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Heisenberg formulation of adiabatic elimination for open quantum systems with two timescales

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Abstract

Consider an open quantum system governed by a Gorini–Kossakowski–Sudarshan–Lindblad master equation with two times-scales: a fast one, exponentially converging towards a linear subspace of quasi-equilibria; a slow one resulting from small decoherence and Hamiltonian dynamics. Usually adiabatic elimination is performed in the Schrödinger picture. We propose here a Heisenberg formulation where the invariant operators attached to the fast decay dynamics towards the quasi-equilibria subspace play a key role. Based on geometric singular perturbations, asymptotic expansions of the Heisenberg slow dynamics and of the fast invariant linear subspaces are proposed. They exploit Carr's approximation lemma from center-manifold and bifurcation theory. Second-order expansions are detailed and shown to ensure preservation, up to second-order terms, of the complete positivity for the slow propagator on a slow timescale. Such expansions can be exploited numerically to derive reduced-order dynamical models.

1. Introduction

In the quantum physics community, adiabatic elimination is widely used to analyze the dynamics of open and dissipative quantum systems (see e.g. [1, 2, 3, 4, 5]. It corresponds in fact to a perturbation technique known in dynamical and control system theory as singular perturbations for slow/fast systems. It is related to the Tikhonov approximation theorem (see, e.g., [6, 7]) and its coordinate-free formulation due to Fenichel [8]. The notion of invariant slow manifolds plays a crucial role for dynamical systems with two timescales dynamics: the fast and exponentially converging ones and the slow ones of reduced dimension. In this context, adiabatic elimination produces low dimensional dynamical models via the derivation from the original slow/fast differential equations of the slow differential equations governing the evolution on the invariant slow manifold. For open quantum system governed by the deterministic Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) master equation, adiabatic elimination is usually performed in the Schrödinger picture.

We perform here adiabatic elimination in the Heisenberg picture where the invariant operators associated to the fast dynamics play a crucial role: they are used to describe the slow dynamics but also to define the equations characterizing the fast invariant linear subspace. As far as we know, such Heisenberg point of view has not been considered in such a systematic and general way, despite the fact that, for stochastic quantum systems, the Heisenberg stochastic evolutions play a central role (see, e.g., [9, 10]). In particular, our derivation relies on very general two timescale assumptions: we only assume an exponentially fast convergence towards a linear subspace of quasi-equilibria those structure does not necessarily correspond to a decoherence free subspace; the slow dynamics can result from arbitrary perturbations either Hamiltonian or Lindbladian; we do not assume a tensor-product structure where the fast decay is due to local decoherence fast dynamics of some sub-systems.

Combining two asymptotic expansions, a first one for the slow dynamics and a complementary one providing the set of linear equations characterizing the exponentially fast decaying subspace, we show how to approximate up to exponentially small corrections the propagator over a slow timescale (see lemma 3). We explain how to compute the order n corrections knowing the correction of order $r < n$. The second-order approximation of the slow dynamics is shown to preserve complete positivity in the following sense: its second-order propagator over a slow timescale corresponds, up to second-order correction, to a Trace Preserving and Completely Positive (TPCP) map (see lemma 4). Such preservation has been shown in specific cases for the second-order as in [3, 4] or for first-order as in [11, 12].

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In section 2, the slow/fast structure of the GKLS differential equations is detailed either in Schrödinger picture with an orthonormal basis $(\widehat{S}_d)_{1\leq d\leq \bar{d}}$ for quasi-equilibria quantum states but also in the Heisenberg picture with the associated basis of quasi-invariant operators $(\widehat{J}_d)_{1\leq d\leq \bar{d}}$. In section 3, we detail the asymptotic expansion of the slow dynamics with a Heisenberg point of view. In section 4, the set of independent linear equations describing the fast invariant subspace is constructed and its approximation at any order is given. Section 5 combines lemma 1 of section 3 and lemma 2 of section 4 to prove lemmas 3 and 4, the approximate TPCP character of slow propagators over a slow timescale.

Throughout this paper, the underlying Hilbert space $\mathscr{H}$ is assumed to be of finite dimension. This ensures uniqueness, existence and convergence of these asymptotic expansions versus the small parameter ε. The calculations below use the language of operators. Thus, they can be used, at least formally, even for an infinite dimensional Hilbert space despite the fact that precise mathematical justifications relying on functional analysis methods are not straightforward.

2. Slow/fast dynamics

2.1. Singular perturbations for finite dimensional, linear and time-invariant systems

Take a linear time-invariant system of finite dimension

$$\frac{d}{dt}\xi = (A_0+\varepsilon A_1)\xi$$

where ξ is a real vector of finite dimension $\bar{D}$, A_0 and A_1 are $\bar{D}\times\bar{D}$ matrices with real entries and ε is a small parameter. Assuming a slow/fast structure means that A_0 can be block diagonalized in two blocks:

$$A_0 = P_0\begin{pmatrix} 0 & 0 \\ 0 & \Gamma_0 \end{pmatrix}P_0^{-1}$$

where P_0 is invertible and Γ_0 is a Hurwitz (stable) matrix of dimension $\bar{D}-\bar{d}>0$, the dimension of the fast dynamics and where $\bar{d}>0$ is the dimension of the slow dynamics. Standard perturbation theory (see [14]) ensures that, for ε small enough, one has a similar block decomposition:

$$A_0+\varepsilon A_1 = P(\varepsilon)\begin{pmatrix} \Delta(\varepsilon) & 0 \\ 0 & \Gamma(\varepsilon) \end{pmatrix}P^{-1}(\varepsilon)$$

where the matrices $P(\varepsilon)$, $\Delta(\varepsilon)$ and $\Gamma(\varepsilon)$ are analytic versus ε with $P(0)=P_0$, $\Delta(0)=0$ and $\Gamma(0)=\Gamma_0$. Geometrically, exists, for ε small enough, two invariant linear subspaces:

- the slow one, of dimension $\bar{d}$, corresponding to the slow evolution governed by the propagator $e^{t\Delta(\varepsilon)}$
- the fast one, of dimension $\bar{D}-\bar{d}$, corresponding to the fast and exponentially stable evolution governed by the propagator $e^{t\Gamma(\varepsilon)}$.

2.2. Slow/fast GKSL quantum dynamics

All the developments below combine the above dynamics structure with non-commutative computations with operators used to describe the decoherence dynamics of open-quantum systems.

Consider the time-varying density operator ρ_t on underlying Hilbert space $\mathscr{H}$ of finite dimension obeying to the following dynamics

$$\frac{d}{dt}\rho_t = \mathscr{L}_0(\rho_t)+\varepsilon\mathscr{L}_1(\rho_t) \tag{1}$$

where ε is a small positive parameter and where the GKSL linear super-operators $\mathscr{L}_0$ and $\mathscr{L}_1$ read ($\sigma = 0,1$)

$$\mathscr{L}_\sigma(\rho) = -i[\widehat{H}_\sigma,\rho]\ldots \\ +\sum_\nu \widehat{L}_{\sigma,\nu}\rho\widehat{L}_{\sigma,\nu}^\dagger - \tfrac{1}{2}\left(\widehat{L}_{\sigma,\nu}^\dagger\widehat{L}_{\sigma,\nu}\rho+\rho\widehat{L}_{\sigma,\nu}^\dagger\widehat{L}_{\sigma,\nu}\right)$$

with $\widehat{H}_\sigma$ Hermitian operator and $\widehat{L}_{\sigma,\nu}$ any operator not necessarily Hermitian.

Assume that for $\varepsilon=0$ and any initial condition ρ_0, ρ_t converges exponentially towards a steady state depending a priori on ρ_0. This means that we have a TPCP map $\overline{\mathscr{K}}_0$ such that for any ρ_0:

$$\lim_{t\mapsto+\infty} e^{t\mathscr{L}_0}(\rho_0) \triangleq \overline{\mathscr{K}}_0(\rho_0) \tag{2}$$

The range of $\overline{\mathscr{K}}_0$ is denoted by $\mathscr{D}_0$, the linear space of equilibria for $\mathscr{L}_0$ corresponding to its kernel. Denote by $\bar{d}$ the dimension of $\mathscr{D}_0$ and consider an orthonormal basis of $\mathscr{D}_0$ made of $\bar{d}$ Hermitian operators $\widehat{S}_1,\ldots,\widehat{S}_{\bar{d}}$ such that $\mathrm{Tr}\left(\widehat{\mathrm{S}}_{\mathrm{d}}\widehat{\mathrm{S}}_{\mathrm{d'}}\right)=\delta_{\mathrm{d,d'}}$. To each $\widehat{S}_d$ is associated an invariant operator $\widehat{J}_d=\lim_{t\mapsto+\infty} e^{t\mathscr{L}_0^*}(\widehat{S}_d)$ being a steady-state of the adjoint dynamics (according to the Frobenius Hermitian product) $\frac{d}{dt}\widehat{J}=\mathscr{L}_0^*(\widehat{J})$ where $\mathscr{L}_0^*$ is the adjoint of $\mathscr{L}_0$ (see, e.g., [13]). For any solution ρ_t of (1) with $\varepsilon=0$, $\mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}}\rho_{\mathrm{t}}\right)$ is constant since $\mathscr{L}_0^*(\widehat{J}_d)=0$ implies that

$$\frac{d}{dt}\mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}}\rho_{\mathrm{t}}\right)=\mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}}\mathscr{L}_0(\rho_{\mathrm{t}})\right)=\mathrm{Tr}\left(\mathscr{L}_0^*(\widehat{\mathrm{J}}_{\mathrm{d}})\rho_{\mathrm{t}}\right)=0.$$

Thus, one has:

$$\lim_{t\mapsto+\infty}\rho_t = \sum_{d=1}^{\bar{d}}\mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}}\rho_0\right)\widehat{\mathrm{S}}_{\mathrm{d}} \triangleq \overline{\mathscr{K}}_0(\rho_0). \tag{3}$$

Moreover $\mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}}\widehat{\mathrm{S}}_{\mathrm{d}'}\right) = \delta_{\mathrm{d},\mathrm{d}'}$ since for any $t > 0$

$$\mathrm{Tr}\left(\mathrm{e}^{t\mathscr{L}_0^*}(\widehat{\mathrm{S}}_{\mathrm{d}})\,\widehat{\mathrm{S}}_{\mathrm{d}'}\right) = \mathrm{Tr}\left(\widehat{\mathrm{S}}_{\mathrm{d}}\,\mathrm{e}^{t\mathscr{L}_0}(\widehat{\mathrm{S}}_{\mathrm{d}'})\right)$$

and $e^{t\mathscr{L}_0}(\widehat{S}_{d'}) = \widehat{S}_{d'}$. Notice that the operator subspace of co-dimension $\bar{d}$ defined by

$$\left\{\rho \mid \forall d \in \{1,\ldots,\bar{d}\},\ \mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}}\rho\right) = 0\right\}$$

corresponds to the set of trajectories exponentially converging to 0, i.e. the fast invariant subspace when $\varepsilon = 0$.

3. Asymptotic expansion of the slow dynamics

For $\varepsilon > 0$ and small, (1) admits also a $\bar{d}$ dimensional linear invariant subspace denoted by $\mathscr{D}_\varepsilon$ and close to $\mathscr{D}_0$ (see [14] for a mathematical justification). This means that the set of real variables $x_1 = \mathrm{Tr}\left(\widehat{\mathrm{J}}_1\rho\right)$, $\ldots$, $x_{\bar{d}} = \mathrm{Tr}\left(\widehat{\mathrm{J}}_{\bar{\mathrm{d}}}\rho\right)$ can be chosen as local coordinates on $\mathscr{D}_\varepsilon$ with the perturbed operator basis $\widehat{S}_1(\varepsilon), \ldots \widehat{S}_{\bar{d}}(\varepsilon)$. If at some time t, the solution ρ_t of the perturbed system (1), belongs to $\mathscr{D}_\varepsilon$, it remains on $\mathscr{D}_\varepsilon$ at any time: $\frac{d}{dt}\rho_t = (\mathscr{L}_0 + \varepsilon\mathscr{L}_1)(\rho_t)$ where $\rho_t = \sum_{d=1}^{\bar{d}} x_d(t)\widehat{S}_d(\varepsilon)$. Thus, for any $(x_1(t),\ldots,x_{\bar{d}}(t)) \in \mathbb{R}^{\bar{d}}$, this invariance property reads

$$\sum_{d=1}^{\bar{d}} \frac{dx_d}{dt}\,\widehat{S}_d(\varepsilon) = (\mathscr{L}_0 + \varepsilon\mathscr{L}_1)\left(\sum_{d=1}^{\bar{d}} x_d\widehat{S}_d(\varepsilon)\right). \quad (4)$$

Thus, for any $d \in \{1,\ldots,\bar{d}\}$, $\frac{dx_d}{dt}$ depends linearly on $x = (x_1,\ldots,x_{\bar{d}})$, i.e.

$$\frac{d}{dt}x_d = \sum_{d'} F_{d,d'}(\varepsilon)x_{d'}$$

where $F_{d,d'}(\varepsilon)$ are real coefficients to be chosen in order to satisfy the invariance conditions:

$$\sum_{d'=1}^{\bar{d}} F_{d',d}(\varepsilon)\widehat{S}_{d'}(\varepsilon) = (\mathscr{L}_0 + \varepsilon\mathscr{L}_1)(\widehat{S}_d(\varepsilon))$$

for all $d \in \{1,\ldots,\bar{d}\}$. With the asymptotic expansion

$$F_{d,d'}(\varepsilon) = \sum_{n\geq 0} \varepsilon^n F_{d,d'}^{(n)}, \quad \widehat{S}_d(\varepsilon) = \sum_{n\geq 0} \varepsilon^n \widehat{S}_d^{(n)}$$

one can compute recursively $F_{d,d'}^{(n)}$ and $\widehat{S}_d^{(n)}$ from $F_{d,d'}^{(m)}$ and $\widehat{S}_d^{(m)}$ with $m < n$. The recurrence relationship is based on the identification of terms with same orders versus ε in the following equations: $\forall d \in \{1,\ldots,\bar{d}\}$

$$\sum_{d'=1}^{\bar{d}} \left(\sum_{n\geq 0} \varepsilon^n F_{d',d}^{(n)}\right)\left(\sum_{n'\geq 0} \varepsilon^{n'} \widehat{S}_{d'}^{(n')}\right)\ldots$$
$$= (\mathscr{L}_0 + \varepsilon\mathscr{L}_1)\left(\sum_{n\geq 0} \varepsilon^n \widehat{S}_d^{(n)}\right).$$

The zero-order term is satisfied with

$$F_{d,d'}^{(0)} = 0, \quad \widehat{S}_d^{(0)} = \widehat{S}_d.$$

First-order conditions read: $\forall d \in \{1,\ldots,\bar{d}\}$

$$\sum_{d''=1}^{\bar{d}} F_{d'',d}^{(1)}\widehat{S}_{d''}^{(0)} = \mathscr{L}_0(\widehat{S}_d^{(1)}) + \mathscr{L}_1(\widehat{S}_d^{(0)}).$$

Left multiplication by operator $\widehat{J}_{d'}$ and taking the trace yields

$$F_{d',d}^{(1)} = \mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}'}\mathscr{L}_1(\widehat{\mathrm{S}}_{\mathrm{d}}^{(0)})\right) \quad (5)$$

since $\mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}'}\widehat{\mathrm{S}}_{\mathrm{d}''}^{(0)}\right) = \delta_{\mathrm{d}',\mathrm{d}''}$ and $\mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}'}\mathscr{L}_0(\widehat{\mathrm{W}})\right) = 0$ for any operator $\widehat{W}$ because $\mathscr{L}_0^*(\widehat{J}_{d'}) = 0$. Thus, $\widehat{S}_d^{(1)}$ is a solution $\widehat{X}$ of the following equation:

$$\mathscr{L}_0(\widehat{X}) = \sum_{d'} \mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}'}\mathscr{L}_1(\widehat{\mathrm{S}}_{\mathrm{d}}^{(0)})\right)\widehat{\mathrm{S}}_{\mathrm{d}'} - \mathscr{L}_1(\widehat{\mathrm{S}}_{\mathrm{d}}^{(0)})\ldots$$
$$= \overline{\mathscr{K}}_0\big(\mathscr{L}_1(\widehat{S}_d^{(0)})\big) - \mathscr{L}_1(\widehat{S}_d^{(0)})$$

where the quantum channel $\overline{\mathscr{K}}_0$ is defined in (2). Following [4], the general solution $\widehat{X}$ is given by the absolutely converging integral,

$$\int_0^{+\infty} e^{s\mathscr{L}_0}\left(\mathscr{L}_1(\widehat{S}_d^{(0)}) - \overline{\mathscr{K}}_0\big(\mathscr{L}_1(\widehat{S}_d^{(0)})\big)\right)\,ds + \widehat{W}$$

where $\widehat{W}$ belongs to $\mathscr{D}_0$ the kernel of $\mathscr{L}_0$. We consider here the solution with $\widehat{W} = 0$ and thus

$$\widehat{S}_d^{(1)} = \int_0^{+\infty} e^{s\mathscr{L}_0}\left(\mathscr{L}_1(\widehat{S}_d^{(0)}) - \overline{\mathscr{K}}_0\big(\mathscr{L}_1(\widehat{S}_d^{(0)})\big)\right)\,ds$$

where for all d', $\mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}'}\widehat{\mathrm{S}}_{\mathrm{d}}^{(1)}\right) = 0$. The super-operator $\overline{\mathscr{R}}_0$ defined for any operator $\widehat{W}$ by

$$\overline{\mathscr{R}}_0(\widehat{W}) = \int_0^{+\infty} e^{s\mathscr{L}_0}\left(\widehat{W} - \overline{\mathscr{K}}_0(\widehat{W})\right)\,ds \quad (6)$$

provides thus the unique solution $\widehat{X} = \overline{\mathscr{R}}_0(\widehat{W})$ of $\mathscr{L}_0(\widehat{X}) = \overline{\mathscr{K}}_0(\widehat{W}) - \widehat{W}$ such that for all d, $\mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}}\widehat{\mathrm{X}}\right) = 0$. To summarize the above calculation:

$$\widehat{S}_d^{(1)} = \overline{\mathscr{R}}_0\big(\mathscr{L}_1(\widehat{S}_d)\big). \quad (7)$$

Take $n \geq 2$ and assume that we have computed all the terms $F^{(r)}_{d',d}$ and $\widehat{S}^{(r)}_{d'}$ of order $r < n$ with $\mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d'}}\widehat{\mathrm{S}}^{(\mathrm{r})}_{\mathrm{d}}\right) = 0$ for all d and d'. Invariance conditions of order n read: $\forall d \in \{1,\ldots,\bar{d}\}$

$$\sum_{d''=1}^{\bar{d}} \sum_{r=1}^{n} F^{(r)}_{d'',d} \widehat{S}^{(n-r)}_{d''} = \mathscr{L}_0(\widehat{S}^{(n)}_d) + \mathscr{L}_1(\widehat{S}^{(n-1)}_d).$$

Left multiplication by operator $\widehat{J}_{d'}$ and taking the trace yields

$$F^{(n)}_{d',d} = \mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d'}} \mathscr{L}_1(\widehat{\mathrm{S}}^{(\mathrm{n-1})}_{\mathrm{d}})\right). \tag{8}$$

Then $\widehat{S}^{(n)}_d$ is as follows

$$\widehat{S}^{(n)}_d = \overline{\mathscr{R}}_0 \left(\mathscr{L}_1(\widehat{S}^{(n-1)}_d) - \sum_{d''=1}^{\bar{d}} \sum_{r=1}^{n} F^{(r)}_{d'',d} \widehat{S}^{(n-r)}_{d''} \right) \tag{9}$$

and satisfies for all d', $\mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d'}}\widehat{\mathrm{S}}^{(\mathrm{n})}_{\mathrm{d}}\right) = 0$.

With such asymptotic expansion, we get an order n approximation of the dynamics on the invariant slow-manifold $\mathscr{D}_\varepsilon$.

Lemma 1. *Take the slow-fast dynamics* (1)*. For ε small enough, it admits a unique invariant slow manifold of dimension $\bar{d}$ with local real coordinates $(x_1,\ldots,x_{\bar{d}})$ and analytic versus ε*

$$\rho = \sum_{d=1}^{\bar{d}} x_d \left(\sum_{n=0}^{+\infty} \varepsilon^n \widehat{S}^{(n)}_d \right)$$

Moreover, the evolution on this invariant slow linear subspace is governed by the following linear system, analytic versus ε

$$\frac{d}{dt}x(t) = \left(\sum_{n=1}^{\infty} \varepsilon^n F^{(n)} \right) x(t) \tag{10}$$

where $F^{(n)}$ is the matrix of real entries $F^{(n)}_{d,d'}$. Here $\widehat{S}^{(n)}_d$ and $F^{(n)}_{d,d'}$ are given recursively by (8) *and* (9) *starting with $\widehat{S}^{(0)}_d = \widehat{S}_d$ and $F^{(0)}_{d,d'} = 0$.*

The precise proof of this lemma is based on the above calculations and on the approximation theorem 5, page 32 of [15]. Uniqueness and analyticity versus ε is automatically guaranteed since (1) is linear, time-invariant and of finite dimension.

4. Asymptotic expansion of invariant fast manifold

Section 3 was devoted to the invariant slow subspace of (1). To compute the equation of the invariant fast subspace, it is in fact enough to search for its invariant operators equations, i.e., $\bar{d}$ independent linear scalar equations on ρ. For $\varepsilon = 0$, the fast subspace corresponds to the solutions ρ_t of $\frac{d}{dt}\rho = \mathscr{L}_0(\rho_t)$ converging to 0. They are characterized by the following $\bar{d}$ linearly independent equations:

$$\forall d \in \{1,\ldots,\bar{d}\}, \quad \mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}}\rho\right) = 0.$$

Thus, for $\varepsilon > 0$, we are looking for the following set of $\bar{d}$ equations,

$$\forall d \in \{1,\ldots,\bar{d}\}, \quad \mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}}(\varepsilon)\rho\right) = 0$$

where $\widehat{J}_d(\varepsilon) = \sum_{n\geq 0} \varepsilon^n \widehat{J}^{(n)}_d$ with $\widehat{J}^{(0)}_d = \widehat{J}_d$. Invariance conditions mean that for all ρ such that $\mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d'}}(\varepsilon)\rho\right) = 0$ for all d', we have,

$$\forall d \in \{1,\ldots,\bar{d}\}, \quad \frac{d}{dt}\mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}}(\varepsilon)(\mathscr{L}_0(\rho) + \varepsilon \mathscr{L}_1(\rho)\right) = 0.$$

Thus, exists a square matrix of entries $G_{d,d'}(\varepsilon)$ depending analytically on ε such that, $\forall d \in \{1,\ldots,\bar{d}\}$,

$$(\mathscr{L}^*_0 + \varepsilon \mathscr{L}^*_1)(\widehat{J}_d(\varepsilon)) = \sum_{d'} G_{d,d'}(\varepsilon)\widehat{J}_{d'}(\varepsilon)$$

Setting $G_{d,d'}(\varepsilon) = \sum_{n\geq 0} \varepsilon^n G^{(n)}_{d,d'}$ and identifying terms of the same power n versus ε give:

- for $n = 0$, $G^{(0)}_{d,d'} = 0$ since $\mathscr{L}^*_0(\widehat{J}_d) = 0$.
- for $n = 1$:

  $$\mathscr{L}^*_0(\widehat{J}^{(1)}_d) + \mathscr{L}^*_1(\widehat{J}_d) = \sum_{d'} G^{(1)}_{d,d'}\widehat{J}_{d'}$$

  Taking the trace with $\widehat{S}_{d'}$ gives $G^{(1)}_{d,d'} = \mathrm{Tr}\left(\widehat{\mathrm{S}}_{\mathrm{d'}}\mathscr{L}^*_1(\widehat{\mathrm{J}}_{\mathrm{d}})\right)$ and

  $$\widehat{J}^{(1)}_d = \overline{\mathscr{R}}^*_0 \left(\mathscr{L}^*_1(\widehat{J}_d) - \sum_{d'} G^{(1)}_{d,d'}\widehat{J}_{d'} \right)$$

  where

  $$\overline{\mathscr{R}}^*_0(\widehat{W}) = \int_0^{+\infty} e^{s\mathscr{L}^*_0} \left(\widehat{W} - \overline{\mathscr{K}}^*_0(\widehat{W}) \right) ds \tag{11}$$

  and $\overline{\mathscr{K}}^*_0 = \lim_{s\mapsto+\infty} e^{s\mathscr{L}^*_0}$.
- for $n \geq 2$, we have

  $$G^{(n)}_{d',d} = \mathrm{Tr}\left(\widehat{\mathrm{S}}_{\mathrm{d'}}\mathscr{L}^*_1(\widehat{\mathrm{J}}^{(\mathrm{n-1})}_{\mathrm{d}})\right). \tag{12}$$

  and

  $$\widehat{J}^{(n)}_d = \overline{\mathscr{R}}^*_0 \left(\mathscr{L}^*_1(\widehat{J}^{(n-1)}_d) - \sum_{d''=1}^{\bar{d}} \sum_{r=1}^{n} G^{(r)}_{d'',d} \widehat{J}^{(n-r)}_{d''} \right). \tag{13}$$

We have thus proved the following lemma

Lemma 2. *For the slow/fast system* (1) *and ε small enough, exists $\bar{d}$ linearly independent Hermitian operators $\widehat{J}_d(\varepsilon)$, analytic versus ε such that the linear subset of Hermitian operators*

$$\left\{\rho \mid \forall d \in \{1,\ldots,\bar{d}\}, \mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}}(\varepsilon)\rho\right) = 0\right\}$$

is invariant. Any trajectory of (1) *starting in this subset converges exponentially to zero with a strictly positive rate independent of ε. Moreover, by construction, $\forall d,d' \in \{1,\ldots,\bar{d}\}$, $\mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}}(\varepsilon)\widehat{\mathrm{S}}_{\mathrm{d}'}\right) = \delta_{\mathrm{d,d'}}$.*

5. Slow propagator and TPCP maps

Take $T > 0$. Then the linear map $\mathscr{K}_{\varepsilon,T}$ on $\mathscr{D}_0$:

$$\mathscr{D}_0 \ni \rho_0 \mapsto \mathscr{K}_{\varepsilon,T}(\rho_0) \triangleq \overline{\mathscr{K}}_0\left(e^{T(\mathscr{L}_0+\varepsilon\mathscr{L}_1)}(\rho_0)\right) \in \mathscr{D}_0$$

is TPCP. Take ε small enough and consider the $\bar{d}\times\bar{d}$ matrix $E(\varepsilon)$ of entries

$$E_{d,d'}(\varepsilon) = \mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}}(\varepsilon)\widehat{\mathrm{S}}_{\mathrm{d}'}(\varepsilon)\right)$$

where $\widehat{S}_{d'}(\varepsilon) = \sum_{n\geq 0}\varepsilon^n \widehat{S}^{(n)}_{d'}$ with $\widehat{S}_{d'} = \widehat{S}^{(0)}_{d'}$ and $\widehat{J}_d(\varepsilon) = \sum_{n\geq 0}\varepsilon^n \widehat{J}^{(n)}_d$ with $\widehat{J}_d = \widehat{J}^{(0)}_d$ are defined in lemma 1 and lemma 2 respectively. Set $\rho_0 = \sum_d x_d(0)\widehat{S}_d$ with $x(0) = (x_1(0),\ldots,x_{\bar{d}}(0)) \in \mathbb{R}^{\bar{d}}$ and consider the solution $z_\varepsilon(0) \in \mathbb{R}^{\bar{d}}$ of the linear system

$$E(\varepsilon)z_\varepsilon(0) = x(0).$$

From $\mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}}\widehat{\mathrm{S}}_{\mathrm{d}'}(\varepsilon)\right) = \mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}}(\varepsilon)\widehat{\mathrm{S}}_{\mathrm{d}'}\right) = \delta_{\mathrm{d,d'}}$ one has

$$E(\varepsilon) = I_{\bar{d}} + \varepsilon^2 C(\varepsilon)$$

where $I_{\bar{d}}$ is the identity matrix of size $\bar{d}$ and the matrix $C(\varepsilon)$ is analytic versus ε. Thus, $E(\varepsilon)$ is invertible for ε small with $E^{-1}(\varepsilon) = I_{\bar{d}} + O(\varepsilon^2)$. Then we have the following lemma

Lemma 3. *For ε small enough, exist $\gamma > 0$ and $M > 0$ such that for any $T > 0$ and any $x(0) \in \mathbb{R}^{\bar{d}}$ we have*

$$\sum_d \left|\mathrm{Tr}\left(\widehat{\mathrm{S}}_{\mathrm{d}}\mathscr{K}_{\varepsilon,\mathrm{T}}(\rho_0)\right) - \mathrm{z}_{\varepsilon,\mathrm{d}}(\mathrm{T})\right| \leq Me^{-\gamma T}\sqrt{\mathrm{Tr}\left(\rho_0^2\right)}$$

where $z_\varepsilon(T) = e^{TF(\varepsilon)}E^{-1}(\varepsilon)x(0)$ and $\rho_0 = \sum_d x_d(0)\widehat{S}_d$.

Proof. It is based on the following arguments inspired by [15, theorem 2, point b, page 4] and by the notion of shadow trajectories around hyperbolic invariant manifold (see, e.g.,[16] and [1]). Since $\overline{\mathscr{K}}^*_0(\widehat{S}_d) = \widehat{J}_d$ one has

$$\mathrm{Tr}\left(\widehat{\mathrm{S}}_{\mathrm{d}}\mathscr{K}_{\varepsilon,\mathrm{T}}(\rho_0)\right) = \mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}}\mathrm{e}^{\mathrm{T}(\mathscr{L}_0+\varepsilon\mathscr{L}_1)}(\rho_0)\right)$$

The initial state on the invariant manifold $\rho_{\varepsilon,0} = \sum_d z_{\varepsilon,d}(0)\widehat{S}_d(\varepsilon)$ and ρ_0 are such that $\rho_0 - \rho_{\varepsilon,0}$ belongs to the invariant fast manifold. Thus, exist $\gamma > 0$ and $M > 0$ such that

$$\left\|e^{T(\mathscr{L}_0+\varepsilon\mathscr{L}_1)}(\rho_0-\rho_{\varepsilon,0})\right\| \leq Me^{-\gamma T}\left\|\rho_0-\rho_{\varepsilon,0}\right\|.$$

We conclude with $e^{T(\mathscr{L}_0+\varepsilon\mathscr{L}_1)}(\rho_{\varepsilon,0}) = \sum_d z_{\varepsilon,d}(T)\widehat{S}_d(\varepsilon)$ and $\mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}}\widehat{\mathrm{S}}_{\mathrm{d}'}(\varepsilon)\right) = \delta_{\mathrm{d,d'}}$. □

Take $\bar{T} > 0$. This lemma implies that for $T = \bar{T}/\varepsilon$ corresponding to a slow timescale, the approximation of $\mathscr{K}_{\varepsilon,\frac{\bar{T}}{\varepsilon}}(\rho_0)$ by $\sum_d z_{\varepsilon,d}(\frac{\bar{T}}{\varepsilon})\widehat{S}_d$ is exponentially precise. Thus, if we restrict the asymptotic expansion to second-order terms and consider evolution over a slow timescale $\frac{\bar{T}}{\varepsilon}$, we are close up to order 2 terms to a TPCP map as stated in the following lemma.

Lemma 4. *Consider the slow/fast system* (1)*, the second-order approximation of the slow dynamics*

$$\frac{d}{dt}x = \left(\varepsilon F^{(1)} + \varepsilon^2 F^{(2)}\right)x$$

and its propagator matrix $\mathscr{G}^{(2)}_{\frac{\bar{T}}{\varepsilon}} = e^{\frac{\bar{T}}{\varepsilon}(\varepsilon F^{(1)}+\varepsilon^2F^{(2)})}$ over the slow timescale $\bar{T}/\varepsilon$. For any $\bar{T} > 0$, exists $M_{\bar{T}} > 0$ such that for any $x(0) \in \mathbb{R}^{\bar{d}}$ we have

$$\left\|\mathscr{K}_{\varepsilon,\frac{\bar{T}}{\varepsilon}}(\rho_0) - \rho(\tfrac{\bar{T}}{\varepsilon})\right\| \leq M_{\bar{T}}\varepsilon^2\|x(0)\|$$

where $\rho_0 = \sum_d x_d(0)\widehat{S}_d$ and $\rho(\frac{\bar{T}}{\varepsilon}) = \sum_d x_d(\frac{\bar{T}}{\varepsilon})\widehat{S}_d$ with $x(\frac{\bar{T}}{\varepsilon}) = \mathscr{G}^{(2)}_{\frac{\bar{T}}{\varepsilon}}x(0)$.

Notice that

$$F^{(1)}_{d',d} = \mathrm{Tr}\left(\widehat{\mathrm{J}}_{\mathrm{d}'}\mathscr{L}_1(\widehat{\mathrm{S}}_{\mathrm{d}})\right)$$

and ($\overline{\mathscr{R}}_0$ defined in (6))

$$F^{(2)}_{d',d} = \mathrm{Tr}\left(\mathscr{L}_1^*(\widehat{\mathrm{J}}_{\mathrm{d}'})\,\overline{\mathscr{R}}_0\left(\mathscr{L}_1(\widehat{\mathrm{S}}_{\mathrm{d}})\right)\right)$$

can be computed from the nominal operators $\widehat{S}_d$ and $\widehat{J}_{d'}$ defined in section 2.

Proof. It combines lemma 3, the fact that $\mathscr{G}^{(2)}_{\frac{\bar{T}}{\varepsilon}} = e^{\frac{\bar{T}}{\varepsilon}F(\varepsilon)} + O(\varepsilon^2)$ since $\frac{\bar{T}}{\varepsilon}F(\varepsilon) = \bar{T}(F^{(1)}+\varepsilon F^{(2)}) + O(\varepsilon^2)$ and that $E^{-1}(\varepsilon)x(0) = x(0) + O(\varepsilon^2)$ since $E^{-1}(\varepsilon) = I_{\bar{d}} + O(\varepsilon^2)$. □

[1] https://spartacus-idh.com/liseuse/116/#page/142

6. Concluding remarks

For infinite dimensional systems, mathematical justifications of such asymptotic expansions are not straightforward (existence, uniqueness and convergence of the series). For infinite dimensional systems constructed with mainly bounded operators, precise mathematical guaranties are available (see, e.g., [17, 18]). One cannot use these available results when the dynamics rely on unbounded operators parameterizing the GKSL dynamical model (1). The mathematical justification of such infinite dimensional extension requires precise functional analysis investigations and assumptions. They will be addressed in future developments.

These asymptotic expansion can be exploited numerically to simulate on a classical computer, composite open quantum systems encountered in quantum error correction (see, e.g., [19] for preliminary results with cat-qubit systems).

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