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Cooperative Control of Constrained Discrete-Time Multi-Train Systems: A Fully Distributed Approach

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Abstract— **The paper investigates fully distributed cooperative control for discrete-time multi-train systems, focusing on managing transient constraints on position differences between neighboring trains, as well as on individual train velocities and inputs. Existing research on this topic is limited and typically requires a linear network structure, with each train having access to network structural information, such as the Laplacian matrix. However, travel routes may vary significantly, and the network structure is prone to change. Consequently, this study introduces a fully distributed control law that solely relies on relative position information from neighboring trains, offering adaptability to changes in the network structure of actual multi-train systems. Technically, the paper ingeniously converts both position and velocity constraints into input constraints, guiding the controller design to achieve the objectives of transient constraints.**

I. INTRODUCTION

Trains have become a popular choice for travel due to their safety and comfort. To accommodate the continuously increasing number of passengers, enhancing the carrying capacity of railway systems is both necessary and urgent. Cooperative control of multiple trains, while ensuring safety, can reduce the operational distance between adjacent trains. It has been demonstrated to effectively increase the capacity of existing railway systems, as documented in references [1], [2]. Therefore, research on the cooperative control of multiple trains has garnered widespread attention.

In the continuous-time domain, significant progress has been made in the cooperative control of multi-train systems. Reference [1] introduces a cooperative control method for multi-train systems under the context of the moving block signaling system, along with corresponding stability criteria. Reference [3] proposes a distributed cooperative control scheme for trains with a rear fence communication topology. To ensure the robustness of train operations: [4] investigates robust distributed control problems under external disturbances; [5] studies distributed fault-tolerant control problems in the case of local actuator failures; [6] explores cooperative control strategies for trains under denial-of-service attacks. Addressing the physical constraints inherent in trains: [5] considers the issue of input saturation; [7] investigates problems where both velocity and input are constrained. To prevent train collisions: [8] explores tracking and anti-collision issues; [2] delves into cooperative predefined performance control problems.

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On the contrary, research on cooperative control of train systems in the discrete-time domain is notably lacking. Due to the widespread use of digital circuits in train systems, both control signals and sensor signals are predominantly in digital form. Therefore, research in the continuous-time domain need be discretized before being applied to real systems, which inevitably introduces numerical errors. While these errors may be tolerable for some linear systems, they pose high risks for trains with complex nonlinear characteristics and stringent safety requirements. This could potentially lead to system instability or significant accidents. To seamlessly apply control strategies to train systems, researchers consider studying discrete-time train models. In this regard, [9], [10] address the tracking control problem with velocity and input constraints in discrete-time train systems, proposing a distributed cooperative tracking control algorithm. However, [9], [10] only consider a single particle train model. Building upon this, [11] extends the aforementioned research to a multi-particle train model, which includes nonlinear coupling between different compartments within a single train. Nevertheless, the above studies do not address collision avoidance, which is crucial for ensuring the safe operation of trains. In addressing this concern, [12] incorporates considerations for collision avoidance among trains based on the aforementioned research.

In the discrete-time domain, although research on cooperative control in multi-train systems has addressed multiple control tasks, several significant issues persist. Firstly, only collision avoidance between trains has been achieved, overlooking situations where trains are too far apart to communicate effectively due to limitations of communication devices. This necessitates constraining the position difference between neighboring trains within upper and lower bounds, a task currently only accomplished in the continuous-time domain, as demonstrated in [2], [13]. Secondly, only research [12] has achieved collision avoidance between adjacent trains in the discrete-time domain. However, [12] relies solely on the braking of the rear train to prevent collisions when the distance between adjacent trains is too close, which may result in energy waste. Clearly, allowing the front train to increase the distance from the rear train by accelerating can enhance the flexibility of adjusting inter-train distances to ensure position difference constraints. Finally, the majority of studies, such as [9]–[12], require the communication network of multi-train systems to be linear and fixed, with each train having access to global network structural information. However, as the operational routes of different trains may vary, the network in a multi-train system is susceptible to changes, and newly added trains may struggle to promptly acquire global network information.

In addressing to the aforementioned issues, this paper achieves cooperative control in the discrete-time domain for a multi-train system. It encompasses three specific objectives: 1) maintaining the distances between neighboring trains within predetermined bounds; 2) ensuring that the velocity and input of each train remain within predefined constraints; 3) achieving the convergence of the distance between neighboring trains to the desired distance, with the velocity of each train converging to the reference velocity. Innovations in this paper are threefold:

1) The paper introduces a fully distributed control law design, exclusively leveraging relative positional information among neighboring trains. This design eliminates the need for global network topology information and avoids imposing a linear network structure, which is required in the studies of [9]–[12].
2) The paper simultaneously considers upper and lower bounds constraints on position difference, velocity, and inputs, with independent and mutually unknown constraint boundaries for different trains. This aspect is either partially addressed or not addressed in existing research on discrete-time domains for trains, such as [9]–[12].
3) An ingeniously transformation of position difference and velocity constraints into input constraints, guiding the design of the controller to simultaneously satisfy the aforementioned triple constraints.

II. PROBLEM FORMULATION

Consider a multi-train system composed of N trains. The dynamics of each train are represented as follows:

$$\begin{aligned} x_i(k+1) =& x_i(k) + v_i(k)T \\ v_i(k+1) =& v_i(k) + u_i(k)T - f_i(k)T,\ i = 1, \cdots, N, \end{aligned} \quad (1)$$

where $x_i(k)$, $v_i(k)$ and $u_i(k) \in \mathbb{R}$ denote the position, velocity, and input of the i-th train, respectively. T is the sampling period, and $f_i(k)$ represents the base resistance. Assuming the neglect of the length of train, each train is treated as a particle. The base resistance is typically given in the following form:

$$f_i(k) = a_i + b_i v_i(k) + c_i v_i^2(k),$$

where the coefficients a_i, b_i, and c_i are empirical parameters obtained from actual train operation data.

From (1), the multi-train system can be regarded as a multi-agent system, where each train is considered as a subsystem. Communication among these subsystems occurs through a predefined network. Below, we establish rules and symbols governing the topology network. In this paper, we assume that the network topology among the agents is described by a fixed undirected weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \cdots, N\}$ denotes the set of indices of each agent, and $\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V}\}$ is the set of edges connecting two distinct agents. The connectivity matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ of $\mathcal{G}$ is defined such that $a_{ij} = a_{ji} = 1$ if $(i, j) \in \mathcal{E}$, and $a_{ij} = a_{ji} = 0$ if $(i, j) \notin \mathcal{E}$. Additionally, we ensure that $a_{ii} = 0$ to disallow self-edges (i, i). In this context, we refer to agent j as a neighbor of agent i if $a_{ij} = a_{ji} = 1$; otherwise, j is not considered a neighbor of i. The set of all neighbors of agent i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. The degree of the i-th agent in the graph $\mathcal{G}$, denoted by $|\mathcal{N}_i|$, represents the number of neighbors of the i-th agent, where $|\cdot|$ denotes the cardinality of a set.

Assumption 2.1: The undirected graph $\mathcal{G}$ is connected. Each agent can only access the relative position information of its neighboring agents and cannot obtain the global network information represented by the connectivity matrix A.

The state of an actual multi-train system is subject to physical constraints. For instance, to avoid collisions, for two adjacent trains, the position of the trailing train must consistently remain behind the position of the leading train; due to communication equipment limitations, the distance between adjacent trains must consistently be less than the maximum effective communication distance to ensure stable communication. Additionally, during operation, due to constraints imposed by different operational states and track conditions, the velocity of trains often needs to be further restricted within specific constraint sets. For example, there may be a minimum velocity requirement for trains in electrified neutral sections. Furthermore, due to the inherent torque limitations of motor outputs, the dynamic model of trains exhibits natural input saturation constraints.

Under the Assumption 2.1, to address the constraints imposed on the practical train system mentioned above, our focus is on designing a fully distributed controller $u_i(k)$ to achieve the following three objectives.

$\mathcal{O}_1$: Ensure that the position difference $x_{ij}(k) = x_i(k) - x_j(k)$ between neighboring trains remains within a specific constraint range throughout the entire operational process, i.e.,

$$\underline{\gamma}_{ij} \leq x_{ij}(k) \leq \overline{\gamma}_{ij},\ \forall k \geq 0,\ (i, j) \in \mathcal{E}. \quad (2)$$

$\mathcal{O}_2$: Ensure that the velocity and input of each train remain within specific constraint ranges throughout the entire operational process, i.e.,

$$\underline{\rho}_i \leq v_i(k) \leq \overline{\rho}_i,\ \underline{\mu}_i \leq u_i(k) \leq \overline{\mu}_i,\ \forall k \geq 0,\ i \in \mathcal{V}. \quad (3)$$

for constants $\underline{\rho}_i < \overline{\rho}_i$ and $\underline{\mu}_i < \overline{\mu}_i$.

$\mathcal{O}_3$: Achieve the asymptotic convergence of the position difference $x_{ij}(k)$ to a desired distance d_{ij}, with the train velocity $v_i(k)$ asymptotically tracking a reference velocity v_r, i.e.,

$$\begin{aligned} &\lim_{k\to\infty} x_{ij}(k) = d_{ij} \\ &\lim_{k\to\infty} v_i(k) = v_r,\ \forall (i, j) \in \mathcal{E},\ i \in \mathcal{V}. \end{aligned} \quad (4)$$

Assumption 2.2: For all $(i, j) \in \mathcal{E}$, $\underline{\gamma}_{ij} < d_{ij} < \overline{\gamma}_{ij}$, $d_{ij} = -d_{ji}$ and $\underline{\gamma}_{ij} = -\overline{\gamma}_{ji}$. The distance d_{ij} satisfies transitivity in the multi-agent system, i.e., $d_{ij} = d_{il} + d_{lj}$, $\forall(i,j),(i,l),(l,j) \in \mathcal{E}$.

Assumption 2.3: For all $i \in \mathcal{V}$, $\underline{\rho}_i < v_r < \overline{\rho}_i$, v_r is known for all trains to ensure safety.

Assumption 2.4: Defining $\underline{f}_i = \min_{v_i(k)\in[\underline{\rho}_i,\overline{\rho}_i]}\{f_i(k)\}$ and $\overline{f}_i = \max_{v_i(k)\in[\underline{\rho}_i,\overline{\rho}_i]}\{f_i(k)\}$, for all $i \in \mathcal{V}$, $\underline{\mu}_i < \underline{f}_i$ and $\overline{f}_i < \overline{\mu}_i$, $[v_r - \overline{\rho}_i]/T > \underline{\mu}_i - \underline{f}_i$ and $[v_r - \underline{\rho}_i]/T < \overline{\mu}_i - \overline{f}_i$.

Remark 2.1: Assumptions 2.2 and 2.3 are foundational, established to ensure the simultaneous achievement of objectives $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ without contradiction. The Assumption 2.4 postulates that the input $u_i(k)$ of train is significantly greater than the foundational resistance $f_i(k)$. This assumption is deemed reasonable and has also been utilized in other rail research, as evidenced in [12]. Additionally, the Assumption 2.4 dictates the relationship between input constraints and velocity constraints, a requirement similar to those widely applied in other studies simultaneously considering input constraints and velocity constraints, as demonstrated in [12], [14], [15].

III. MAIN RESULTS

Given the transitivity property of the distance d_{ij}, as known from Assumption 2.1, there exists a set of constant d_i, $1,\cdots,N$ such that $d_{ij} = d_i - d_j$, $\forall(i,j) \in \mathcal{E}$. Therefore, we introduce a coordinate transformation: $y_i(k) = x_i(k) - d_i$. To facilitate controller design, we further introduce the following transformations: $\overline{y}_i(k) = y_i(k) - v_r kT$ and $\overline{v}_i(k) = v_i(k) - v_r$, and $\overline{u}_i(k) = u_i(k) - f_i(k)$. The system (1) is then equivalent to:

$$\begin{aligned}\overline{y}_i(k+1) =& \overline{y}_i(k) + \overline{v}_i(k)T\\ \overline{v}_i(k+1) =& \overline{v}_i(k) + \overline{u}_i(k)T,\ i = 1,\cdots,N.\end{aligned} \tag{5}$$

Correspondingly, the objectives $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ are rewritten as the following three objectives:

$\mathcal{O}_1'$: Defining $\overline{y}_{ij}(k) = \overline{y}_i(k) - \overline{y}_j(k)$, then objective $\mathcal{O}_1$ is equivalent to:

$$\underline{\gamma}_{ij} - d_{ij} \le \overline{y}_{ij}(k) \le \overline{\gamma}_{ij} - d_{ij},\ \forall k \ge 0,\ (i,j) \in \mathcal{E}. \tag{6}$$

$\mathcal{O}_2'$: Under Assumption 2.4, the objective $\mathcal{O}_2$ can be achieved when the following conditions hold:

$$\begin{aligned}&\underline{\rho}_i - v_r \le \overline{v}_i(k) \le \overline{\rho}_i - v_r,\\ &\underline{\mu}_i - \underline{f}_i \le \overline{u}_i(k) \le \overline{\mu}_i - \overline{f}_i,\ \forall k \ge 0,\ i \in \mathcal{V}.\end{aligned} \tag{7}$$

$\mathcal{O}_3'$: Based on the above coordinate transformation, the objective $\mathcal{O}_3$ is equivalent to:

$$\begin{aligned}&\lim_{k\to\infty} \overline{y}_{ij}(k) = 0\\ &\lim_{k\to\infty} \overline{v}_i(k) = 0,\ \forall(i,j) \in \mathcal{E},\ i \in \mathcal{V}.\end{aligned} \tag{8}$$

Furthermore, for analytical convenience, system (5) can be equivalently represented as:

$$\begin{aligned}\overline{y}_i(k+2) =& \overline{y}_i(k+1) + \overline{v}_i(k+1)T\\ \overline{v}_i(k+1) =& \overline{v}_i(k) + \overline{u}_i(k)T,\ i = 1,\cdots,N.\end{aligned} \tag{9}$$

Let $\overline{y} = [\overline{y}_1,\cdots,\overline{y}_N]^\intercal$. Before designing the controller, we make the following definitions:

$$\begin{aligned}\underline{\alpha}_i(\overline{y}(k+1)) =& \max_{j\in\mathcal{N}_i}\left\{-\frac{\overline{y}_{ij}(k+1) - (\underline{\gamma}_{ij} - d_{ij})}{2}\right\}\\ \overline{\alpha}_i(\overline{y}(k+1)) =& \min_{j\in\mathcal{N}_i}\left\{\frac{\overline{\gamma}_{ij} - d_{ij} - \overline{y}_{ij}(k+1)}{2}\right\}\\ \underline{\beta}_i =& (\underline{\rho}_i - v_r)T\\ \overline{\beta}_i =& (\overline{\rho}_i - v_r)T\\ \underline{\eta}_i =& \overline{\rho}_i T + (\underline{\mu}_i - \underline{f}_i)T^2 - v_r T\\ \overline{\eta}_i =& \underline{\rho}_i T + (\overline{\mu}_i - \overline{f}_i)T^2 - v_r T\\ \underline{\delta}_i(\overline{y}(k+1)) =& \max\{\underline{\alpha}_i(\overline{y}(k+1)), \underline{\beta}_i, \underline{\eta}_i\}\\ \overline{\delta}_i(\overline{y}(k+1)) =& \min\{\overline{\alpha}_i(\overline{y}(k+1)), \overline{\beta}_i, \overline{\eta}_i\}\\ G_i(\overline{y}(k+1)) =& -\frac{1}{2|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i}\overline{y}_{ij}(k+1).\end{aligned} \tag{10}$$

Slightly abusing the notation, we employ $\underline{\delta}_i = \underline{\delta}_i(\overline{y}(k+1))$, $\overline{\delta}_i = \overline{\delta}_i(\overline{y}(k+1))$, and $G_i = G_i(\overline{y}(k+1))$ in the following sections. Next, we denote $v_{ij}(k) = v_i(k) - v_j(k)$ and define a saturation function as:

$$\mathrm{sat}(\underline{\xi}, \overline{\xi}, g) = \begin{cases}\overline{\xi},\ \text{if}\ g \ge \overline{\xi}\\ g,\ \text{if}\ \underline{\xi} < g < \overline{\xi}\\ \underline{\xi},\ \text{if}\ g \le \underline{\xi}\end{cases}$$

for $\underline{\xi} \le \overline{\xi}$. Let $\omega = 1/T$.

The controller for each agent can be designed as:

$$\overline{u}_i(k) = -\omega\overline{v}_i(k) + \omega^2\,\mathrm{sat}(\underline{\delta}_i, \overline{\delta}_i, G_i) \tag{11}$$

for $i \in \mathcal{V}$. Substituting (11) into (9), we have:

$$\overline{y}_i(k+2) = \overline{y}_i(k+1) + \mathrm{sat}(\underline{\delta}_i, \overline{\delta}_i, G_i),\ i = 1,\cdots,N. \tag{12}$$

Now, it is ready to state the main result in the following theorem.

Theorem 1: Consider the multi-train system (1) with controller $u_i(k) = \overline{u}_i(k) + f_i(k)$, where $\overline{u}_i(k)$ is given by (11). Under Assumptions 2.1 to 2.4, and with the initial state satisfying

$$\begin{aligned}&\underline{\gamma}_{ij} \le x_{ij}(0) + v_{ij}(0)T \le \overline{\gamma}_{ij}\\ &\underline{\rho}_i \le v_i(0) \le \overline{\rho}_i,\ \forall(i,j) \in \mathcal{E},\ i \in \mathcal{V},\end{aligned} \tag{13}$$

the objectives $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ can be achieved in the sense of (2), (3) and (4), respectively.

Proof: To establish the objectives $\mathcal{O}_1$ and $\mathcal{O}_2$, we can do so by proving the objectives $\mathcal{O}_1'$ and $\mathcal{O}_2'$. From the initial state of system (1) satisfying (13), we can infer that the initial state of (9) has:

$$\begin{aligned}&\underline{\gamma}_{ij} - d_{ij} \le \overline{y}_{ij}(1) \le \overline{\gamma}_{ij} - d_{ij}\\ &\underline{\rho}_i - v_r \le \overline{v}_i(0) \le \overline{\rho}_i - v_r,\ \forall(i,j) \in \mathcal{E},\ i \in \mathcal{V}.\end{aligned}$$

Next, in order to prove objectives $\mathcal{O}_1'$ and $\mathcal{O}_2'$, we demonstrate the following implications hold for all $k \ge 0$, $(i,j) \in$

$\mathcal{E}$, $i \in \mathcal{V}$:

$$\begin{cases}\underline{\gamma}_{ij} - d_{ij} \leq \overline{y}_{ij}(k+1) \leq \overline{\gamma}_{ij} - d_{ij} \\ \underline{\rho}_i - v_r \leq \overline{v}_i(k) \leq \overline{\rho}_i - v_r\end{cases} \Longrightarrow$$
$$\begin{cases}\underline{\gamma}_{ij} - d_{ij} \leq \overline{y}_{ij}(k+2) \leq \overline{\gamma}_{ij} - d_{ij} \\ \underline{\rho}_i - v_r \leq \overline{v}_i(k+1) \leq \overline{\rho}_i - v_r \\ \underline{\mu}_i - \underline{f}_i \leq \overline{u}_i(k) \leq \overline{\mu}_i - \overline{f}_i\end{cases}.$$

Due to $\underline{\gamma}_{ij} - d_{ij} \leq \overline{y}_{ij}(k+1) \leq \overline{\gamma}_{ij} - d_{ij}$, we have $\underline{\alpha}_i(\overline{y}(k+1)) \leq 0 \leq \overline{\alpha}_i(\overline{y}(k+1))$. From Assumptions 2.3 and 2.4, we can verify that $\underline{\beta}_i, \underline{\eta}_i < 0$ and $\overline{\beta}_i, \overline{\eta}_i > 0$. Therefore, we have $\underline{\delta}_i \leq 0$ and $\overline{\delta}_i \geq 0$. Then, it can be obtained from (10) that

$$\underline{\alpha}_i(\overline{y}(k+1)) \leq \underline{\delta}_i \leq \operatorname{sat}(\underline{\delta}_i, \overline{\delta}_i, G_i)$$
$$\overline{\alpha}_i(\overline{y}(k+1)) \geq \overline{\delta}_i \geq \operatorname{sat}(\underline{\delta}_i, \overline{\delta}_i, G_i).$$

Noting $\overline{y}_{ij} = -\overline{y}_{ji}$, $\forall (i,j) \in \mathcal{E}$, from (12) and Assumption 2.2, we have:

$$\begin{aligned}\overline{y}_{ij}(k+2) =& \overline{y}_{ij}(k+1) + \operatorname{sat}(\underline{\delta}_i, \overline{\delta}_i, G_i) - \operatorname{sat}(\underline{\delta}_j, \overline{\delta}_j, G_j) \\ \geq& \overline{y}_{ij}(k+1) - \frac{\overline{y}_{ij}(k+1) - (\underline{\gamma}_{ij} - d_{ij})}{2} \\ &- \frac{\overline{\gamma}_{ji} - d_{ji} - \overline{y}_{ji}(k+1)}{2} \\ \geq& \underline{\gamma}_{ij} - d_{ij}.\end{aligned}$$

Similarly, we have $\overline{y}_{ij}(k+2) \leq \overline{\gamma}_{ij} - d_{ij}$.

From (12), we deduce that

$$\underline{\beta}_i \leq \overline{y}_i(k+2) - \overline{y}_i(k+1) \leq \overline{\beta}_i.$$

Therefore, we have $\overline{v}_i(k+1) = (\overline{y}_i(k+2) - \overline{y}_i(k+1))/T \geq \underline{\rho}_i - v_r$. Similarly, we know that $\overline{v}_i(k+1) \leq \overline{\rho}_i - v_r$.

From (12), we know that

$$\underline{\eta}_i \leq \overline{y}_i(k+2) - \overline{y}_i(k+1) \leq \overline{\eta}_i.$$

Therefore, we have

$$\begin{aligned}\overline{u}_i(k) =& \frac{\overline{v}_i(k+1) - \overline{v}_i(k)}{T} \\ \geq& \frac{\overline{y}_i(k+2) - \overline{y}_i(k+1)}{T^2} - \frac{\overline{\rho}_i - v_r}{T} \\ \geq& \frac{\overline{\rho}_i T + (\underline{\mu}_i - \underline{f}_i)T^2 - v_r T}{T^2} - \frac{\overline{\rho}_i - v_r}{T} \\ \geq& \underline{\mu}_i - \underline{f}_i.\end{aligned}$$

Similarly, we can deduce that $\overline{u}_i(k) \leq \overline{\mu}_i - \overline{f}_i$. Therefore, the objectives $\mathcal{O}_1$ and $\mathcal{O}_2$ have been demonstrated.

To prove the objective $\mathcal{O}_3$, we can establish the validity of objective $\mathcal{O}_3'$. We make the following definitions:

$$\begin{aligned}\overline{y}_{min}(k) =& \min_{i \in \mathcal{V}} \{\overline{y}_i(k)\} \\ \overline{y}_{max}(k) =& \max_{i \in \mathcal{V}} \{\overline{y}_i(k)\} \\ \phi(k) =& \{i \in \mathcal{V} \mid \overline{y}_i(k) = \overline{y}_{min}(k)\} \\ \psi(k) =& \{i \in \mathcal{V} \mid \overline{y}_i(k) > \overline{y}_{min}(k)\} \\ V(k) =& \overline{y}_{max}(k) - \overline{y}_{min}(k) \\ \Delta V(k) =& V(k+1) - V(k).\end{aligned}$$

Below, we divide the proof of objective $\mathcal{O}_3'$ into four steps.

Step 1: $\forall k \geq k_0 + 1$ and $k_0 \geq 0$, we have $\overline{y}_{\min}(k) \geq \overline{y}_{\min}(k_0+1)$ and $\overline{y}_{\max}(k) \leq \overline{y}_{\max}(k_0+1)$. If $\overline{y}_i(k_0+1) > \overline{y}_{\min}(k_0+1)$, then $\overline{y}_i(k) > \overline{y}_{\min}(k_0+1)$, $\forall k \geq k_0+1$. If $\overline{y}_i(k_0+1) < \overline{y}_{\max}(k_0+1)$, then $\overline{y}_i(k) < \overline{y}_{\max}(k_0+1)$, $\forall k \geq k_0+1$.

From (12), we deduce that

$$\begin{aligned}\overline{y}_i(k+2) =& \overline{y}_i(k+1) + \operatorname{sat}(\underline{\delta}_i, \overline{\delta}_i, G_i) \\ \geq& \overline{y}_i(k+1) - \frac{1}{2|\mathcal{N}_i|}|\mathcal{N}_i|[\overline{y}_i(k+1) - \overline{y}_{\min}(k+1)] \\ \geq& \frac{1}{2}[\overline{y}_i(k+1) + \overline{y}_{\min}(k+1)] \\ \geq& \overline{y}_{\min}(k+1).\end{aligned}$$

Through iteration, we know that $\overline{y}_{\min}(k) \geq \overline{y}_{\min}(k_0+1)$, $\forall k \geq k_0+1$. Based on a similar analysis, we know that $\overline{y}_{\max}(k) \leq \overline{y}_{\max}(k_0+1)$, $\forall k \geq k_0+1$. If $\overline{y}_i(k+1) > \overline{y}_{\min}(k+1)$, we can deduce:

$$\overline{y}_i(k+2) \geq \frac{1}{2}[\overline{y}_i(k+1) + \overline{y}_{\min}(k+1)] > \overline{y}_{\min}(k+1).$$

Due to $\overline{y}_i(k_0+1) > \overline{y}_{\min}(k_0+1)$, through iteration, we can deduce $\overline{y}_i(k) > \overline{y}_{\min}(k_0+1)$, $\forall k \geq k_0+1$. Using a similar analytical approach, we can infer that if $\overline{y}_i(k_0+1) < \overline{y}_{\max}(k_0+1)$, then $\overline{y}_i(k) < \overline{y}_{\max}(k_0+1)$, $\forall k \geq k_0+1$.

Define the function: $W(\overline{y}(k+1)) > 0$ if $\overline{y}_{\min}(k+1) < \overline{y}_{\max}(k+1)$; $W(\overline{y}(k+1)) = 0$ if $\overline{y}_{\min}(k+1) = \overline{y}_{\max}(k+1)$.

Step 2: $\forall k_0 \geq 0$, $\Delta V(k_0+1) \leq 0$. If $\Delta V(k_0+1) < 0$, then $\Delta V(k_0+1) \leq -W(\overline{y}(k_0+1))$. If $\Delta V(k_0+1) = 0$ and $\overline{y}_{\min}(k_0+1) < \overline{y}_{\max}(k_0+1)$, there exists $1 \leq k_1 \leq N-1$ such that $\Delta V(k_0+1+k_1) \leq -W(\overline{y}(k_0+1+k_1)$.

From Step 1, it is evident that $\Delta V(k_0+1) \leq 0$. It is easy to verify that the system described in (12) is an autonomous system. Therefore, if $\Delta V(k_0+1) < 0$, we have $\Delta V(k_0+1) \leq -W(\overline{y}(k_0+1))$. If $\Delta V(k_0+1) = 0$, then both $\overline{y}_{min}(k_0+2) = \overline{y}_{min}(k_0+1)$ and $\overline{y}_{max}(k_0+2) = \overline{y}_{max}(k_0+1)$. Considering agent i, it satisfies $i \in \phi(k_0+1)$ and $\mathcal{N}_i \cap \psi(k_0+1) \neq \varnothing$. From the definition of (10), it is evident that $G_i(\overline{y}(k_0+1)) > 0$. Due to $\overline{y}_{ij}(k_0+1) \leq 0$, $\forall j \in \mathcal{N}_i$, it follows that $\overline{\alpha}_i(\overline{y}(k_0+1)) > 0$, and furthermore, $\overline{\delta}_i(\overline{y}(k_0+1)) > 0$. From (12), we know that $\overline{y}_i(k_0+2) > \overline{y}_i(k_0+1)$. According to Step 1, we know that $\overline{y}_i(k+2) > \overline{y}_{min}(k_0+1)$, $\forall k \geq k_0$. Due to $|\phi(k)| \leq N-1$, there exists $1 \leq k_1 \leq N-1$ such that $\overline{y}_{min}(k_0+2+k_1) > \overline{y}_{min}(k_0+1+k_1)$, implying $\Delta V(k_0+1+k_1) < 0$. Based on the aforementioned inference, it follows that $\Delta V(k_0+1+k_1) \leq -W(\overline{y}(k_0+1+k_1)$.

Step 3: $\lim_{k \to \infty} V(k) = 0$.

Since $V(k) \geq 0$ and $\Delta V(k) \leq 0$, we have $\lim_{k \to \infty} V(k) = V^*$ for some constant $V^* \geq 0$. The following employs proof by contradiction to establish $V^* = 0$.

Assuming $V^* > 0$, we have $W(\overline{y}(k+1)) \geq w(V^*) > 0$, where $w(V^*)$ is a constant related to V^*. If $\Delta V(k) < 0$, then $\Delta V(k) \leq -w(V^*)$, which contradicts $\lim_{k \to \infty} V(k) = V^*$. If $\Delta V(k) = 0$, according to Step 2, there exists an $1 \leq k_1 \leq$

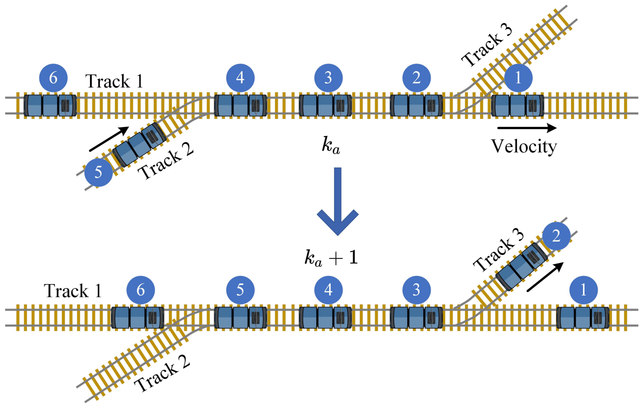

Fig. 1. Multi-train system example: We consider the distributed cooperative control of a multi-train system on Track 1. At time k_a, the multi-train system on Track 1 is composed of Trains 1, 2, 3, 4, and 6. Due to changes in the train routes, at time k_a+1, Train 2 moves from Track 1 to Track 3, and Train 5 moves from Track 2 to Track 1. Consequently, the composition of the multi-train system on Track 1 becomes Trains 1, 3, 4, 5, and 6.

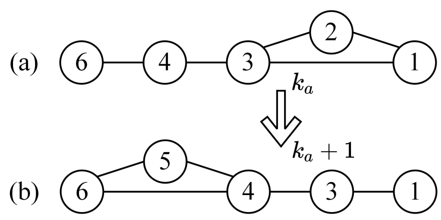

Fig. 2. The communication network corresponding to the multi-train system in Fig. 1, where the topology network changes from topology (a) to topology (b) at time k_a.

$N-1$ such that $\Delta V(k+k_1) \leq -w(V^*)$, which contradicts $\lim_{k\to\infty} V(k) = V^*$. Therefore, we can conclude that $V^* = 0$.

Step 4: The objective $\mathcal{O}_3'$ can be achieved in the sense of (8).

Since $\lim_{k\to\infty} V(k) = 0$ from Step 3, it is evident that $\lim_{k\to\infty} \overline{y}_{ij}(k) = 0$, $\forall (i,j) \in \mathcal{E}$; furthermore, $\lim_{k\to\infty} G_i(\overline{y}(k+1)) = 0$. Then, we know that $\lim_{k\to\infty} \overline{u}_i(k) = -\omega \overline{v}_i(k)$ from (11). Substituting $\overline{u}_i(k)$ into (9), we have $\lim_{k\to\infty} \overline{v}_i(k) = 0$, $\forall i \in \mathcal{V}$. Hence, objective $\mathcal{O}_3'$ has been proven, implying the verification of objective $\mathcal{O}_3$ as well. Therefore, the proof is complete. ■

IV. NUMERICAL SIMULATION

In this section, we present a numerical simulation to validate the effectiveness of the controller scheme proposed in Theorem 1. We use an example of a multi-train system with changing communication networks as a simulation scenario, denoted as Fig. 1. Fig. 2 represents the communication network of the multi-train system corresponding to the example in Fig. 1. In Fig. 2, due to changes in the train communication network, the controller schemes based on global topology network information or requiring a linear network, as proposed in [12], are no longer applicable. Fortunately, the fully distributed control scheme proposed in this paper is independent of network information and can seamlessly achieve control for general network systems.

The network change depicted in Fig. 2 occurs at time $k_a = 1000\text{s}$. The dynamic model of each train is (1), with the sampling period $T = 1\text{s}$, and the coefficients of $f_i(k)$ are specified as: $a_i = 1.176 \times 10^{-2}\text{N/kg}$, $b_i = 7.7616 \times 10^{-4}\text{N s/(m kg)}$, and $c_i = 1.6 \times 10^{-5}\text{N s}^2/(\text{m}^2\text{ kg})$. The constraints for the objective $\mathcal{O}_1$ are set as: $\underline{\gamma}_{i-1,i} = 100\text{m}$, $\overline{\gamma}_{i-1,i} = 10000\text{m}$, $i = 2, \cdots, 6$, and $\underline{\gamma}_{j-1,j+1} = 5100\text{m}$, $\overline{\gamma}_{j-1,j+1} = 15000\text{m}$, $j = 2, 5$. Similarly, the constraints for the objective $\mathcal{O}_2$ are set as: $\underline{\rho}_i = 75\text{m/s}$, $\overline{\rho}_i = 85\text{m/s}$ and $\underline{\mu}_i = -11\text{m/s}^2$, $\overline{\mu}_i = 11\text{m/s}^2$, $i = 1, \cdots, 6$. In the objective $\mathcal{O}_3$, the desired distance and reference velocity are respectively set as: $d_{i-1,i} = 5000\text{m}$, $i = 2, \cdots, 6$, $d_{j-1,j+1} = 10000\text{m}$, $j = 2, 5$, and $v_r = 80\text{m/s}$. Certainly, it is clear that Assumptions 2.1 to 2.4 are all satisfied. Taking the initial position and velocity as: $x_1(0) = 24000\text{m}$, $x_2(0) = 21000\text{m}$, $x_3(0) = 13000\text{m}$, $x_4(0) = 12000\text{m}$, $x_5(1000) = 89000\text{m}$, $x_6(0) = 0\text{m}$, $v_1(0) = 85\text{m/s}$, $v_2(0) = 82\text{m/s}$, $v_3(0) = 81\text{m/s}$, $v_4(0) = 81\text{m/s}$, $v_5(1000) = 80\text{m/s}$ and $v_6(0) = 76\text{m/s}$. Clearly, the requirements for the initial state specified in (13) are also fulfilled.

Fig. 3 and 4 depict the evolution trajectories of train positions and position differences. Note that the position differences between neighboring trains are within the constraint range throughout the entire operation, validating the achievement of objective $\mathcal{O}_1$. Fig. 5 and 6 illustrate the evolution trajectories of train velocities and inputs. The velocity and input of each train remain within the constraint range throughout the entire operation, confirming the accomplishment of objective $\mathcal{O}_2$. As observed from Fig. 4 and 5, the neighboring position differences and velocities of the trains converge to the desired values, providing evidence of the realization of objective $\mathcal{O}_3$. The simulation results align precisely with the anticipated outcomes, demonstrating the remarkable efficacy of the control scheme proposed in this paper.

V. CONCLUSION

In this paper, we have successfully addressed the fully distributed cooperative control problem of discrete-time multi-train systems, where the position difference between neighboring trains, as well as the velocities and inputs of each train, are subject to transient constraints. The paper explores a fully distributed control law that only relies on the relative position information of neighboring trains, providing flexibility to adapt to the network structural changes in actual multi-train systems. Future research may explore extending this approach to tackle increasingly complex scenarios and handle more intricate discrete-time systems.

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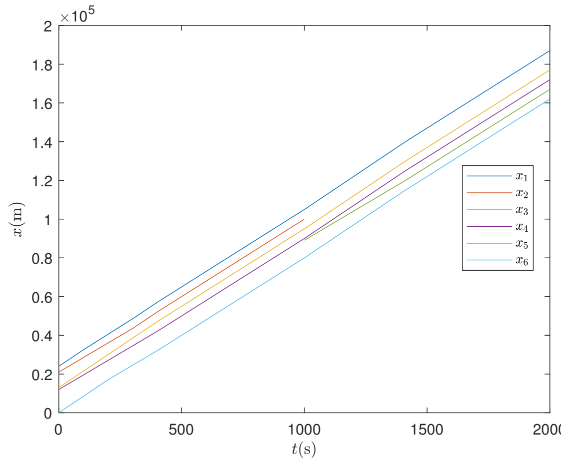

Fig. 3. Evolution trajectories of positions x_i, $i = 1, \cdots, 6$ during operation, including only the trains present in the network topology depicted in Fig. 2.

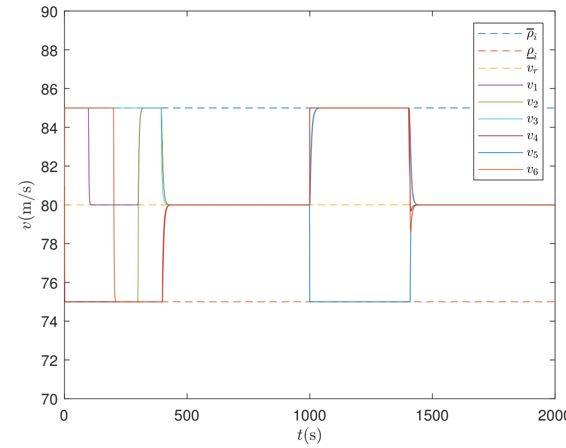

Fig. 5. The velocity evolution trajectories v_i, $i = 1, \cdots, 6$, along with the identical reference velocity v_r and constraint boundaries $\underline{\rho}_i$ and $\overline{\rho}_i$.

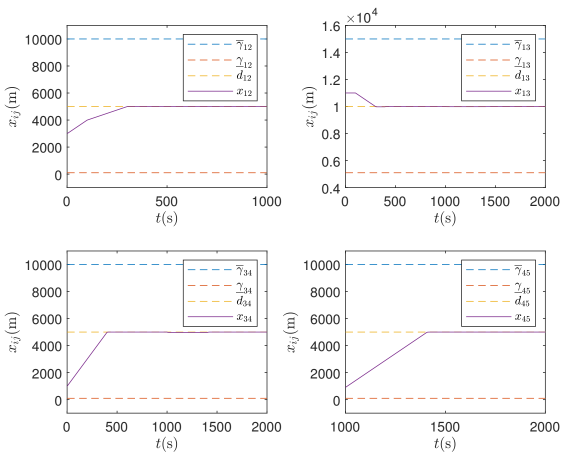

Fig. 4. Evolution trajectory of the position difference x_{ij}, $(i,j) \in \mathcal{E}$, along with its corresponding desired distance d_{ij} and constraint boundaries $\underline{\gamma}_{ij}$ and $\overline{\gamma}_{ij}$.

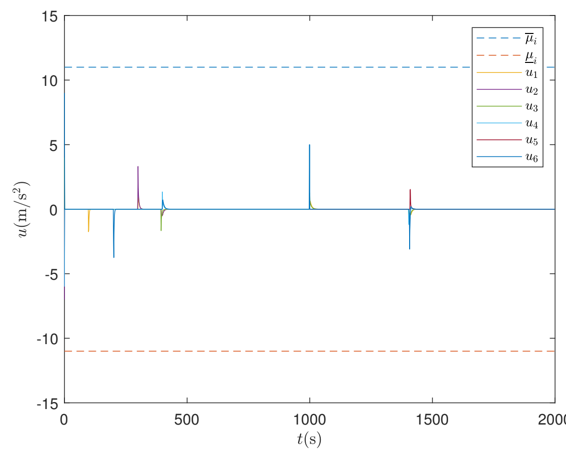

Fig. 6. Inputs u_i, $i = 1, \cdots, 6$, along with their constraint boundaries, which are identical.

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