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Event-Triggered Distributed Optimization Algorithm over Directed Networks: A Nonsingular Estimator Approach

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Abstract— **This paper investigates the event-triggered distributed optimization problems (ETDOPs) over strongly connected directed networks. By assigning an additional scalar state variable to each agent and utilizing diminishing time-varying gain/step-size, a class of modified event-triggered distributed optimization algorithms (ETDOAs) is proposed, which can address the ETDOPs well and can avoid the inverse operation of some estimators in the existing literature. Compared with the existing DOAs, this paper gives a new idea to solve the DOPs under weighted-unbalanced digraphs and continuous communication of agent networks is avoided. Finally, numerical simulations are given to illustrate the effectiveness of the proposed ETDOAs.**

I. INTRODUCTION

Distributed optimization problem (DOP) generally refers to minimizing the sum of local cost functions through cooperation and coordination among multiple agents [4]-[5], while each agent only has access to one local cost function. In the past two decades, the theory and applications of distributed optimization have attracted more and more attention, and gradually penetrated into almost all fields of scientific research, engineering applications and social life, such as smart power grids, sensor networks, social networks, and cyber-physical systems [1]-[3].

The DOP was originally proposed in [4], in which the authors designed a discrete-time distributed optimization algorithm (DOA) consisting of a consistency term and a negative gradient term, the so-called distributed subgradient method. This basic algorithm was yet modified to apply to more general networks [5]. Later, continuous-time dynamic DOAs based on auxiliary variables were proposed [6] to overcome the disadvantages caused by the use of decay step size in the discrete-time algorithms [4]-[5]. Although the auxiliary variable algorithm abandons the disadvantages caused by the attenuation step, the introduction of auxiliary variables will inevitably lead to the increase of communication and computation, and then increase the consumption of network resources. Then, two classes of continuous-time zero-gradient-sum DOAs were proposed [7]-[8], so as to solve the problem caused by decay step size and auxiliary variables.

Note that the aforementioned results mainly concentrate on the DOPs under undirected communication topologies. Until now, some literature also consider this problem over directed networks. Two kinds of proportional-integral DOAs were proposed for DOPs under weight-balanced digraphs [9]-[10]. Then, the authors proposed a fully distributed adaptive optimization algorithm [11] over strongly connected digraphs while the left eigenvector corresponding to the zero eigenvalues of the Laplacian matrix must be known. By utilizing surplus-based method, a directed-distributed projected subgradient algorithm [12] was proposed to solve a constrained optimization problem. Further, discrete-time and continuous-time DOAs [13]-[14] were designed for weight-unbalanced digraphs by combining with additional estimators. Recently, the authors also investigated the DOPs for time-varying directed graphs [15].

Look back at the literature mentioned above, although much research has been carried out in DOPs, much of the existing literature invokes the assumptions that (i) the communication topology among agents is bidirectional or directed but weighted-balanced and (ii) the communication transmission is continuous (see [4]-[6], [9]-[10]). However, these assumptions are not always practical in real-world distributed systems, where communication may be unidirectional and the agent's power supply and communication channels are limited (e.g., agents in outdoor wireless network use broadcast-based communication schemes).

Motivated by these limitations, this paper investigates the event-triggered distributed optimization problems (ETDOPs) under strongly connected directed networks. A modified event-triggered distributed optimization algorithm (ETDOA) is proposed, which is suitable for implementation in the case where the communication between agents is described in terms of a weighted unbalanced graph. The main contributions of this paper distinguishing from the existing literature [4]-[15] are described as follows:

First, the communication topology studied in this paper focuses on weighted-unbalanced digraphs. Compared with most existing works [4]-[6], [9]-[10], the scenarios considered in this paper are much more general and thus more applicable in practice. By assigning an additional scalar state variable to each agent, a class of modified ETDOAs is proposed, which can deal with the unbalance of the graph well with the cost of knowing the out-degree information of each agent. Compared with the existing directed network results [13], [14], the initial conditions of additional scalar

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state variable can be chosen arbitrarily, and the proposed ETDOA can effectively avoid the inverse operation of the estimator and reduce the the dimension of auxiliary variable.

Second, the communication transmission mode adopted in this paper is event-triggered. It is known to all that there is few results about ETDOAs over weighted-unbalanced directed networks. By designing appropriate event triggering mechanisms, continuous communication of agent networks is avoided and Zeno behavior is excluded.

Third, a diminishing time-varying gain/step-size is embedded in the ETDOA, which will eventually make the system converge to the accurate optimal solution slowly without missing it. On the other hand, the diminishing gains/step-sizes could be designed without having any global information about the network or the aggregate/local cost functions. This is in contrast to the algorithms which work with a constant gain/step-size [4]-[6], [9]-[15].

The reminder of this paper is organized as follows. In Section II, some preliminaries and the problem statement are presented. In Section III, a modified ETDOA over directed networks is proposed. Then, numerical simulations are provided to illustrate the obtained results in Section IV. Finally, Section V summarizes this paper.

Notations: R^n denotes n-dimension real vectors and $R^{n\times n}$ denotes $n\times n$-dimension real matrices. Notation I_n denotes the $n\times n$ identity matrix. $\mathbf{1}$ and $\mathbf{0}$ are the column vectors, respectively with all elements being 1 and 0. $\otimes$ is the Kronecker product. For a matrix $M\in R^{n\times n}$, M^T denotes its transpose. Let ∇h be the gradient of a function h. For a vector $\theta=(\theta_1,\theta_2,\cdots,\theta_n)^T\in R^n$, let $\|\theta\|$ be the Euclidean norm of θ.

For a continuously differentiable function $h: R^n\to R$, ∇h represents its gradient. h is strongly convex on R^n if there exists a positive constant μ such that $(y-x)^T(\nabla h(y)-\nabla h(x))\geq \mu\|y-x\|^2,\forall x,y\in R^n,x\neq y$. h is locally Lipschitz at $x\in R^n$ if there exists a neighborhood $\mathscr{W}$ and a positive constant ω such that $\|h(y)-h(x)\|\leq\omega\|y-x\|,\forall x,y\in\mathscr{W}$. h is globally Lipschitz on R^n if it is locally Lipschitz at x for all $x\in R^n$.

II. Preliminaries and Problem Statement

A. Graph theory

A weighted digraph is described by $\mathscr{G}(\mathscr{V},\mathscr{E},\mathscr{A})$ to represent information communication among nodes, which consists of a node set $\mathscr{V}=\{v_1,v_2,\ldots,v_N\}$ and a directed edge set $\mathscr{E}\subseteq\mathscr{V}\times\mathscr{V}$ and a weighted adjacency matrix $\mathscr{A}=[a_{ij}]_{N\times N}$. For a directed network, an edge $e_{ji}\in\mathscr{E}$ implies that node v_j can receive information from v_i. All elements in $\mathscr{A}$ are nonnegative, and $a_{ji}>0$ if and only if $e_{ji}\in\mathscr{E}$. A directed network is strongly connected if there is a directed path between any two vertexes. Let $\mathscr{N}_i^i=\{j\in\mathscr{V}:a_{ij}>0\}$ and $\mathscr{N}_i^o=\{j\in\mathscr{V}:a_{ji}>0\}$ denote the set of in-neighbors and out-neighbors of node v_i, respectively. Correspondingly, $d_i^{in}=\sum_{j\in\mathscr{N}_i^i}a_{ij}$ represents the in-degree of node v_i and $d_i^{out}=\sum_{j\in\mathscr{N}_i^o}a_{ji}$ represents the out-degree of node v_i. The Laplacian matrix $\mathscr{L}=[l_{ij}]_{N\times N}$ of a directed network $\mathscr{G}$ is defined in this paper by $\mathscr{L}=\mathscr{D}^{out}-\mathscr{A}$, where $\mathscr{D}^{out}=\mathrm{diag}(d_1^{out},\ldots,d_N^{out})$. Note that $\mathbf{1}_N^T\mathscr{L}=\mathbf{0}_N^T$ and $\mathscr{L}$ is column diagonally dominant, thus, all the eigenvalues of $\mathscr{L}$ have strictly positive real parts except for the zero eigenvalue(s). A directed network is weight-balanced if and only if $d_i^{out}=d_i^{in}$.

Assumption 1: This paper considers the strongly connected directed networks.

Since $\mathscr{G}$ is assumed to be strongly connected, we can further deduce that (1) Zero is a simple eigenvalue of $\mathscr{L}$. (2) There exists a unique positive vector $\phi=(\phi_1,\ldots,\phi_N)^T$, such that $\mathscr{L}\phi=\mathbf{0}_N$ and $\mathbf{1}_N^T\phi=1$. (3) $\mathscr{L}\Phi$ is the Laplacian matrix for a strongly connected and weight-balanced network with the adjacency matrix $\mathscr{A}\Phi$, where $\Phi=\mathrm{diag}(\phi_1,\phi_2,\cdots,\phi_N)$, matrix $\hat{\mathscr{L}}=\Phi\mathscr{L}^T+\mathscr{L}\Phi$ is positive semi-definite, and 0 is a simple eigenvalue of $\hat{\mathscr{L}}$. (4) $\lim_{t\to\infty}\exp(-\mathscr{L}t)=\phi\mathbf{1}_N^T$.

Assumption 2: Each agent can know its own out-degree information.

B. Problem statement

This paper considers the following distributed optimization problem under strongly connected directed graphs:

$$\min_{X\in R^{Nn}}\tilde{h}(X)=\min_{x_i\in R^n}\sum_{i=1}^N h_i(x_i),\quad \text{s.t. } x_i=x_j,i,j\in\mathscr{V},\qquad (1)$$

where $x_i\in R^n$ is the state of agent i, $\tilde{h}(X)=\sum_{i=1}^N h_i(x_i)$, $X=(x_1^T,\ldots,x_N^T)^T$ and $h_i(x_i):R^n\to R$, is a local cost function available only for agent i.

Assumption 3: The local cost function $h_i(x_i)$, $i=1,\ldots,N$, is continuously differentiable, strongly convex, and $\nabla h_i(x_i)$ is globally Lipschitz on R^n.

The objective of this paper is to design ETDOAs under strongly connected directed networks, so as to solve the DOP (1) and reduce communication cost.

Before moving on, a basic lemma is proposed as follows:

Lemma 1: Consider the following estimator that does not use an event-triggered communication strategy,

$$\dot{\gamma}_i=-d_i^{out}\gamma_i+\sum_{j\in\mathscr{N}_i^i}a_{ij}\gamma_j,\qquad (2)$$

where γ_i is the state of estimator i with initial condition $\sum_{i=1}^N\gamma_i(0)=1$. Then, γ_i will eventually estimate the elements $\phi_1,\ldots,\phi_N$ of the right eigenvector corresponding to the zero eigenvalue of the Laplacian matrix $\mathscr{L}$, i.e., $\lim_{t\to\infty}\gamma(t)=\phi$.

***Proof*:** The compact form of (2) is $\dot{\gamma}=-\mathscr{L}\gamma$, where $\gamma=\left(\gamma_1^T,\ldots,\gamma_N^T\right)^T\in R^N$. The solution of the above equation is $\gamma(t)=e^{-\mathscr{L}t}\gamma(0)$, where $\gamma(0)=\left(\gamma_1^T(0),\ldots,\gamma_N^T(0)\right)^T$. Note that $\mathscr{L}=\mathscr{D}^{out}-\mathscr{A}$, and $\lim\limits_{t\to\infty}e^{-\mathscr{L}t}=\phi\mathbf{1}_N^T$, thus, one can obtain that

$$\lim_{t\to\infty}\gamma(t)=\phi\mathbf{1}_N^T\gamma(0)=\begin{bmatrix}\phi_1 & \cdots & \phi_1\\ \vdots & \ddots & \vdots\\ \phi_N & \cdots & \phi_N\end{bmatrix}\gamma(0).$$

Therefore, if one selects $\sum_{i=1}^N\gamma_i(0)=1$, then $\lim\limits_{t\to\infty}\gamma(t)=\phi$. This completes the proof. ■

Remark 1: The initial conditions $\gamma_i(0)$ can be chosen arbitrarily, as long as $\gamma_i(0)\geq 0$, and $\gamma_i(0)\neq 0$ for all $i=1,2,\ldots,N$. Therefore, the convergence of $\gamma(t)$ can be expressed more generally as $\lim_{t\to\infty}\gamma(t)=m\phi$, where $m=\sum_{i=1}^{N}\gamma_i(0)>0$.

III. ETDOA OVER DIRECTED NETWORKS

In this section, a modified ETDOA over directed networks is designed as follows:

$$\dot{x}_i=-\alpha(t)\nabla h_i(x_i)-d_i^{out}\gamma_i\tilde{x}_i+\sum_{j\in\mathcal{N}_i^i}a_{ij}\gamma_j\tilde{x}_j, \tag{3a}$$

$$\dot{\gamma}_i=-d_i^{out}\tilde{\gamma}_i+\sum_{j\in\mathcal{N}_i^i}a_{ij}\tilde{\gamma}_j, \tag{3b}$$

where $\tilde{x}_i(t)=x_i(t_k^i)$, $\forall t\in\left[t_k^i,t_{k+1}^i\right)$ is the sampled state at k-th triggering instant of agent i and it will remain unchanged until the next triggering instant. $\tilde{\gamma}_i(t)=\gamma_i(\tau_k^i)$, $\forall t\in\left[\tau_k^i,\tau_{k+1}^i\right)$. $\alpha(t)$ is a continuous and monotonically non-increasing function which satisfies $\alpha(t)>0$, $\int_0^\infty\alpha(t)\,dt=\infty$, and $\lim\limits_{t\to\infty}\alpha(t)=0$. The initial conditions $x_i(0)$ are chosen arbitrarily, and $\gamma_i(0)$ are selected as $\sum_{i=1}^{N}\gamma_i(0)=1$ for the convenience of analysis. For example, one can initialize one of the estimators in the network to 1 and set the rest to 0.

Remark 2: The introduction of estimator $\gamma_i(t)$ gives a new idea to solve the DOPs under weighted-unbalanced digraphs, which has three obvious advantages: First, it will avoid the inverse operation of the estimator adopted in references [13], [14] and so on. Second, compared with references [13], [14], in which the estimators $z_i(t)$ must be initialized at the corresponding location $z_i^j(0)=0,\forall i\neq j$, $\quad z_i^i(0)=1,\forall i\in\mathcal{V}$ (this scheme will inevitably lead to the problem of agent numbering, that is, each agent must know the numbering of its own and its neighbors), the initialization scheme adopted in this paper is more flexible. Third, form lemma 1, one can see that the requirement of $\gamma_i(0)$ in this paper is more general. At the same time, the dimension of auxiliary variable $\gamma_i(t)$ is smaller than the estimators in [14], which will reduce the amount of computation in practical application.

The triggering instants t_k^i and τ_k^i are determined by the following event-triggered conditions:

$$t_{k+1}^i=\inf\{t>t_k^i:\|e_{xi}\|^2-\mu_{1i}e^{-\upsilon_{1i}t}\geq 0\}, \tag{4a}$$

$$\tau_{k+1}^i=\inf\{t>\tau_k^i:\left\|e_{\gamma i}\right\|^2-\mu_{2i}e^{-\upsilon_{2i}t}\geq 0\}, \tag{4b}$$

where $e_{xi}(t)=\tilde{x}_i(t)-x_i(t)$ and $e_{\gamma i}(t)=\tilde{\gamma}_i(t)-\gamma_i(t)$ are measurement errors of the i-th agent and measurement errors of the i-th compensator, and μ_{1i}, υ_{1i} and μ_{2i}, υ_{2i} are positive constants. Information flows among agents are generated only when the triggering conditions (4) are met. At the same time, $e_{xi}(t)$ and $e_{\gamma i}(t)$ will reset to zero.

Let $x=\left(x_1^T,\ldots,x_N^T\right)^T$, $\Lambda_d=\operatorname{diag}\{\gamma_1,\ldots,\gamma_N\}$, $\nabla h(x)=\left(\nabla h_1(x_1)^T,\ldots,\nabla h_N(x_N)^T\right)^T$, then, system (3) can be written into the following compact form:

$$\dot{x}=-\alpha(t)\nabla h(x)-(\mathcal{L}\Lambda_d\otimes I_n)\tilde{x}, \tag{5a}$$

$$\dot{\gamma}=-\mathcal{L}\tilde{\gamma}. \tag{5b}$$

Lemma 2: The event-triggered estimator (5b) can converge to the unique right eigenvector associated with the zero eigenvalue of $\mathcal{L}$ without existing Zeno behavior.

***Proof*:** Define a tracking error between the estimator and the right eigenvector as $\eta=\gamma-\phi$, where $\gamma=\left(\gamma_1^T,\cdots,\gamma_N^T\right)^T$, $\phi=\left(\phi_1^T,\ldots,\phi_N^T\right)^T$.

Then, the dynamics of tracking error η is

$$\dot{\eta}=\dot{\gamma}=-\mathcal{L}\left(\eta+\phi+e_\gamma\right), \tag{6}$$

where $e_\gamma=\left[e_{\gamma 1}^T,\ldots,e_{\gamma N}^T\right]^T$. Due to $\mathcal{L}\phi=\mathbf{0}_N$, one has $\dot{\eta}=-\mathcal{L}\left(\eta+e_\gamma\right)$.

Construct the following candidate Lyapunov function $V_1=\eta^T\left(\Phi^T\otimes I_N\right)\eta$, where $\Phi=\operatorname{diag}(\phi_1,\ldots,\phi_N)$. Taking the time derivative of V_1 along the trajectories of (6), one has

$$\begin{aligned}
\dot{V}=&\dot{\eta}^T\left(\Phi^T\otimes I_N\right)\eta+\eta^T\left(\Phi^T\otimes I_N\right)\dot{\eta}\\
=&-\left(\eta+e_\gamma\right)^T\left(\mathcal{L}^T\Phi^T\otimes I\right)\eta-\eta^T\left(\Phi^T\mathcal{L}\otimes I\right)\left(\eta+e_\gamma\right)\\
=&-\eta^T\left(\left(\mathcal{L}\Phi+\Phi\mathcal{L}^T\right)^T\otimes I\right)\eta-\eta^T\left(\Phi^T\mathcal{L}\otimes I\right)e_\gamma\\
&-e_\gamma^T\left(\mathcal{L}^T\Phi^T\otimes I\right)\eta\\
=&-\eta^T\left(\left(\mathcal{L}\Phi+\Phi\mathcal{L}^T\right)^T\otimes I\right)\eta-2\eta^T\left(\Phi^T\mathcal{L}\otimes I\right)e_\gamma\\
\leq&-\eta^T\left(\left(\mathcal{L}\Phi+\Phi\mathcal{L}^T\right)^T\otimes I\right)\eta+\alpha\eta^T\left(\mathcal{L}\Phi\Phi\mathcal{L}^T\otimes I\right)\eta\\
&+\frac{1}{\alpha}e_\gamma^T\left(\mathcal{L}\Phi\Phi\mathcal{L}^T\otimes I\right)e_\gamma\\
\leq&-\lambda_2\left(\hat{\mathcal{L}}\right)\eta^T\eta+\beta\lambda_{\max}\left(\tilde{\mathcal{L}}\right)\eta^T\eta+\frac{1}{\beta}\lambda_{\max}\left(\tilde{\mathcal{L}}\right)e_\gamma^Te_\gamma\\
\leq&-\left[\lambda_2\left(\hat{\mathcal{L}}\right)-\beta\lambda_{\max}\left(\tilde{\mathcal{L}}\right)\right]\eta^T\eta+\frac{1}{\beta}\lambda_{\max}\left(\tilde{\mathcal{L}}\right)e_\gamma^Te_\gamma,
\end{aligned} \tag{7}$$

where $\Phi=\Phi^T$, $\hat{\mathcal{L}}=\Phi\mathcal{L}^T+\mathcal{L}\Phi$, $\lambda_2\left(\hat{\mathcal{L}}\right)$ is the minimum nonzero eigenvalue of matrix $\hat{\mathcal{L}}$. $\tilde{\mathcal{L}}=\mathcal{L}\Phi\Phi\mathcal{L}^T$, $\lambda_{\max}\left(\tilde{\mathcal{L}}\right)$ is the maximum eigenvalue of matrix $\tilde{\mathcal{L}}$. Then, by selecting $\beta<\frac{\lambda_2(\hat{\mathcal{L}})}{\lambda_{\max}(\tilde{\mathcal{L}})}$ and utilizing event-triggering condition $\left\|e_\gamma\right\|^2-\mu_2e^{-\upsilon_2t}\leq 0$, it can be obtained that

$$\dot{V}_1\leq-\left[\lambda_2\left(\hat{\mathcal{L}}\right)-\beta\lambda_{\max}\left(\tilde{\mathcal{L}}\right)\right]\eta^T\eta+\frac{\lambda_{\max}\left(\tilde{\mathcal{L}}\right)}{\beta}\mu_2e^{-\upsilon_2t}. \tag{8}$$

Then, it follows from the Barbalat's lemma that the tracking error $\eta(t)$ can converge to 0. Consequently, one can deduce that the event-triggered estimator (5b) can converge to the unique right eigenvector associated with the zero eigenvalue of $\mathcal{L}$, i.e., $\lim_{t\to\infty}\gamma(t)=\phi$ and $\lim_{t\to\infty}\Lambda_d=\Phi$.

In the following, it's shown that there is no Zeno behavior among the event-triggered estimators. First, the time derivative of $\|e_{\gamma i}(t)\|$ over the interval $\left[\tau_k^i,\tau_{k+1}^i\right)$ satisfies

$$\begin{aligned}
\frac{d}{dt}\left\|e_{\gamma i}(t)\right\|\leq&\frac{\left\|e_{\gamma i}^T\right\|}{\left\|e_{\gamma i}\right\|}\left\|\dot{e}_{\gamma i}\right\|=\left\|-d_i^{out}\tilde{\gamma}_i+\sum_{j\in\mathcal{N}_i^i}a_{ij}\tilde{\gamma}_j\right\|\\
\leq&\left\|e_{\gamma i}(t)\right\|+\left\|-d_i^{out}\tilde{\gamma}_i+\sum_{j\in\mathcal{N}_i^i}a_{ij}\tilde{\gamma}_j\right\|\\
\leq&\left\|e_{\gamma i}(t)\right\|+\mathcal{J}_i,
\end{aligned} \tag{9}$$

where $\frac{d}{dt}\left\|e_{\gamma i}(t)\right\|$ denotes the right-hand derivative of $\left\|e_{\gamma i}(t)\right\|$ at $t=\tau_k^i$, and $\mathcal{J}_i=\max\limits_{t\in\left[\tau_k^i,\tau_{k+1}^i\right)}\left\|-d_i^{out}\tilde{\gamma}_i+\sum\limits_{j\in\mathcal{N}_i^i}a_{ij}\tilde{\gamma}_j\right\|$.

Then, it follows that

$$\left\|e_{\gamma i}(t)\right\| \leq \mathscr{J}_i\left(e^{t-\tau_k^i}-1\right), \tag{10}$$

and $e_{\gamma i}\left(\tau_k^i\right)=0$. Recalling the triggering function (4b), one has $\left\|e_{\gamma i}\right\|^2 \leq \mu_{2i} e^{-\upsilon_{2i} t}$ between any two triggering instants. Therefore, a lower bounded T_1 of $\tau_{k+1}^i-\tau_k^i$ can be determined by solving $\mu_{2i} e^{-\upsilon_{2i}\left(\tau_k^i+T_1\right)} \leq \mathscr{J}_i^2\left(e^{T_1}-1\right)^2$, which yields $\tau_{k+1}^i-\tau_k^i \geq T_1 \geq \ln\left(\frac{\sqrt{\mu_{2i}}}{\mathscr{J}_i} e^{-\frac{\upsilon_{2i}}{2}\left(\tau_k^i+T_1\right)}+1\right)>0$. Therefore, there is a positive lower bound between any two triggering intervals and there does not exist Zeno behavior in the event-triggered balanced compensator. This completes the proof.

Theorem 1: Under Assumptions 1-3, by adopting the modified ETDOA (3) with triggering conditions (4), choosing suitable time-varying gain $\alpha(t)$, and initialing $\sum_{i=1}^N \gamma_i(0)=1$, then, the states of each agent asymptotically converge to the unique global minimizer $x^* \in R^n$ of the problem (1) over any strongly connected directed networks. Besides, the communication cost is reduced and the system does not exist Zeno behavior.

***Proof*:** The proof of the main results is divided into two parts: optimality analysis and convergence analysis.

A. Optimality analysis

By making a change of variable, one has

$$\begin{aligned}\dot{x} &= -\alpha(t)\nabla h(x)-\left(\mathscr{L}\Lambda_d \otimes I_n\right)\tilde{x} \\ &= -\alpha(t)\nabla h(x)-\left(\mathscr{L}\Phi \otimes I_n\right)\tilde{x}-\left(\mathscr{L}\left(\Lambda_d-\Phi\right)\otimes I_n\right)\tilde{x}.\end{aligned} \tag{11}$$

Hence, the equilibrium point $\bar{x}=\left(\bar{x}_1^T,\ldots,\bar{x}_N^T\right)^T \in R^{Nn}$ of (5a) satisfies

$$-\alpha(t)\nabla h(\bar{x})-\left(\mathscr{L}\Phi\otimes I_n\right)\bar{x}-\left(\mathscr{L}\left(\Lambda_d-\Phi\right)\otimes I_n\right)\bar{x}=0. \tag{12}$$

Left multiplying (12) by $\mathbf{1}_N^T$ and using the fact $\mathbf{1}_N^T\mathscr{L}\Phi=\mathbf{0}_N^T, \alpha(t)>0$, and $\lim_{t\to\infty}\Lambda_d=\Phi$, one has

$$\sum_{i=1}^N \nabla h_i\left(\bar{x}_i\right)=0, \tag{13}$$

thus, based on the strong convexity of the local cost function h_i, one is able to derive that the equilibrium point $\bar{x}$ of (5a) is the optimal solution of problem (1) and $\bar{x}=\mathbf{1}_N \otimes x^\star$.

B. Convergence analysis

Introduce state transition variable $\xi=x-\bar{x}$, and the dynamics of ξ is expressed as

$$\begin{aligned}\dot{\xi}=&-\alpha(t)\left(\nabla h(\xi+\bar{x})-\nabla h(\bar{x})\right)-\left(\mathscr{L}\Phi\otimes I_n\right)\left(\xi+e_x\right) \\ &-\left(\mathscr{L}\left(\Lambda_d-\Phi\right)\otimes I_n\right)\left(\xi+e_x\right).\end{aligned} \tag{14}$$

Since $\lim_{t\to\infty}\Lambda_d=\Phi$ according to Lemma 2 and following from Lemma 6 in [17], one can obtain that analyzing the convergence of the above system is equivalent to analyzing the convergence of the following nominal system:

$$\dot{\xi}=-\alpha(t)\left(\nabla h(\xi+\bar{x})-\nabla h(\bar{x})\right)-\left(\mathscr{L}\Phi\otimes I_n\right)\left(\xi+e_x\right). \tag{15}$$

Next, consider a candidate Lyapunov function $V_2=\frac{1}{2}\xi^T\xi$, therefore, the derivative of the Lyapunov function with respect to time is

$$\dot{V}_2=-\alpha(t)\xi^T\left(\nabla h(\xi+\bar{x})-\nabla h(\bar{x})\right)-\xi^T\left(\mathscr{L}\Phi\otimes I_n\right)\left(\xi+e_x\right). \tag{16}$$

Then, by exploiting the Assumption 2 and μ-strong convexity of the local cost functions, the following inequality is obtained, $-\alpha(t)\xi^T\left(\nabla h(\xi+\bar{x})-\nabla h(\bar{x})\right) \leq \mu\alpha(t)\xi^T\xi$.

The second term of Lyapunov function derivative can be treated as follows:

$$\begin{aligned}&-\xi^T\left(\mathscr{L}\Phi\otimes I_n\right)\left(\xi+e_x\right) \\ =&-\frac{1}{2}\xi^T\left(\left(\Phi\mathscr{L}^T+\mathscr{L}\Phi\right)\otimes I_n\right)\xi-\xi^T\left(\mathscr{L}\Phi\otimes I_n\right)e_x \\ \leq&-\frac{1}{2}\lambda_2\left(\hat{\mathscr{L}}\right)\xi^T\xi+\frac{\delta}{2}\xi^T\xi+\frac{1}{2\delta}e_x^T\left(\mathscr{L}\Phi\Phi\mathscr{L}^T\otimes I_n\right)e_x \\ \leq&-\frac{1}{2}\left(\lambda_2\left(\hat{\mathscr{L}}\right)-\delta\right)\xi^T\xi+\frac{1}{2\delta}\lambda_{\max}\left(\tilde{\mathscr{L}}\right)e_x^Te_x.\end{aligned} \tag{17}$$

Therefore,

$$\begin{aligned}\dot{V}_2 \leq&-\frac{1}{2}\left(\lambda_2\left(\hat{\mathscr{L}}\right)-\delta-2\mu\alpha(t)\right)\xi^T\xi+\frac{1}{2\delta}\lambda_{\max}\left(\tilde{\mathscr{L}}\right)e_x^Te_x \\ \leq&-\frac{1}{2}\left(\lambda_2\left(\hat{\mathscr{L}}\right)-\delta-2\mu\alpha(t)\right)V_2+\frac{1}{2\delta}\lambda_{\max}\left(\tilde{\mathscr{L}}\right)\mu_1 e^{-\upsilon_1 t} \\ \leq&-\theta_1(t)V_2+\kappa\mu_1 e^{-\upsilon_1 t},\end{aligned} \tag{18}$$

where the last inequality is obtained by using the event-triggering condition $\|e_x\|^2-\mu_1 e^{-\upsilon_1 t} \leq 0$, $\theta_1(t)=\frac{1}{2}\left(-2\mu\alpha(t)-\delta+\lambda_2\left(\hat{\mathscr{L}}\right)\right)$, and $\kappa=\frac{1}{2\delta}\lambda_{\max}\left(\tilde{\mathscr{L}}\right)$.

From the Comparison Lemma [[16], Lemma 3.4], it follows that

$$V_2 \leq V_2(0)e^{-\int_0^t\theta_1(\tau)d\tau}+\kappa\mu_1\int_0^t e^{-\int_\tau^s\theta_1(s)ds}e^{-\upsilon_1\tau}d\tau. \tag{19}$$

Obviously, $\int_0^\infty\theta_1(t)dt=\infty$, therefore, as $t\to\infty$, the first term on the right-hand side vanishes. Note that if one selects suitable time-varying gain $\alpha(t)$ such that 1) $\limsup_{t\to\infty}\frac{\kappa\mu_1 e^{-\upsilon_1 t}}{\theta_1(t)}<\infty$. 2) there exists a finite time $t_0>0$, and $\theta_1(t)\geq 0$ for all $t\geq t_0$.

Therefore, there exists a time $t_1>t_0$ and a positive constant Γ such that $\frac{\kappa\mu_1 e^{-\upsilon_1 t}}{\theta_1(t)}<\Gamma$ for all $t>t_1$. Furthermore,

$$\kappa\mu_1\int_0^t e^{-\int_\tau^s\theta_1(s)ds}e^{-\upsilon_1\tau}d\tau<\Gamma\int_0^t e^{-\int_\tau^s\theta_1(s)ds}\theta_1(\tau)d\tau<\Gamma,$$

for all $t>t_1$. Similarly, there exists a time $t_2>0$ such that $V_2(0)e^{-\int_0^t\theta_1(\tau)d\tau}<\Gamma$ for all $t>t_2$. Let $t':=\max\{t_1,t_2\}$, and $\hat{\Gamma}:=\max_{0\leq t\leq t'}V_2(t)$. Therefore, one has $V_2(t)<\max\left\{\hat{\Gamma},2\Gamma\right\}$ for all $t>0$. Then, in light of $V_2(t)\geq\frac{1}{2}\|\xi(t)\|^2$, one has that $\xi(t)$ exponentially converges to a bounded set, i.e., the states of each agent exponentially converge to a small neighborhood of the unique global minimizer x^* of the problem (1).

Next, it is proved that there is no Zeno phenomenon in the execution of the ETDOA.

The time derivative of $e_{xi}(t)$ over the interval $\left[t_k^i, t_{k+1}^i\right)$ satisfies

$$\dot{e}_{xi}(t) = -\alpha(t)\nabla h_i(x_i) - d_i^{out}\gamma_i\tilde{x}_i + \sum_{j\in\mathcal{N}_i^i} a_{ij}\gamma_j\tilde{x}_j, \tag{20}$$

According to $e_{xi}\left(t_k^i\right) = 0$, the solution of $e_{xi}(t)$ is

$$e_{xi}(t) = \int_{t_s^i}^{t}\left\{-\alpha(\tau)\nabla h_i(x_i(\tau)) - d_i^{out}\gamma_i x_i\left(t_s^i\right) + \sum_{j\in\mathcal{N}_i^i} a_{ij}\gamma_j x_j\left(t_q^j\right)\right\}d\tau, \tag{21}$$

where t_s^i and t_q^j are the latest triggering instants of agent i and agent j, respectively. Let t_{s+1}^i be the next triggering instant of agent i and $\sigma = t - t_s^i$. Then, we will show that $t_{s+1}^i - t_s^i$ has a positive lower bound, implying that no Zeno behavior exists.

Above, it is proved that the states of each agent can asymptotically converge to the global optimal point x^* of the problem (1). Therefore, one can has the boundedness of $\nabla h_i(x_i)$ and $-d_i^{out}\gamma_i x_i + \sum_{j\in\mathcal{N}_i^i} a_{ij}\gamma_j x_j$. Then, define $\Delta_1 = \max_{i\in 1,\cdots,N}\{\|\nabla h_i(x_i)\|\}$, $\Delta_2 = \max_{i\in 1,\cdots,N}\left\{\left\| - d_i^{out}\gamma_i x_i + \sum_{j\in\mathcal{N}_i^i} a_{ij}\gamma_j x_j\right\|\right\}$, and $\mathscr{F}(\sigma) = (\Delta_1 + \Delta_2)\sigma$. Note that $\mathscr{F}(\sigma) > 0$ if and only if $\sigma > 0$. Since $\|e_{xi}(t)\| \le \mathscr{F}(\sigma)$, $\forall t \ge t_s^i$, the interval between two adjacent events $t_{s+1}^i - t_s^i$ is larger than or equal to the implicit solution of $\mathscr{F}(\bar{\sigma}) = \sqrt{\mu_{1i}}e^{-\frac{\upsilon_{1i}}{2}\left(\bar{\sigma}+t_s^i\right)}$. And the right-hand side of the above equation is always strictly positive, which implies that $t_{s+1}^i - t_s^i \ge \bar{\sigma} > 0$, thus the intervals between any two adjacent events for the i-th agent are strictly positive, i.e., no Zeno behavior exists. This completes the proof. ■

Corollary 1: If one changes the time-varying gain $\alpha(t)$ to be a fixed positive constant gain α, the ETDOP can be also addressed and $\xi(t)$ will exponentially converge to 0.

***Proof*:** Under this case, the ETDOA is now designed as

$$\dot{x}_i = -\alpha\nabla h_i(x_i) - d_i^{out}\gamma_i\tilde{x}_i + \sum_{j\in\mathcal{N}_i^i} a_{ij}\gamma_j\tilde{x}_j, \tag{22a}$$

$$\dot{\gamma}_i = -d_i^{out}\tilde{\gamma}_i + \sum_{j\in\mathcal{N}_i^i} a_{ij}\tilde{\gamma}_j. \tag{22b}$$

It is easy to find that the proof of optimality is similar to Theorem 1, we only analyze the convergence of the system (22a). The dynamics of ξ is now expressed as

$$\begin{aligned}\dot{\xi} =& -\alpha(\nabla h(\xi+\bar{x}) - \nabla h(\bar{x})) - (\mathscr{L}\Phi\otimes I_n)(\xi+e_x)\\ &-(\mathscr{L}(\Lambda_d - \Phi)\otimes I_n)(\xi+e_x).\end{aligned} \tag{23}$$

Chose the same Lyapunov function $V_2 = \frac{1}{2}\xi^T\xi$, therefore,

$$\dot{V}_2 = -\alpha\xi^T(\nabla h(\xi+\bar{x}) - \nabla h(\bar{x})) - \xi^T(\mathscr{L}\Phi\otimes I_n)(\xi+e_x). \tag{24}$$

Then, one has

$$\dot{V}_2 \le -\frac{1}{2}\left(\lambda_2\left(\hat{\mathscr{L}}\right) - \delta - 2\mu\alpha\right)V_2 + \kappa\mu_1 e^{-\upsilon_1 t}, \tag{25}$$

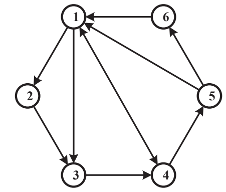

Fig. 1: The communication network among agents.

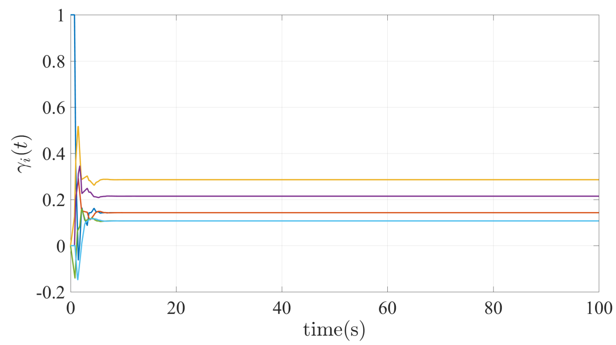

Fig. 2: The evolution of the state trajectory of each estimator.

Then, if one selects a small α satisfying $\alpha < \frac{\lambda_2(\hat{\mathscr{L}})-\delta}{2\mu}$, then, it follows from the Comparison Lemma that $\xi(t)$ exponentially converges to 0, i.e., the states of each agent exponentially converge to the unique global minimizer x^* of the problem (1) with zero errors.

Remark 3: The advantage of using decreasing time-varying gain/step $\alpha(t)$ in this paper is that the system can slowly converge to the exact optimal solution without missing it as the system gradually approaches the optimal solution. Besides, the diminishing gains/step-sizes could be designed without having any global information about the network or the aggregate/local cost functions, while the cost of utilizing $\alpha(t)$ is a slower convergence rate of system. For convenience, a suitable constant gain α can also be selected when designing the ETDOAs, and the proof procedure can be simplified.

IV. Simulations

Consider a group of six agents and the underlying communication network is shown in Fig. 1. Each agent is

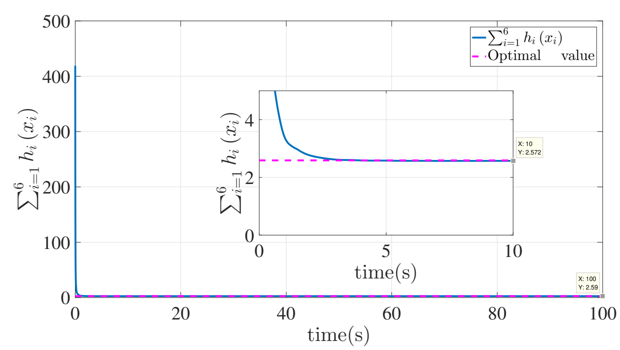

Fig. 3: The evolution of the sum of local cost functions $\sum_{i=1}^{6} h_i(x_i)$ under ETDOA (3).

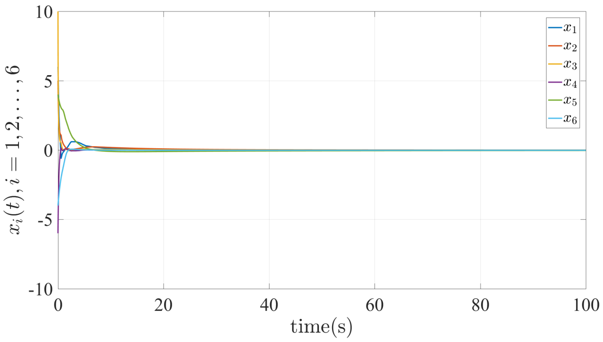

Fig. 4: The evolution of the state trajectory of each agent.

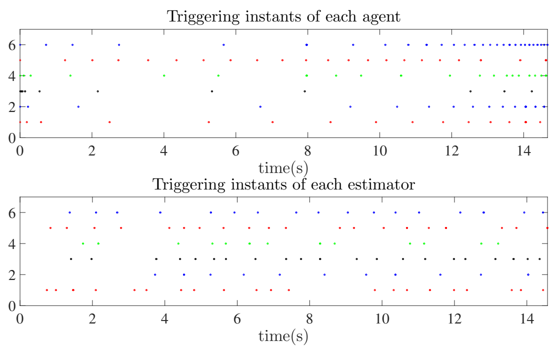

Fig. 5: Triggering instants of each estimator and agent.

assigned a local cost function $h_i, i = 1, 2, \cdots, 6$, which is represented by $h_1 = 0.5e^{-0.5x_1} + 0.4e^{0.3x_1}, h_2 = 0.5x_2^2 \ln\left(1+x_2^2\right), h_3 = 0.5x_3^2 \ln\left(1+x_3^2\right) + x_3^2, h_4 = x_4^2 + e^{0.1x_4}, h_5 = \ln\left(e^{-0.1x_5} + e^{0.3x_5}\right) + 0.1x_5^2, h_6 = \frac{x_6^2}{\ln\left(2+x_6^2\right)}$. The initial values $x_i(0)$ are chosen as $[6; 5; 10; -6; 4; -4]$ and the initial values $\gamma_i(0)$ are set to $[1; 0; 0; 0; 0; 0]$. μ_{1i}, υ_{1i}, μ_{2i}, υ_{2i} are set to the appropriate positive parameter. And the time-varying gain $\alpha(t)$ is chosen as $\frac{5}{t+2}$ for all $t > 0$. In Fig. 2, the evolution of the state trajectory of each estimator is presented, which demonstrates that $\gamma_i(t), i = 1, \cdots, 6$, converge to unique right eigenvector ψ associated with the zero eigenvalue of $\mathscr{L}$. Fig. 3 gives the evolution of the sum of local cost functions $\sum_{i=1}^{6} h_i(x_i)$ under ETDOA (3). In Fig. 4, the evolution of the state trajectory of each agent under ETDOA (3) is shown. And Fig. 5 depicts the triggering instants of each agent and each estimator, which indicates that continuous communication is avoided. These simulation results demonstrates that the ETDOP under strongly connected directed graphs is solved, i.e., the sum of the local cost functions is minimized, the states of each agent asymptotically converge to the unique global minimizer x^*, and the continuous communication is avoided.

V. Conclusions

In this paper, a class of modified ETDOAs is proposed to deal with the ETDOPs under the weighted-unbalanced digraphs. Compared with the existing literature, the communication topology and communication mechanism in this paper have more practical significance. Future work will be on the event-triggered distributed constrained optimization problems under weighted-unbalanced digraphs.

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