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Consensus Control Driven by Value Exchange

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***Abstract*— A multi-agent system is a system consisting of multiple agents that can make a global decision autonomously. So far, many studies have been conducted under the assumption that all agents behave cooperatively. On the other hand, in a large-scale multi-agent system such as a connected vehicle network, each agent must be owned by different owners. Thus, in such a system, the agents do not necessarily work cooperatively and each agent pursues its utility. Such a multi-agent system is modeled as a system driven by the exchange of certain values. In this paper, we address a consensus problem for a multi-agent system driven by the exchange between tokens and information. Unlike the typical consensus control, we assume that each agent has tokens and collects information from its neighbors in exchange for tokens. To this system, we derive a necessary and sufficient condition for the system to achieve consensus. This condition is characterized by the number of tokens and the network structure. Moreover, we disclose that the consensus value is given by the left eigenvector of the Perron matrix associated with the initial token distribution. Finally, we discuss the convergence property for several specific network structures.**

I. INTRODUCTION

A multi-agent system is a system consisting of multiple agents that can make a global decision autonomously. So far, many studies have been conducted under the assumption that all agents behave cooperatively. For example, the formation of vehicles and the computation of statistics in sensor networks have been actively studied so far [1]–[11].

On the other hand, in a large-scale multi-agent system such as a connected vehicle network, each agent must be owned by different owners [12]. Thus, in such a system, the agents do not necessarily work cooperatively and each agent pursues its utility. Such a multi-agent system is sometimes modeled as a system driven by the exchange of certain values.

In game theory, such systems have been studied (see, e.g., [13], [14]). Moreover, in the field of mechanism design, many rules of value exchange have been developed [15]. These studies have analyzed the convergence property of the system driven by the value exchange and proposed exchange rules to achieve the desired states under certain dynamics of agents. However, as far as we know, consensus control driven by value exchange has never been studied so far.

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In this paper, we address a consensus problem for a multi-agent system driven by the exchange between tokens and information. Unlike the typical consensus control [10], [11], we assume that each agent has tokens and collects information from its neighbors in exchange for tokens. If there is a sufficient number of tokens in the system, the agents interact infinitely many times, which results in consensus. However, if not, some agent does not obtain new information after a while. Therefore, the number of tokens to achieve consensus is of interest to us. The contributions of this paper are summarized as follows.

First, we derive a necessary and sufficient condition for the system to achieve consensus. This condition is characterized by the number of tokens and the network structure. In particular, we show that the consensus is achieved under the property that the tokens continuously circulate through the agents. This property is called here the *token cyclicity*.

Second, we disclose that the consensus value is given by the left eigenvector of the Perron matrix associated with the initial token distribution.

Finally, we discuss the convergence property for several specific network structures.

Notation: Let $\mathbf{R}$, $\mathbf{R}_+$, and $\mathbf{Z}_{0+}$ be the real number field, the set of positive real numbers, and the set of nonnegative integers, respectively. For a finite set $\mathbf{S}$, $|\mathbf{S}|$ denotes the cardinality of $\mathbf{S}$. Next, consider a directed graph $G = (\mathbf{V}, \mathbf{E})$ with the node set $\mathbf{V}$ and the edge set $\mathbf{E} \subseteq \mathbf{V} \times \mathbf{V}$. For a node $i \in \mathbf{V}$, we use two neighbor sets:

$$\mathbf{N}_i^{in} := \{j \in \mathbf{V} : (j, i) \in \mathbf{E}\},$$
$$\mathbf{N}_i^{out} := \{j \in \mathbf{V} : (i, j) \in \mathbf{E}\}.$$

We denote by $\Delta(G)$ the maximum value of $|\mathbf{N}_i^{in}|$ with respect to $i \in \mathbf{V}$. Let L and P be the graph Laplacian and the Perron matrix of G. Note that

$$P := I - \varepsilon L$$

for $\varepsilon \in \mathbf{R}$. Consider the graphs $G_1 = (\mathbf{V}_1, \mathbf{E}_1)$ and $G_2 = (\mathbf{V}_2, \mathbf{E}_2)$. We define the product of graphs G_1 and G_2 as

$$G_1 G_2 := (\mathbf{V}_1 \cup \mathbf{V}_2, \mathbf{E}_1 \mathbf{E}_2),$$

where $\mathbf{E}_1 \mathbf{E}_2 \in (\mathbf{V}_1 \cup \mathbf{V}_2) \times (\mathbf{V}_1 \cup \mathbf{V}_2)$ is the set of edges (i, j) satisfying either of following conditions:

(i) $(i, j) \in \mathbf{E}_1$ or $(i, j) \in \mathbf{E}_2$.
(ii) There exists an $l \in (\mathbf{V}_1 \cup \mathbf{V}_2) \backslash \{i, j\}$ such that $(i, l) \in \mathbf{E}_1$ and $(l, j) \in \mathbf{E}_2$.

Finally, we use $\mathrm{diag}(c_1, c_2, \ldots, c_n)$ to represent the diagonal matrix, whose diagonal elements are $c_1, c_2, \ldots, c_n$.

II. PROBLEM FORMULATION

As stated in Section I, we consider the consensus control of multi-agent systems driven by the value exchange between tokens and information. In this section, we formulate the system and problem to be studied.

A. System Description

Consider a network system Σ with n agents. The network is represented by a directed graph $G = (\mathbf{V}, \mathbf{E})$ with the index set of the agents, i.e., $\mathbf{V} = \{1, 2, \ldots, n\}$, and the edge set, i.e., $\mathbf{E} \subseteq \mathbf{V} \times \mathbf{V}$.

The dynamics of agent $i \in \mathbf{V}$ is given by

$$\begin{cases} x_i(t+1) = x_i(t) + u_i(t), \\ y_i(t+1) = y_i(t) + v_i(t). \end{cases} \tag{1}$$

Here, $x_i(t) \in \mathbf{R}$ and $y_i(t) \in \mathbf{R}_+$ are the states representing the information memory and the credit of tokens. Moreover, $u_i(t) \in \mathbf{R}$ is the incoming information and $v_i(t) \in \mathbf{R}$ is the income of tokens.

For this system, we consider a consensus problem by the exchange between information and tokens. Assume that if agent i has a sufficient number of tokens, agent i pays each neighbor one token and receives the information about the difference of the state, otherwise agent i does nothing. We use $s_i(y_i(t))$ to represent the ability of the exchange, i.e.,

$$s_i(y_i(t)) = \begin{cases} 1, & |\mathbf{N}_i^{in}| \geq 1 \text{ and } y_i(t) \geq |\mathbf{N}_i^{in}|, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where $\mathbf{N}_i^{in}$ is the neighbor of agent i and the condition $y_i(t) \geq |\mathbf{N}_i^{in}|$ implies that agent i has a sufficient number of tokens.

Then, the input $u_i(t)$ is given by

$$u_i(t) = -s_i(y_i(t)) \left(\varepsilon \sum_{j \in \mathbf{N}_i^{in}} (x_i(t) - x_j(t)) \right), \tag{3}$$

where $\varepsilon \in \mathbf{R}_+$ is a positive number such that $\varepsilon < 1/\Delta(G)$. This input is the typical consensus controller when $s_i(y_i(t)) = 1$.

On the other hand, the input $v_i(t)$ is given by

$$v_i(t) = -s_i(y_i(t)) |\mathbf{N}_i^{in}| + |\tilde{\mathbf{N}}_i^{out}(t)|, \tag{4}$$

where $\tilde{\mathbf{N}}_i^{out}(t)$ is the set defined as

$$\tilde{\mathbf{N}}_i^{out}(t) = \mathbf{N}_i^{out} \cap \tilde{\mathbf{V}}(t). \tag{5}$$

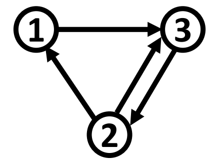
Here, $\mathbf{N}_i^{out}$ is the index set of neighbors that can pay a token to agent i, while

$$\tilde{\mathbf{V}}(t) = \{ i \in \mathbf{V} : s_i(y_i(t)) = 1 \}, \tag{6}$$

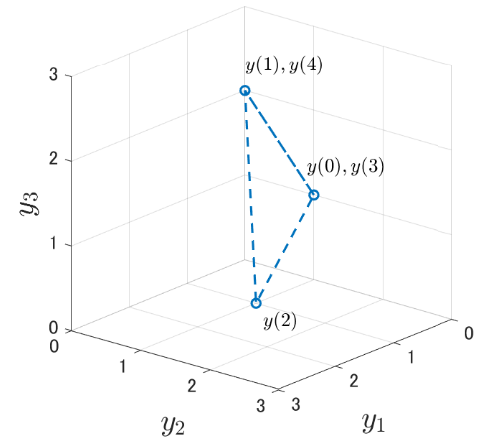
which is the index set of agents that pay tokens at time t. The first and second terms of (4) are the outgo and income of agent i at time t.

The following example shows the dynamics of the system Σ.

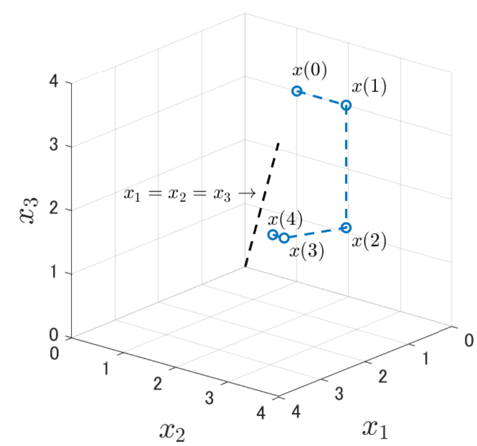
Example 1: Consider the system Σ with the network in Fig. 1 (a) and $\varepsilon = 1/2.1$. The initial state and initial token distribution are given by $x(0) = [0\ 1\ 3]^\top$ and $y(0) = [0\ 1\ 1]^\top$, where the i-th elements of $x(0)$ and $y(0)$ correspond to the initial state and token of agent i, respectively.



(a) Network structure of Example 1.



(b) Token distribution $y(t)$ at $t = 0, 1, 2, 3, 4$.



(c) State $x(t)$ at $t = 0, 1, 2, 3, 4$.

Fig. 1: Example of value exchange.

At $t = 0$, we have $s_2(y_2(0)) = 1$ since $y_2(0) \geq |\mathbf{N}_2^{in}| = 1$. Therefore, agent 2 pays one token to agent 3. On the other hand, $s_1(y_1(0)) = s_3(y_3(0)) = 0$ since $y_1(0) < |\mathbf{N}_1^{in}| = 1$ and $y_3(0) < |\mathbf{N}_3^{in}| = 2$. Thus, we obtain $y(1) = [0\ 0\ 2]^\top$ and $x(1) = [0\ 1.95\ 3]^\top$.

Next, at $t = 1$, we obtain $s_1(y_1(1)) = 0$, $s_2(y_2(1)) = 0$, and $s_3(y_3(1)) = 1$. Therefore, it follows that $y(2) = [1\ 1\ 0]^\top$ and $x(2) = [0\ 1.95\ 1.07]^\top$.

Similarly, we obtain $y(3) = [0\ 1\ 1]^\top$, $y(4) = [0\ 0\ 2]^\top$, $x(3) = [0.93\ 1.53\ 1.07]^\top$, and $x(4) = [0.93\ 1.31\ 1.07]^\top$. Fig. 1 (b) and (c) show the trajectories of $y(t)$ and $x(t)$ at $t = 0, 1, 2, 3, 4$ on the three-dimensional state spaces, respectively. From (c), we see that the state $x_i(t)$ $(i = 1, 2, 3)$ approaches the consensus, i.e., $x_1 = x_2 = x_3$. □

B. Consensus Problem

The total number of tokens is expressed as

$$\bar{y} = \sum_{i \in \mathbf{V}} y_i(0). \tag{7}$$

Here, we are interested in a consensus problem under the assumption that the total number of tokens $\bar{y}$ is prespecified but the initial token distribution $y(0)$ can not be specified. This assumption corresponds to the case that each agent is owned by different owners and each owner can only determine the initial tokens of its agents. Unlike the typical consensus condition depending on the network structure [11], the condition depends on both the network structure and the total number of tokens.

Example 2: Consider a multi-agent system Σ with 4 agents. The network structure is given as Fig. 2 and $\varepsilon = 1/4$. The initial states are given as $x_1(0) = 1$, $x_2(0) = 6$, $x_3(0) = 5$, and $x_4(0) = 8$.

First, assume $y(0) = [1\ 0\ 0\ 0]^\top$, i.e., $\bar{y} = 1$. Fig. 3 (a) shows the time evolution of $x_i(t)$. In this case, we see that the system does not achieve consensus.

Next, assume $y(0) = [1\ 1\ 0\ 0]^\top$, i.e., $\bar{y} = 2$. Fig. 3 (b) depicts the time evolution of $x_i(t)$, which exhibits that the system achieves consensus.

These indicate that the achievement of consensus depends on $\bar{y}$. □

Motivated by this fact, we address the following problem.

Problem 1: Consider the multi-agent system Σ. Then, find $\bar{y} \in \mathbf{R}_+$ such that

$$\lim_{t \to \infty} (x_i(t) - x_j(t)) = 0, \quad \forall (i,j) \in \mathbf{V} \times \mathbf{V} \tag{8}$$

for any $x_i(0) \in \mathbf{R}$ and $y_i(0) \in \mathbf{R}_+$ ($i = 1, 2, \ldots, n$) satisfying (7). □

III. CONDITION FOR CONSENSUS

A. Token Cyclicity

To derive a solution to Problem 1, we introduce the notion of token cyclicity.

Definition 1: Consider the system Σ. The pair $(G, \bar{y})$ is said to be **token cyclic** if

$$\bigcup_{t=0}^{\infty} \tilde{\mathbf{V}}(t) = \mathbf{V} \tag{9}$$

holds for all $y_i(0) \in \mathbf{R}_+$ ($i = 1, 2, \ldots, n$) satisfying (7). □

The left-hand side of (9) denotes the list of i such that there exists a $t \in \mathbf{Z}_{0+}$ satisfying $s_i(y_i(t)) = 1$. Thus, (9) implies that all agents pay tokens at least once on the time interval $\mathbf{Z}_{0+}$. Moreover, if (9) holds for all $y_i(0) \in \mathbf{R}_+$ ($i = 1, 2, \ldots, n$) satisfying (7), then, for any $t_0 \in \mathbf{Z}_{0+}$, there exists a $T \in \mathbf{Z}_{0+}$ such that

$$\bigcup_{t=t_0}^{t_0+T} \tilde{\mathbf{V}}(t) = \mathbf{V}. \tag{10}$$

Therefore, the token cyclicity implies that the tokens continuously circulate through the agents.

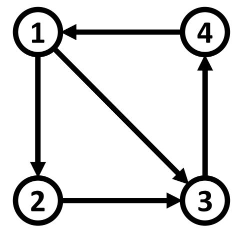
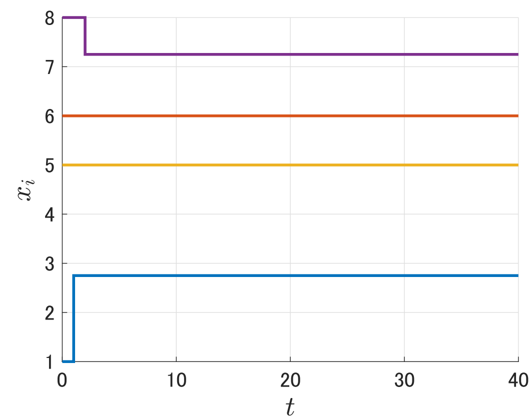
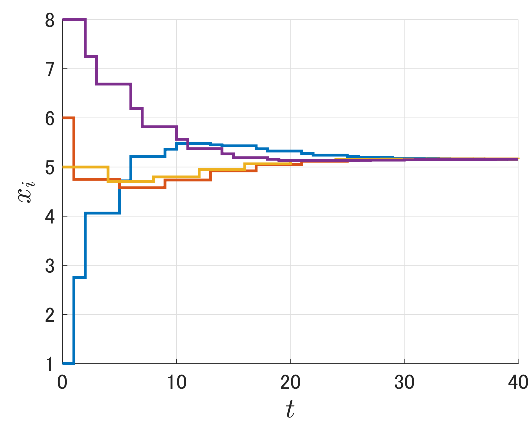


Fig. 2: Network structure of Example 2.



(a) $y(0) = [1\ 0\ 0\ 0]^\top$.



(b) $y(0) = [1\ 1\ 0\ 0]^\top$.

Fig. 3: Simulation result of Example 2.

Example 3: Consider the case with the network structure G in Fig. 4 (a) and $\bar{y} = 1$. Then, $(G, \bar{y})$ is not token cyclic, since $s_i(y_i(t)) = 0$ for all $(i, t) \in \mathbf{V} \times \mathbf{Z}_{0+}$ if $y(0) = [0\ 0\ 0\ 1]^\top$. In fact, for the network structure G, $(G, \bar{y})$ is not token cyclic not only for the above $\bar{y}$ but also for all $\bar{y} \in \mathbf{R}_+$.

Next, consider the case with the network structure G in Fig. 4 (b). Then, $(G, \bar{y})$ for $\bar{y} = 1$ is not token cyclic, since $s_i(y_i(t)) = 0$ for all $(i, t) \in \mathbf{V} \times \mathbf{Z}_{0+}$ if $y(0) = [0\ 1\ 0\ 0]^\top$. On the other hand, $(G, \bar{y})$ for $\bar{y} = 2$ is token cyclic, since (9) holds for all $y_i(0) \in \mathbf{R}_+$ ($i = 1, 2, \ldots, n$) satisfying (7). □

The following result provides a necessary and sufficient condition for token cyclicity.

Lemma 1: Consider the system Σ. The pair $(G, \bar{y})$ is token cyclic if and only if G is strongly connected and

$$\bar{y} \geq \sum_{i \in \mathbf{V}} (|\mathbf{N}_i^{in}| - 1) + 1 \tag{11}$$

holds. □

By using Lemma 1, we can determine the total number of tokens $\bar{y}$ such that the pair $(G, \bar{y})$ is token cyclic.

B. Solution to Problem 1

Using the notion of token cyclicity, we can obtain a solution to Problem 1.

Theorem 1: Consider the system Σ. Then, the system achieves consensus for any $x_i(0) \in \mathbf{R}$ and $y_i(0) \in \mathbf{R}_+$ $(i = 1, 2, \ldots, n)$ satisfying (7) if and only if $(G, \bar{y})$ is token cyclic. □

Proof: From (1) and (3), the system Σ is expressed as

$$x(t+1) = P(t)x(t), \tag{12}$$

where

$$P(t) = I - \varepsilon \mathrm{diag}(s_1(y_1(t)), s_2(y_2(t)), \ldots, s_n(y_n(t)))L, \tag{13}$$

which is the Perron matrix of the graph $G(t) = (\mathbf{V}, \mathbf{E}(t))$ for $\mathbf{E}(t) = \{(i,j) \in \mathbf{E} : j \in \tilde{\mathbf{V}}(t)\}$ and $\varepsilon \in \mathbf{R}_+$.

Since $\bar{y}$ is finite, there exist a $t_0 \in \mathbf{Z}_{0+}$ and $T \in \mathbf{Z}_{0+}$ such that

$$y(t_0) = y(T + t_0). \tag{14}$$

Now, let $\bar{P}(y(t_0)) \in \mathbf{R}^{n \times n}$ be the matrix defined as

$$\bar{P}(y(t_0)) = P(T - 1 + t_0)P(T - 2 + t_0) \cdots P(t_0), \tag{15}$$

which is the Perron matrix of the graph

$$\bar{G} = G(t_0) \cdots G(T - 2 + t_0)G(T - 1 + t_0) \tag{16}$$

for a positive real number $\bar{\varepsilon} < 1/\Delta(\bar{G})$ [11]. Moreover, for $\tau \in \mathbf{Z}_{0+}$, we define the state $z(\tau)$ as

$$z(\tau) = x(\tau T + t_0). \tag{17}$$

Then, by using $\bar{P}(y(t_0))$ and (12), we can obtain the difference equation of $z(\tau)$ as

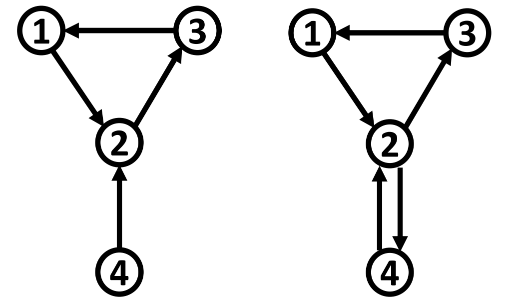
$$z(\tau + 1) = \bar{P}(y(t_0))z(\tau), \quad z(0) = x(t_0), \tag{18}$$

which is the dynamics (12) at $t \in \{t_0, T + t_0, 2T + t_0, \ldots\}$.

It is known that the system (18) achieves consensus for any $z(0) \in \mathbf{R}^n$ if and only if the network $\bar{G}$ has a directed spanning tree [11]. Thus, Theorem 1 can be obtained by showing that $\bar{G}$ has a spanning tree for any $y(0) \in \mathbf{R}_+^n$ satisfying (7) if and only if $(G, \bar{y})$ is token cyclic. □

From Lemma 1 and Theorem 1, the system Σ achieves consensus for any initial states by determining the number of tokens $\bar{y}$ so that (11) is satisfied.

Example 4: Consider again the system in Example 2 with the network structure G in Fig. 2 and $\bar{y} = 2$. From Lemma 1, $(G, \bar{y})$ is token cyclic because G is strongly connected and $\Sigma_{i \in \mathbf{V}}(|\mathbf{N}_i^{in}| - 1) + 1 = 0 + 0 + 1 + 0 + 1 = 2 \leq \bar{y}$. Thus, it follows from Theorem 1 that the system Σ achieves consensus for any $y(0) \in \mathbf{R}_+^n$ satisfying (7). This result agrees with Example 2. □



(a) (b)

Fig. 4: Example graphs of token cyclicity.

Especially, we obtain the following corollaries that determine the minimum number of tokens for cycle graphs and complete graphs to achieve consensus for any initial states.

Corollary 1: Consider the system Σ with a cycle graph G. Then the system achieves consensus for any $x_i(0) \in \mathbf{R}$ and $y_i(0) \in \mathbf{R}_+$ $(i = 1, 2, \ldots, n)$ satisfying (7) if and only if $\bar{y} \geq 1$. □

Proof: From Lemma 1, the pair of a cycle graph G and $\bar{y} = 1$ is token cyclic. From this fact and Theorem 1, we obtain Corollary 1. □

Corollary 2: Consider the system Σ with a complete graph G. Then the system achieves consensus for any $x_i(0) \in \mathbf{R}$ and $y_i(0) \in \mathbf{R}_+$ $(i = 1, 2, \ldots, n)$ satisfying (7) if and only if $\bar{y} \geq (n-1)^2$. □

Proof: When G is a complete graph, $|\mathbf{N}_i^{in}| = n - 1$ for all $i \in \mathbf{V}$. Then, from Lemma 1, the pair $(G, \bar{y})$ is token cyclic if and only if

$$\bar{y} \geq \sum_{i \in \mathbf{V}} (|\mathbf{N}_i^{in}| - 1) + 1 = (n-1)^2. \tag{19}$$

From this fact and Theorem 1, we obtain Corollary 2. □

C. Consensus Value

When the system Σ achieves consensus, then the consensus value is obtained as follows.

Theorem 2: Consider the system Σ. Suppose that $x_i(0)$ and $y_i(0)$ $(i = 1, 2, \ldots, n)$ are given. Assume that $(G, \bar{y})$ is token cyclic. If there exists a $T \in \mathbf{Z}_{0+}$ such that

$$y(0) = y(T), \tag{20}$$

then the consensus value $\alpha \in \mathbf{R}$ is obtained as

$$\alpha = \frac{\sum_{i=1}^{n} v_i(y(0))x_i(0)}{\sum_{i=1}^{n} v_i(y(0))}, \tag{21}$$

where $v(y(0)) \in \mathbf{R}^n$ is the left eigenvector of $\bar{P}(y(0))$ in (15) associated with the eigenvalue one, and $v_i(y(0)) \in \mathbf{R}$ is its i-th element. □

Proof: Consider the system (18). If (20) holds, i.e., $t_0 = 0$ satisfies (14), then $z(0) = x(0)$. From the property of the consensus value of the system (18) [2], we obtain (21). □

Theorem 2 implies that the consensus value of the system Σ can be characterized by the weighted graph determined from the graph G and the initial token distribution $y(0)$.

Example 5: Consider the system in Example 2 with the network structure G in Fig. 2 and $y(0) = [1\ 1\ 0\ 0]^\top$. We obtain the consensus value as 5.16.

In fact, since $y(1) = [1\ 0\ 0\ 1]^\top$, $y(2) = [0\ 0\ 1\ 1]^\top$, $y(3) = [0\ 0\ 2\ 0]^\top$, and $y(4) = [1\ 1\ 0\ 0]^\top = y(0)$, there exists a $T \in \mathbf{Z}_{0+}$ satisfying (20). Then, from (15), we obtain

$$\bar{P}(y(0)) = P(3)P(2)P(1)P(0) = \begin{bmatrix} 0.5625 & 0 & 0 & 0.4375 \\ 0.2500 & 0.7500 & 0 & 0 \\ 0.2031 & 0.1875 & 0.5000 & 0.1094 \\ 0 & 0 & 0.4375 & 0.5625 \end{bmatrix}, \tag{22}$$

with the left eigenvector of the eigenvalue one as

$$v(y(0)) = [0.4663\ 0.3917\ 0.5223\ 0.5969]. \tag{23}$$

Note that $P(t)$ $(t = 0, 1, 2, 3)$ is the Perron matrix of $G(t)$ shown in Fig. 5 and $\bar{P}(y(0))$ is the Perron matrix of the weighted graph $\bar{G}$ shown in Fig. 6.

Finally, from (21), we can obtain the consensus value α as

$$\alpha = \frac{0.4663 \times 1 + 0.3917 \times 6 + 0.5223 \times 5 + 0.5969 \times 8}{0.4663 + 0.3917 + 0.5223 + 0.5969} = 5.16, \tag{24}$$

which agrees with the simulation result. □

IV. Convergence Property of System Σ with Not Strongly Connected Graphs

In this section, we discuss the convergence property of the system Σ with not strongly connected graphs such as directed trees.

Consider the case that G is a directed tree. Then, we obtain the following proposition, which indicates that the system with directed trees never achieves consensus even if we can determine not only the total number of tokens $\bar{y}$ but also the token distribution $y(0)$.

Proposition 1: Consider the system Σ with a directed tree G. Assume that the number of tokens $\bar{y}$ is given. Then, for any $x_i(0) \in \mathbf{R}$ and $y_i(0) \in \mathbf{R}_+$ $(i = 1, 2, \ldots, n)$ satisfying (7), there exist an $i \in \mathbf{V}$ and $T \in \mathbf{Z}_{0+}$ such that

$$y_j(t) = \begin{cases} \bar{y} & (j = i), \\ 0 & (j \in \mathbf{V}\backslash\{i\}), \end{cases} \quad t \geq T \tag{25}$$

holds. □

Proposition 1 implies that the circulation of tokens in the system Σ with a directed tree stops after a finite time and one of the agents collects all tokens.

Finally, we consider the following example that shows the dynamics of the system Σ with a not strongly connected graph. In this example, we see that the system achieves partial consensus, although the entire system does not achieve consensus.

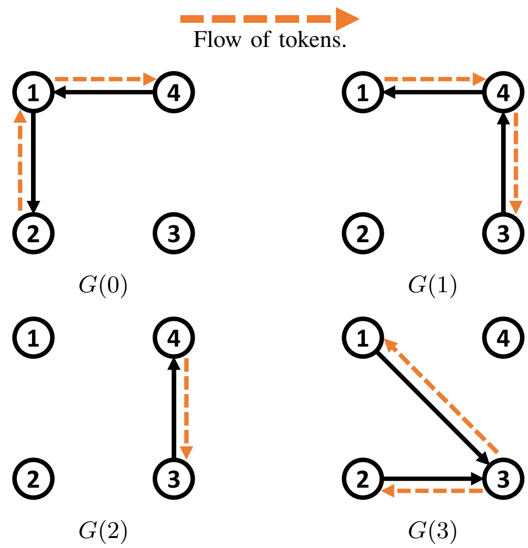


Fig. 5: Network structure $G(t)$ at $t = 0, 1, 2, 3$.

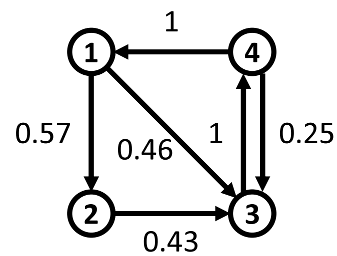


Fig. 6: Weighted graph associated with $\bar{P}(y(0))$ in (22).

Example 6: Consider the system Σ consisting of 9 agents. The network structure G is shown in Fig. 7. This network structure can be expressed as the union of the following 4 graphs:

Complete graph $G_1 = (\mathbf{V}_1, \mathbf{E}_1)$:
$\mathbf{V}_1 = \{1, 2, 3\}$,
$\mathbf{E}_1 = \{(1, 2), (2, 3), (3, 1), (2, 1), (3, 2), (1, 3)\}$.
Cycle graph $G_2 = (\mathbf{V}_2, \mathbf{E}_2)$:
$\mathbf{V}_2 = \{6, 7, 8\}$, $\mathbf{E}_2 = \{(6, 7), (7, 8), (8, 6)\}$.
Directed tree $G_3 = (\mathbf{V}_2, \mathbf{E}_2)$, $G_4 = (\mathbf{V}_4, \mathbf{E}_4)$:
$\mathbf{V}_3 = \{3, 4, 5\}$, $\mathbf{E}_3 = \{(3, 4), (3, 5)\}$,
$\mathbf{V}_4 = \{8, 9\}$, $\mathbf{E}_4 = \{(8, 9)\}$.

Now, we assume that the initial states and the initial token distribution are given as $x(0) = [0\ 3\ 5\ 8\ 9\ 11\ 16\ 18\ 20]^\top$, and $y(0) = [0\ 0\ 0\ 1\ 3\ 0\ 0\ 0\ 1]^\top$, respectively.

Fig. 8 (a) shows the time evolution of the states $x_i(t)$ $(i = 1, 2, \ldots, 9)$ and Fig. 8 (b) shows the time evolution of the tokens $y_i(t)$ $(i = 3, 4, 8, 9)$. From Fig. 8 (a), we see that the states of $x_i(t)$ $(i = 1, 2, 3)$ converge to 1.5 and $x_i(t)$ $(i = 6, 7, 8)$ converges to 14.4, which implies that among the agents $\{1, 2, 3\}$ and $\{6, 7, 8\}$ achieve consensus, respectively, while the entire system does not achieve consensus.

In fact, from Proposition 1, all tokens of agents 4 and 5 are paid to agent 3 and the agents $\{3, 4, 5\}$ never achieve consensus since G_3 is a directed tree. On the other hand, the agents $\{1, 2, 3\}$ obtain four tokens. Then, from Corollary 2, we see that the system Σ with the network G_1 achieves consensus since G_1 is a complete graph with three agents.

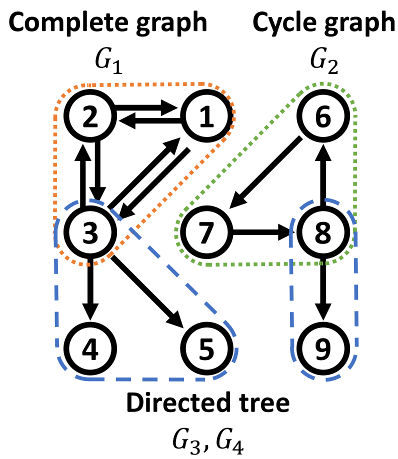
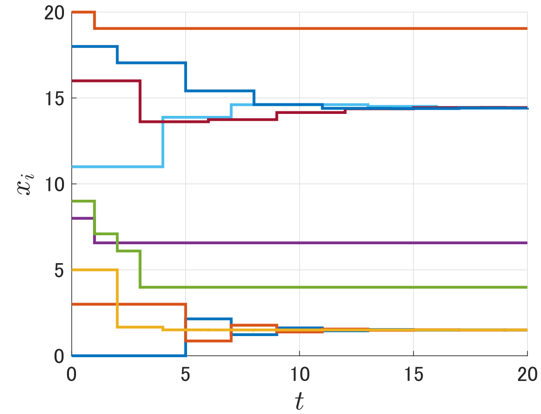


Fig. 7: Network structure of Example 6.

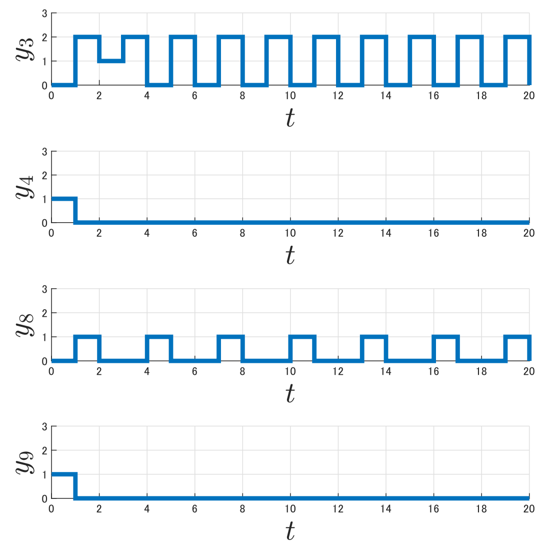
Likewise, from Proposition 1, agent 9 pays all tokens to agent 8 and the agents $\{8, 9\}$ never achieve consensus. On the other hand, the agents $\{6, 7, 8\}$ obtain one token. From Collonary 1, we see that the system Σ with the network G_2 achieves consensus since G_2 is a cycle graph. $\square$

V. Conclusion

In this paper, we have addressed a consensus problem for a multi-agent system driven by the exchange between tokens and information. We have clarified that the system achieves consensus for any initial token distributions if and only if the system satisfies the condition of token cyclicity. Moreover, we have shown that the consensus value is determined from the weighted graph characterized by the network structure and initial token distributions.



(a) Time evolution of the states $x_i(t)$ $(i = 1, 2, \dots, 9)$.



(b) Time evolution of the token $y_i(t)$ $(i = 3, 4, 8, 9)$.

Fig. 8: Simulation result of Example 6.

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