine systems of the form

$$\dot{x}(t) = A_{\sigma(t)}x(t) + b_{\sigma(t)}, \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{N}_N$ is the switching signal, and $A_i \in \mathbb{R}^{n\times n}$, $b_i \in \mathbb{R}^n$ for all $i \in \mathbb{N}_N$. We aim to establish a switching control strategy based on a polytopic Lyapunov function to achieve stabilization of an arbitrary operating point $x_e \in \mathbb{R}^n$. Note that given x_e, we can shift (1) through the change of variables $\tilde{x} := x - x_e$ to obtain an equivalent problem of stabilizing the origin in the variable $\tilde{x}$. Thus, in order to simplify the analysis, we will consider $x_e = 0$ without any loss of generality. Let us introduce the following assumption, which is standard for achieving this goal using quadratic Lyapunov functions (refer to [14] and the references therein). Define the set $\Lambda := \{\lambda \in [0,1]^N : \lambda'\bar{\mathbf{1}} = 1\}$, and for each $\lambda \in \Lambda$, the matrices

$$A_\lambda := \sum_{i=1}^{N} \lambda_i A_i, \quad b_\lambda := \sum_{i=1}^{N} \lambda_i b_i. \tag{2}$$

Assumption 1: There exists $\lambda \in \Lambda$ such that A_λ is Hurwitz and $b_\lambda = 0$.

Under Assumption 1, the existence of a quadratic Lyapunov function for the average system

$$\dot{x}(t) = A_\lambda x(t) + b_\lambda = A_\lambda x(t) \tag{3}$$

is guaranteed. In [14], a switching strategy based on the value of an arbitrary quadratic Lyapunov function for the average system (3) is proven to ensure Uniform Global Asymptotic Stability (UGAS) [19] under arbitrarily fast switching, and practical UGAS in the case of uniform dwell-time switching. The strategy consists of flowing on the current subsystem whenever the derivative of the Lyapunov function along the trajectories is sufficiently negative, and when it is not, switching to a subsystem that minimizes the derivative. The success of this strategy lies in the fact that for any state, there always exists a subsystem for which the derivative of the Lyapunov function is sufficiently negative.

Assumption 1 additionally ensures the existence of a polytopic Lyapunov function for the average system (3) [15]. The problem addressed in this paper is the following: will the strategy in [14] be successful if the quadratic Lyapunov function is replaced by an arbitrary polytopic Lyapunov function for the average dynamics?

B. Formulation with polytopic Lyapunov functions

To address this question, we first establish a necessary and sufficient condition for a symmetric polytopic function to be a Lyapunov function. Given $\Gamma \in \mathbb{R}^{r\times r}$, we define the associated Metzler[1] matrix $\mathcal{M}(\Gamma) \in \mathbb{R}^{r\times r}$ as the matrix whose entries satisfy

$$\mathcal{M}(\Gamma)_{i,k} = \begin{cases} \Gamma_{i,k} & \text{if } i = k, \\ |\Gamma_{i,k}| & \text{if } i \neq k. \end{cases} \tag{4}$$

The following lemma provides a characterization for symmetric polytopic Lyapunov functions. The proof is omitted due to space limitations.

Lemma 1: Let $A \in \mathbb{R}^{n\times n}$ and let $R \in \mathbb{R}^{r\times n}$ be full column rank. The following statements are equivalent:

i) $\Psi(x) := \|Rx\|_\infty$ is a Lyapunov function for $\dot{x} = Ax$;
ii) there exist $\Gamma \in \mathbb{R}^{r\times r}$ and $\beta > 0$ such that

$$\Gamma R = RA, \quad \mathcal{M}(\Gamma)\bar{\mathbf{1}} \preceq -\beta\bar{\mathbf{1}}. \tag{5}$$

Unlike quadratic functions, polytopic functions are not differentiable everywhere. Therefore, we will employ the upper-right Dini derivative instead of the classical derivative.

Remark 2.1: From the definition of beta-contractiveness in [20], condition (5) ensures that $D^+\Psi(x) \leq -\beta\Psi(x)$ for all $x \in \mathbb{R}^n$, where $D^+\Psi(x)$ is the upper-right Dini derivative of $\Psi(x) = \|Rx\|_\infty$ along the solutions of $\dot{x} = Ax$.

[1] A matrix $M \in \mathbb{R}^{r\times r}$ is Metzler if its off-diagonal entries are nonnegative, i.e. if $M_{i,j} \geq 0$ whenever $i \neq j$.

With these results in mind, let us extend the approach presented in [14]. Consider $\lambda \in \Lambda$ such that Assumption 1 holds and let $\Psi(x) := \|Rx\|_\infty$ be a polytopic Lyapunov function for the average system (3). Let $D_k^+\Psi(x)$ denote the upper-right Dini derivative of $\Psi(x)$ along solutions to (1) at a time instant t when $\sigma(t) = k$. Then, we have

$$D_k^+\Psi(x)=\begin{cases}\max\limits_{i\in\mathcal{I}_R(x)} \operatorname{sign}(R_{i,:}x)R_{i,:}(A_kx+b_k), & x\neq 0\\ \max\limits_{i\in\mathbb{N}_r} |R_{i,:}(A_kx+b_k)|, & x=0\end{cases} \tag{6}$$

where $R_{i,:}$ is the i-th row of R and

$$\mathcal{I}_R(x) := \{i \in \mathbb{N}_r : |R_{i,:}x| = \|Rx\|_\infty\}. \tag{7}$$

Replacing the quadratic Lyapunov function of [14] by the polytopic one $\Psi(x)$, we define the hybrid system $\mathcal{H} := (\mathcal{C}, f, \mathcal{D}, g)$, as follows:

$$\mathcal{H}:\begin{cases}\begin{bmatrix}\dot{x}' & \dot{\sigma}\end{bmatrix}' = f(x,\sigma), & (x,\sigma)\in\mathcal{C}\\ \begin{bmatrix}x^{+\prime} & \sigma^+\end{bmatrix}' \in g(x,\sigma), & (x,\sigma)\in\mathcal{D}\end{cases} \tag{8}$$

where the flow and jump maps f and g are

$$f(x,\sigma) := \begin{bmatrix}A_\sigma x + b_\sigma\\ 0\end{bmatrix} \tag{9a}$$

$$g(x,\sigma) := \begin{bmatrix}x\\ \arg\min_{k\in\mathbb{N}_N} D_k^+\Psi(x)\end{bmatrix} \tag{9b}$$

and the flow and jump sets $\mathcal{C}$ and $\mathcal{D}$ are defined by

$$\mathcal{C} := \left\{(x,\sigma) : D_\sigma^+\Psi(x) \le -\eta\beta\Psi(x)\right\} \tag{10a}$$

$$\mathcal{D} := \left\{(x,\sigma) : D_\sigma^+\Psi(x) \ge -\eta\beta\Psi(x)\right\} \tag{10b}$$

with $\eta \in (0,1)$ being a design parameter. In [14], the expressions on the right-hand side of the inequalities in (10) are equal to η times an upper bound on the derivative of the quadratic Lyapunov function along solutions of the average system. In the present case, the upper bound becomes $-\beta\Psi(x)$, as mentioned in Remark 2.1.

Note that by definition of $\mathcal{C}$ in (10a), the value of the Lyapunov function Ψ decreases whenever the solution of $\mathcal{H}$ flows. Additionally, as per (9b), the state x remains unchanged during jumps, as does the value of $\Psi(x)$. Therefore, the value of the Lyapunov function cannot increase while the solution exists, and consequently, divergence of solutions is not possible for system $\mathcal{H}$. However, in the following, we will show that this straightforward approach fails to provide the same stability guarantees presented in [14].

III. Challenges using polytopic functions

In this section, we will show that requiring Ψ to be a polytopic Lyapunov function for the average dynamics is not sufficient to ensure UGAS of system (8). First, we will show that there might exist states $x \neq 0$ for which there is no subsystem to which the solution can jump and subsequently flow. This may result in solutions that keep switching between subsystems infinitely, causing continuous time to stop flowing. This kind of solutions are usually referred to as eventually discrete [19]. The following numerical example illustrates this case.

Example 3.1 (Eventually discrete solutions): Consider a SAS (1) characterized by the matrices

$$A_1 := \begin{bmatrix}-1 & 2\\ -2 & -1\end{bmatrix}, \quad A_2 := \begin{bmatrix}-1 & -2\\ 2 & -1\end{bmatrix}, \tag{11}$$

and vectors $b_1 := b_2 := 0$. Note that both A_1 and A_2 are Hurwitz and that the origin is a common equilibrium point. Therefore, no switching is actually required to stabilize this system and it could be expected that any Lyapunov-function-based switching strategy should be successful. However, we next show that this is not so straightforward when polytopic Lyapunov functions are employed. Choose $\lambda := [0.5,\, 0.5]'$ so that $b_\lambda = 0$, the origin is an equilibrium point of the average system (3), and $A_\lambda = -I$ is Hurwitz. Therefore, Assumption 1 is satisfied. Given that $A := A_\lambda = -I$, we can select $R := I$ for which (5) holds with $\Gamma = -I$ and $\beta = 1$. Therefore, by Lemma 1, $\Psi(x) := \|Rx\|_\infty = \|x\|_\infty$ is a Lyapunov function for the average system. Define the set $\mathcal{S} := \{x \in \mathbb{R}^2\backslash\{0\} : |x_1| = |x_2|\}$. Then, for all $x \in \mathcal{S}$ we have $|R_{1,:}x| = |R_{2,:}x|$, so that $\mathcal{I}_R(x) = \{1, 2\}$ for all $x \in \mathcal{S}$. From (6) and (11), the Dini derivative of the Lyapunov function along the trajectories of subsystem 1 for all $x \in \mathcal{S}$ results

$$\begin{aligned}D_1^+\Psi(x) &= \max_{i\in\{1,\,2\}} \operatorname{sign}(R_{i,:}x)R_{i,:}\begin{bmatrix}-x_1+2x_2\\ -2x_1-x_2\end{bmatrix}\\ &= \max\{\operatorname{sign}(x_1)(-x_1+2x_2),\, \operatorname{sign}(x_2)(-2x_1-x_2)\}\\ &= \max\{-|x_1| + 2\operatorname{sign}(x_1)x_2,\, -|x_2| - 2\operatorname{sign}(x_2)x_1\}\\ &= -\frac{1}{\sqrt{2}}\|x\| + 2\max\{\operatorname{sign}(x_1)x_2,\, -\operatorname{sign}(x_2)x_1\}\\ &= -\frac{1}{\sqrt{2}}\|x\| + \frac{2}{\sqrt{2}}\|x\| = \frac{1}{\sqrt{2}}\|x\|.\end{aligned} \tag{12}$$

The fourth equality follows from the fact that, for all $x \in \mathcal{S}$, we have $\|x\| = \sqrt{x_1^2+x_2^2} = \sqrt{2}|x_1| = \sqrt{2}|x_2|$. The fifth equality is derived from analyzing the cases $\operatorname{sign}(x_1) \neq \operatorname{sign}(x_2)$ and $\operatorname{sign}(x_1) = \operatorname{sign}(x_2)$, and taking into account that $x \in \mathcal{S}$. Following an analogous procedure for subsystem 2, for all $x \in \mathcal{S}$ then

$$D_2^+\Psi(x) = \max_{i\in\{1,2\}} \operatorname{sign}(R_{i,:}x)R_{i,:}A_2x = \frac{1}{\sqrt{2}}\|x\|. \tag{13}$$

Then, $\min_{k\in\mathbb{N}_2} D_k^+\Psi(x) = \frac{1}{\sqrt{2}}\|x\| > 0$ for all $x \in \mathcal{S}$. Hence, by the definition of $\mathcal{C}$ in (10a), $(\mathcal{S}\times\{1,2\})\cap\mathcal{C} = \emptyset$. Consequently, if $x \in \mathcal{S}$ there is no subsystem to which the solution can jump in order to continue flowing. Moreover, by the definition of $\mathcal{D}$ in (10b), we have $(\mathcal{S}\times\{1,2\}) \subseteq \mathcal{D}$. Thus, any solution of $\mathcal{H}$ that reaches the set $\mathcal{S}$ will jump infinitely, whether or not σ changes, without advancement of continuous time, indicating the existence of eventually discrete solutions. ◦

The issue illustrated in the previous example also arises when the system given by matrices (11) is affine with $b_1 \neq 0$ and $b_2 \neq 0$. Indeed, it is easy to prove that in such a case $D_1^+\Psi(x) > 0$ and $D_2^+\Psi(x) > 0$ for all $x \in \mathbb{R}^2$ such that $\|x\| > \sqrt{2}\max_{k\in\mathbb{N}_2}\|b_k\|_\infty$. To formalize the occurrence of this issue for a general case, let us define the flow sets for each subsystem k as

$$\mathcal{C}_k := \{x \in \mathbb{R}^n : (x, k) \in \mathcal{C}\}. \tag{14}$$

Then, the eventually discrete solutions appear whenever the union $\cup_{k\in\mathbb{N}_N}\mathcal{C}_k$ fails to cover $\mathbb{R}^n\backslash\{0\}$.

One possible approach to avoid an infinite number of jumps at the same time instant is to impose a minimum dwell time T [5], [11], [14]. To do this, a new hybrid control strategy will be defined. In this new scheme, the jumps that would have occurred according to the previous logic are omitted if a dwell-time period has not elapsed since the last jump. Therefore, after a jump, the solution flows within the same subsystem for at least T units of time. However, in the following example we will show that this strategy can also fail, leading to unstable solutions.

Example 3.2 (Instability): Consider the SAS of Example 3.1 and add a third subsystem defined by $A_3 := \frac{1}{2}I$ and $b_3 := 0$. Selecting $\lambda := [0.5,\ 0.5,\ 0]'$ we obtain the same average system as in Example 3.1 and Assumption 1 is satisfied. Further, condition (5) holds with the same matrices $R = I$, $\Gamma = -I$ and $\beta = 1$ as in Example 3.1, and $\Psi(x) := \|x\|_\infty$ is a Lyapunov function for the average system. Consider the set $\mathcal{S}$ from Example 3.1. Thus, from (12) and (13), we have

$$D_1^+\Psi(x) = D_2^+\Psi(x) = \frac{1}{\sqrt{2}}\|x\| \tag{15}$$

for all $x \in \mathcal{S}$. For the third subsystem, for all $x \in \mathcal{S}$ then

$$\begin{aligned} D_3^+\Psi(x) &= \max_{i\in\{1,2\}} \operatorname{sign}(R_{i,:}x)R_{i,:}\frac{1}{2}x \\ &= \frac{1}{2}\max\{|x_1|, |x_2|\} = \frac{1}{2\sqrt{2}}\|x\|. \end{aligned} \tag{16}$$

In the last equality we used the fact that $\|x\| = \sqrt{2}|x_1| = \sqrt{2}|x_2|$ for all $x \in \mathcal{S}$. Hence, we have $\min_{k\in\mathbb{N}_3} D_k^+\Psi(x) = \frac{1}{2\sqrt{2}}\|x\| > 0$ for all $x \in \mathcal{S}$. Then, as in Example 3.1, any solution of $\mathcal{H}$ that reaches the set $\mathcal{S}$ will be eventually discrete. In an attempt to overcome this, let us consider the time regularized hybrid system $\mathcal{H}_T$ defined as

$$\mathcal{H}_T : \begin{cases} \begin{bmatrix}\dot{x}' & \dot{\sigma} & \dot{\tau}\end{bmatrix}' = \begin{bmatrix} f(x,\sigma) \\ 1 \end{bmatrix}, & (x,\sigma,\tau) \in \mathcal{C}_T \\ \begin{bmatrix}x^{+\prime} & \sigma^+ & \tau^+\end{bmatrix}' \in \begin{bmatrix} g(x,\sigma) \\ 0 \end{bmatrix}, & (x,\sigma,\tau) \in \mathcal{D}_T \end{cases} \tag{17}$$

where the maps f and g are as defined in (9), and the flow and jump sets $\mathcal{C}_T$ and $\mathcal{D}_T$ are defined by the regions

$$\mathcal{C}_T := \{(x,\sigma,\tau) : (x,\sigma) \in \mathcal{C}\ \vee\ \tau \in [0, T]\} \tag{18a}$$
$$\mathcal{D}_T := \{(x,\sigma,\tau) : (x,\sigma) \in \mathcal{D}\ \wedge\ \tau \in [T, \infty)\} \tag{18b}$$

with $\mathcal{C}$ and $\mathcal{D}$ as in (10), and $T > 0$ the predefined dwell time. According to this new approach, after a jump, the timer τ must reach the minimum dwell-time T before the solution can jump again. From (15) and (16), we have $\arg\min_{k\in\mathbb{N}_3} D_k^+\Psi(x) = 3$ for all $x \in \mathcal{S}$, indicating that from any state $x \in \mathcal{S}$, if the dwell time has expired, we will jump to the third subsystem. Further, the solution to the third subsystem starting from $x_0 \in \mathbb{R}^n$ is $x(t) = x_0 e^{t/2}$. Hence, if $x_0 \in \mathcal{S}$, then $x(t) \in \mathcal{S}$ for all $t \geq 0$, implying that $\mathcal{S}$ is invariant under the dynamics of the third subsystem. Consequently, any solution to $\mathcal{H}_T$ that reaches the set $\mathcal{S}$ with $\tau \geq T$ will jump to the third subsystem and will flow according to its dynamics. Due to the invariance of $\mathcal{S}$, the solution will remain in $\mathcal{S}$. After T units of time, the solution will jump, remaining in the third subsystem but restarting the timer, which leads to an analogous situation as the initial condition. This behaviour repeats itself, ultimately resulting in the divergence of the solution. ◦

Note that imposing a dwell time as in $\mathcal{H}_T$, while eliminating eventually discrete solutions, may introduce divergent solutions which cannot occur in $\mathcal{H}$.

IV. Suitable polytopic Lyapunov functions

In this section, we present conditions under which eventually discrete solutions are avoided for the system $\mathcal{H}$. To introduce these conditions, we first present an example where an *ad hoc* selection of the matrix R, defining the polytopic Lyapunov function, satisfies the mentioned conditions.

Example 4.1 (Suitable Lyapunov function): Consider a SAS (1) with matrices A_1 and A_2 defined as in (11), and vectors $b_1 := [2.5,\ 0]' =: -b_2$. Select the same λ as in Example 3.1, then we have the same average system, and Assumption 1 holds. Now, instead of choosing R as the identity matrix, select

$$R := \begin{bmatrix} 1 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}'. \tag{19}$$

Since $A := A_\lambda = -I$, condition (5) holds with $\Gamma = -I$ and $\beta = 1$. Then, $\Psi(x) := \|Rx\|_\infty$ is a Lyapunov function for the average system.

We aim to ensure that the union of the flow sets $\cup_{k\in\mathbb{N}_2}\mathcal{C}_k$ covers $\mathbb{R}^2\backslash\{0\}$, with $\mathcal{C}_k$ as in (14), to avoid the eventually discrete solutions that arise in Example 3.1. To achieve this, we will show that

$$\min_{k\in\mathbb{N}_2} D_k^+\Psi(x) \leq -\beta\Psi(x) \tag{20}$$

for all $x \neq 0$. In other words, this means that for all $x \neq 0$ there exists a subsystem $k \in \mathbb{N}_2$ such that $D_k^+\Psi(x) \leq -\beta\Psi(x) < -\eta\beta\Psi(x)$ for every $\eta \in (0, 1)$, implying that $x \in \mathcal{C}_k$. First, let us show that the aforementioned holds true for all $x \neq 0$ such that $\mathcal{I}_R(x)$ contains only one element. Let $x \in \mathbb{R}^2 \setminus \{0\}$ and let $\mathcal{I}_R(x) = \{i\}$. From (6), we obtain

$$\begin{aligned} \min_{k\in\mathbb{N}_2} D_k^+\Psi(x) &= \min_{k\in\mathbb{N}_2} \operatorname{sign}(R_{i,:}x)R_{i,:}(A_k x + b_k) \\ &= \min_{\lambda\in\Lambda} \operatorname{sign}(R_{i,:}x)R_{i,:}(A_\lambda x + b_\lambda) \leq -\beta\Psi(x) \end{aligned} \tag{21}$$

The second equality follows from the fact that for all $x \in \mathbb{R}^2$ results $\min_{k\in\mathbb{N}_2} x_k = \min_{\lambda\in\Lambda} \sum_{k=1}^{2} \lambda_k x_k$. The last inequality is derived from Remark 2.1 and the fact that the selected λ belongs to Λ. Therefore, (20) is satisfied for all $x \neq 0$ such that $\mathcal{I}_R(x)$ contains only one element. Now, let us examine whether that condition holds true for the set of states $x \neq 0$ where $\mathcal{I}_R(x)$ has more than one element, denoted as $\hat{\mathcal{S}}$. In a two-dimensional SAS, $\hat{\mathcal{S}}$ is one-dimensional. Consequently, it can be parameterized by a single real variable, α, simplifying the verification of the fulfillment of (20) over $\hat{\mathcal{S}}$. Based on this, we express $\hat{\mathcal{S}}$ as

$$\hat{\mathcal{S}} := \{x = \alpha v : \alpha > 0,\, v \in \mathcal{V}\}, \text{ where} \quad (22)$$
$$\mathcal{V} := \{v : |R_{i,:}v| = |R_{j,:}v| = \|Rv\|_\infty = 1,\; i \neq j\} \quad (23)$$

is the set of vertices $v \in \mathbb{R}^2$ of the Lyapunov function's level curve given by $\Psi(x) = 1$. Additionally, define the set

$$\mathcal{A} := \{\alpha > 0 : \min_{k\in\mathbb{N}_2} D_k^+\Psi(\alpha v) \leq -\beta\Psi(\alpha v)\, \forall v \in \mathcal{V}\}. \quad (24)$$

Note that (20) holds for all $x \in \hat{\mathcal{S}}$ if and only if $\mathcal{A} = \mathbb{R}_{>0}$. Then, we can verify the condition over $x \in \hat{\mathcal{S}}$ by computing $\mathcal{A}$. To simplify the computation, define

$$d_{k,i}(v,\alpha) := \operatorname{sign}(R_{i,:}v)R_{i,:}(A_k v\alpha + b_k). \quad (25)$$

Then, from (24) we can express

$$\begin{aligned}\mathcal{A} &= \bigcap_{v\in\mathcal{V}} \bigcup_{k\in\mathbb{N}_2} \{\alpha > 0 : D_k^+\Psi(\alpha v) \leq -\beta\Psi(\alpha v)\} \\ &= \bigcap_{v\in\mathcal{V}} \bigcup_{k\in\mathbb{N}_2} \bigcap_{i\in\mathcal{I}_R(v)} \{\alpha > 0 : d_{k,i}(v,\alpha) \leq -\beta\alpha\}. \quad (26)\end{aligned}$$

The first equality follows from the facts that requiring the inequality in (24) to be met for all $v \in \mathcal{V}$ is equivalent to its fulfillment at the intersection, and ensuring that the minimum over $k \in \mathbb{N}_2$ satisfies the inequality is equivalent to its fulfillment at the union. The second equality arises from (6) and (25), with the substitution of the maximum in (6) by the intersection, and considering that $\mathcal{I}_R(\alpha v) = \mathcal{I}_R(v)$ and $\Psi(\alpha v) = \alpha$ for all $\alpha > 0$ and $v \in \mathcal{V}$. Then, we can compute $\mathcal{A}$ from (26) by solving linear inequalities and operating on their solutions. It can be checked through symbolic calculus that $\mathcal{A} = \mathbb{R}_{>0}$, which implies (20) holds for all $x \in \hat{\mathcal{S}}$. Combining this with the previous result, we have that (20) holds for all $x \in \mathbb{R}^2\backslash\{0\}$. Then, the union of flow sets covers $\mathbb{R}^2\backslash\{0\}$, and consequently, from any $(x,\sigma) \in \mathcal{D}$ with $x \neq 0$ we jump to $(x,\sigma^+) \in \mathcal{C}$. Thus, from (9b) and (20), after the jump we have $D_{\sigma^+}^+\Psi(x) \leq -\beta\Psi(x)$. Moreover, it can be proved that there exists a neighborhood $\mathcal{N}$ of x such that

$$D_{\sigma^+}^+\Psi(y) < -\eta\beta\Psi(y) \quad (27)$$

for all $y \in \mathcal{N}$ and $\eta \in (0,1)$. The proof of this assertion uses the fact that for all $x \neq 0$ there exists $\mathcal{N}$ such that $\mathcal{I}_R(y) \subseteq \mathcal{I}_R(x)$ for all $y \in \mathcal{N}$, and is omitted for the sake of brevity. Thus, from (27) and the definition of $\mathcal{D}$ in (10b), it follows that the solution must flow for some time before jumping again. Therefore, there are no eventually discrete solutions outside the origin for $\mathcal{H}$. Note that to prove UGAS, Zeno solutions still have to be ruled out, which could be verified by means of the persistent flow property [19]. ○

The condition (20) can be generalized for any SAS as

$$\min_{k\in\mathbb{N}_N} D_k^+\Psi(x) \leq -\beta\Psi(x) \quad (28)$$

for all $x \in \mathbb{R}^n\backslash\{0\}$. Then, following the same reasoning as in the example, (28) leads to the absence of eventually discrete solutions outside the origin for the system $\mathcal{H}$. Furthermore, the conclusion derived from (21), asserting that the bound in (28) holds for all $x \neq 0$ such that $\mathcal{I}_R(x)$ contains only one element, can be easily extended to any SAS. Therefore, in order to avoid eventually discrete solutions, our focus should be on checking the fulfillment of the bound in (28) for the states x where the cardinality of $\mathcal{I}_R(x)$ is greater than one. As illustrated in Example 4.1, in the case of two-dimensional SAS, this can be checked via the computation of $\mathcal{A}$. Algorithm 1 summarizes the procedure of computing $\mathcal{A}$ for any two-dimensional SAS and any matrix R defining the Lyapunov function, allowing us to verify the fulfillment of (28), and consequently the absence of eventually discrete solutions outside the origin for those SAS.

Algorithm 1: Computation of $\mathcal{A}$ (for $n = 2$)

Input: R, β, and A_k, b_k for $k \in \mathbb{N}_N$

```
Compute V from R
A ← R_{>0}
forall v ∈ V do
    A_v ← ∅
    forall k ∈ N_N do
        A_{v,k} ← R_{>0}
        forall i ∈ I_R(v) do
            A_{v,k} ← A_{v,k} ∩ {α > 0 : d_{k,i}(v, α) ≤ −βα}
        A_v ← A_v ∪ A_{v,k}
    A ← A ∩ A_v
return A
```

V. Simulation Results

In this section, the assessment of the examples previously presented is shown in simulation.

In Examples 3.1 and 4.1, the initial conditions are set equal to $x(0) = [0,\, -2.25]'$ with $\sigma(0) = 2$, while for Example 3.2 the initial conditions are $x(0) = [0,\, -2.25]'$ with $\sigma(0) = 2$ and $\tau(0) = 0$. Parameter $\eta = 0.99$ has been chosen for the three examples.

In Example 3.1, we have shown that choosing matrix R as the identity matrix yields an eventually discrete solution. Indeed, in Fig. 1, it is shown that the state stops flowing (dashed blue line) as soon as the set $\mathcal{S}$ is reached due to the discrete behavior defined in (9). Fig. 2 shows that as soon as the state reaches $\mathcal{S}$, the switching variable $\sigma(t)$ randomly switches between indices 1 and 2 due to $D_1^+\Psi(x) = D_2^+\Psi(x)$, while the continuous time variable t stops flowing. Note that when $x \in \mathcal{S}$, infinite solutions are possible since both $\sigma^+ = 1$ and $\sigma^+ = 2$ are acceptable. Hence, the solution shown in Fig. 2 is only one of the possible solutions.

Regarding Example 3.2, selecting the dwell time $T = 0.1$ and matrix R again as the identity matrix, generates an unstable behavior. Specifically, as proved in Section III, the set $\mathcal{S}$ is an invariant set for the solution. Indeed, as illustrated again in Fig. 1, once the state trajectory (red) enters the set $\mathcal{S}$, it remains on it and diverges along the direction of the set $\mathcal{S}$ itself. As detailed in Example 3.2, the third subsystem is always selected due to its lower value of the Dini derivative (red plot in Fig. 2).

Finally, as for Example 4.1, choosing R as in (19), the state trajectory converges towards the origin switching among the modes with increasing frequency, thus avoiding the ill behavior from the previous examples. This behavior can also be appreciated from the solution of the switching variable (yellow in Fig. 2).

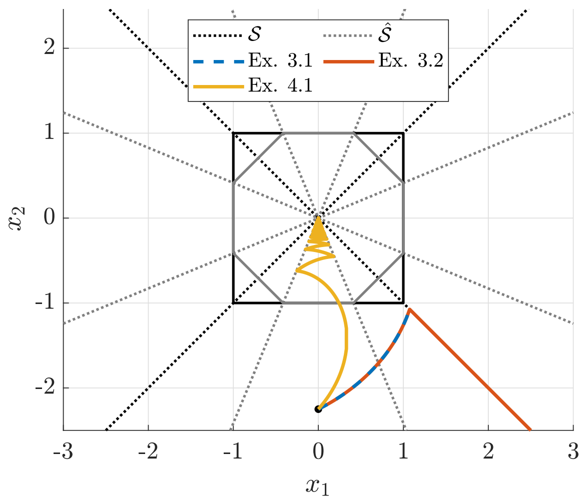

Fig. 1. Solution of the state for Example 3.1 (blue dashed line), Example 3.2 (red line) and Example 4.1 (yellow line). The represented level sets are $\|Rx\|_\infty = 1$ for $R = I$ (black square) and $\|Rx\|_\infty = 1$ for R as in (19) (grey octagon), with the set $\mathcal{S}$ (black dotted lines) and the set $\hat{\mathcal{S}}$ (grey dotted lines).

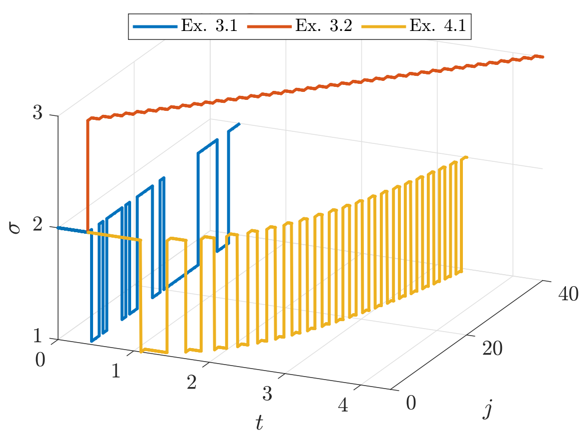

Fig. 2. Solution of the switching variable $\sigma(t, j)$ for Example 3.1 (blue line), Example 3.2 (red line) and Example 4.1 (yellow line), where t is continuous time and j the number of switches or hybrid jumps.

VI. Conclusions

This paper contributes to the literature of switched affine systems relying on switching strategies based on polytopic Lyapunov functions. Making reference to the proposed approaches in [14], where quadratic Lyapunov functions are adopted, in this paper we have shown that the extension to polytopic functions is not straightforward. To this end, we have theoretically discussed three examples. We have also provided sufficient conditions that guarantee the absence of eventually discrete or divergent solutions.

Future research will be devoted to finding more general conditions to ensure stability in the case of higher-dimensional systems.

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