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Generalized Dynamic Observer Design for the Nonlinear Parameter Varying Systems: Exploiting Matrix-Multiplier-based LMIs

Shivaraj Mohite[1], Marouane Alma[2] and Naim Bajcinca[1]

Abstract— **This paper deals with the design of a generalized dynamic observer for a class of Nonlinear Parameter-Varying (NLPV) systems. The primary goal is to formulate a less conservative Linear Matrix Inequality (LMI) condition than the existing ones, ensuring the exponential stability of the error dynamics in the proposed observer. Through the incorporation of the reformulated Lipschitz property, Young's inequalities and a generalized matrix multiplier, two new LMI conditions are established. Due to the deliberate use of these mathematical tools, the obtained LMI conditions contain a larger number of decision variables than existing LMI conditions. As a result, these LMIs have enhanced feasibility than the one presented in the literature. The effectiveness of the newly designed LMI-based generalized dynamic observer is highlighted through a numerical example.**

I. Introduction

Over the past few decades, Linear Parameter-Varying (LPV) systems have been deployed to model the dynamic nature of nonlinear systems [1]. Though this approach is beneficial in the system analysis, it has several limitations, as follows: 1) The linearization of the nonlinear systems leads to a diminution of the overall generality of the systems. 2) One can not employ this method if the nonlinearities of the systems are not known. One of the potential solutions for this issue is to consider a certain amount of nonlinearities inside system dynamics. It facilitates the mitigation of the loss of generality and yields realistic results. This idea resulted in the establishment of a new system class known as Nonlinear Parameter-Varying (NLPV) systems [2].

In the literature, the topic of observer design for LPV systems has been extensively investigated [3], [4]. Among these various observer techniques, the generalized dynamic observer approach presented in [4] is considered in this article. Recently, the development of observers for Nonlinear Parameter-Varying (NLPV) systems has garnered significant attention from researchers in control system engineering [5]. The authors of [6] employed the standard Lipschitz property to handle nonlinearities, while a reformulated Lipschitz property was used in [7] for the same task. In this paper, a nonlinear generalized dynamic observer based on the reformulated Lipschitz property is established.

Due to the advancement in computation technology, LMI-based methods are widely used in the control system domain for observer synthesis purposes [6], [8]. Though all these approaches provide an efficient solution for the state estimations, there is a scope for further enhancements. Recently, the authors of [9] introduced a new matrix-multiplier-based LMI technique for nonlinear observer design. This letter aims to establish a novel LMI-based proportional observer methodology for a class of NLPV systems which is motivated by the methods presented in [9]. Through the judicious deployment of a more generalized matrix multiplier, Young inequalities and the well-known LPV approach, two new LMI conditions are derived in this article. These developed LMIs contain more degrees of freedom from a feasibility point of view and are less conservative than the existing ones [6], [7].

The remainder of the paper is structured as follows: Section II consists of the prerequisites related to LMI-based observer design. The problem statement of this article is illustrated in Section III. Further, in Section IV, the synthesis of the two novel LMI conditions is showcased. Section V emphasises the effectiveness of the proposed methodology with a numerical example. Some conclusions and future perspectives related to this research are outlined in Section VI.

[1]Rhineland-Palatinate Technical University of Kaiserslautern-Landau, Kaiserslautern, Germany. shivaraj.mohite@mv.rptu.de

[2]University of Lorraine, 186, rue de Lorraine, CRAN CNRS UMR 7039, F-54400, Cosnes et Romain, France.

II. Notations and Preliminaries

A. Nomenclature

Throughout the paper, we have utilised the following notations: The initial value of $e(t)$ at $t=0$ is symbolised by e_0. A vector of the canonical basis of $\mathbb{R}^s$ is described as: $e_s(i) = (\underbrace{0,\dots,0,\overbrace{1}^{i^{\text{th}}},0,\dots,0}_{s\text{ components}})^\top \in \mathbb{R}^s,\ s \geq 1$. The terms $\mathbb{I}$ and $\mathbb{O}$ depict the identity matrix and the null matrix, respectively. $A^\top$ and $A^\dagger$ denote the transpose and generalized inverse of matrix A, respectively. $A \in \mathbf{S}^n$ implies that $A = A^\top \in \mathbb{R}^{n\times n}$. The symbol $(\star)$ is used within a symmetric matrix to signify the repeated blocks. For the aforementioned matrix A, $A>0$ ($A<0$) indicates that it is a positive definite matrix (a negative definite matrix). Similarly, a positive semi-definite matrix (a negative semi-definite matrix) is showcased by $A \geq 0$ ($A \leq 0$). $A = \text{block-diag}(A_1,\dots,A_n)$ depicts a block-diagonal matrix having elements $A_1,\dots,A_n$ in the diagonal.

B. Prerequisites

In this section, we have provided the fundamental mathematical tools and background results related to the LMI-based observer design.

Lemma 1 (Young's Inequalities): If there exist two vectors $X,Y \in \mathbb{R}^n$ and a matrix $Z>0 \in \mathbf{S}^n$. Then, the ensuing matrix inequalities are true:

$$X^\top Y + Y^\top X \leq X^\top Z^{-1} X + Y^\top Z Y, \tag{1}$$

$$X^\top Y + Y^\top X \leq (X+ZY)^\top (2Z)^{-1} (X+ZY). \tag{2}$$

The inequality (1) is known as Standard Young's inequality. Further, the new variant of Young's inequality is represented by (2).

Lemma 2 ([10]): Let us consider

$$\mathbb{X}^\top = \begin{bmatrix} a_1\mathbb{I}_n & a_2\mathbb{I}_n & \dots & a_n\mathbb{I}_n \end{bmatrix}, \tag{3}$$

$$\mathbb{Y}^\top = \begin{bmatrix} b_1\mathbb{I}_n & b_2\mathbb{I}_n & \dots & b_n\mathbb{I}_n \end{bmatrix}, \tag{4}$$

along with

$$Z = \begin{bmatrix} Z_1 & Z_{a_1^2} & \dots & Z_{a_1^n} \\ \star & Z_2 & \dots & Z_{a_2^n} \\ \star & \star & \ddots & \vdots \\ \star & \star & \dots & Z_n \end{bmatrix}, \tag{5}$$

where $0 \le a_i \le b_i \; \forall i \in \{1,\dots,n\}$ and $Z_i > 0 \in \mathbf{S}^n$, $Z_{a_i^j} \ge 0 \in \mathbf{S}^n \; \forall\, i \in \{1,\dots,n\}$, $j \in \{2,\dots,n\}$ so that $Z > 0$. Then, the subsequent inequality is fulfilled:

$$\mathbb{X}^\top Z\mathbb{X} - \mathbb{Y}^\top Z\mathbb{Y} \le 0. \tag{6}$$

In sequel, we have illustrated the problem formulation and main results of the article.

III. Articulating the Problem Statement

For enhanced comprehensibility, we have divided this section into two parts. The first segment is concentrated on depicting the form of NLPV systems, and the structure of the generalized dynamic observer. In the second part, we showcased the parameterization of the gain matrices.

A. System and observer structure

We have considered the subsequent NLPV systems for the purpose of observer design:

$$\begin{aligned} \dot{x} &= A(\theta(t))x + B(\theta(t))u + G(\theta(t))f(x), \\ y &= Cx, \end{aligned} \tag{7}$$

where $x \in \mathbb{R}^n$ depicts the states of the systems, $u \in \mathbb{R}^s$ represents the input and $y \in \mathbb{R}^p$ denotes the system's output. The $\theta(t) \in \mathbb{R}^{\bar{\theta}}$ is a vector of time-varying parameters, and assumed to be known and bounded, i.e., it holds:

$$\theta_{\bar{k}_{\min}} \le \theta_{\bar{k}} \le \theta_{\bar{k}_{\max}}, \forall \bar{k} = \{1,\dots,\bar{\theta}\}. \tag{8}$$

The matrices $A(\cdot)$, $B(\cdot)$, $G(\cdot)$ are fixed function of the vector $\theta(t)$. C is a constant matrix of appropriate dimension. The nonlinear function inside the system dynamics is symbolized by $f(x)$. We have presumed that $f(x)$ is a global Lipschitz function. Further, it can be expressed in the following manner:

$$f(x) = \begin{bmatrix} f_1(\underbrace{H_1 x}_{\nu_1})) \\ \vdots \\ f_m(\nu_m) \end{bmatrix}, \tag{9}$$

where $H_i \in \mathbb{R}^{\bar{n}\times n}$.

From (8), the parameter varying matrices of (7) can be represented in the subsequent form:

$$\mathcal{X}(\theta) = \mathcal{X}_0 + \sum_{\bar{k}=1}^{n_\theta} \theta_{\bar{k}} \mathcal{X}_{\bar{k}}, \tag{10}$$

where $\mathcal{X} \in \{A,B,G\}$. The matrices $\mathcal{X}_{\bar{k}} \in \mathbb{R}^{n\times n}, \forall \bar{k} = \{0,\dots,n_\theta\}$ are known and constant. It implies that system (7) can be reformulated as a polytopic system. The coordinates of this polytopic system are denoted by $\rho(\theta(t))$ and vary into the following convex set:

$$\begin{aligned} \mathcal{V}_\rho = \big\{&\rho(\theta(t)) \in \mathbb{R}^{2^{\bar{\theta}}}, \rho(\theta(t)) = \begin{bmatrix} \rho_1(\theta(t)) & \cdots & \rho_{2^{\bar{\theta}}}(\theta(t)) \end{bmatrix}^\top, \\ &\rho_k(\theta(t)) \ge 0, \; \sum_{k=1}^{2^{\bar{\theta}}} \rho_k(\theta(t)) = 1 \big\}. \end{aligned}$$

From [4], the system (7) is written in the following polytopic representation form:

$$\begin{aligned} \dot{x} &= \sum_{k=1}^{2^{\bar{\theta}}} \rho_k(\theta(t)) \big(A_k x + B_k u + G_k f(x)\big), \\ y &= Cx, \end{aligned} \tag{11}$$

where $\rho_k(\theta(t)) \in \mathcal{V}_\rho$. In addition to this, $\rho_k(\theta(t)) = \Pi_{\bar{k}=1}^{n_\theta} \rho_{\bar{k}}^i(\theta_{\bar{k}}(t))$, $\bar{\mathcal{X}}_k = \mathcal{X}_0 + \sum_{\bar{k}=1}^{n_\theta} \theta_{\bar{k}}^i \mathcal{X}_{\bar{k}}$, where $\bar{\mathcal{X}} \in \{A,B,G\}$ and i is equal to 1 or 2 depending on the partition of the j^{th} parameter (ρ_j^1 or ρ_j^2).

For state estimation of (11), the generalized dynamic observer is used and it is described as:

$$\begin{aligned} \dot{\eta}(t) &= \sum_{k=1}^{2^{\bar{\theta}}} \rho_k(\theta(t)) \big(\mathbb{M}_k \eta(t) + \mathbb{X}_k \zeta(t) + \mathbb{Q}_k y \\ &\quad + \mathbb{T}B_k u + \mathbb{T}G_k f(\hat{x}(t))\big), \end{aligned} \tag{12a}$$

$$\dot{\zeta}(t) = \sum_{k=1}^{2^{\bar{\theta}}} \rho_k(\theta(t)) \big(\mathbb{N}_k \eta(t) + \mathbb{Y}_k \zeta(t) + \mathbb{S}_k y\big), \tag{12b}$$

$$\hat{x}(t) = \mathbb{P}\eta(t) + \mathbb{L}y, \tag{12c}$$

where $\eta(t) \in \mathbb{R}^{o_1}$, $\zeta(t) \in \mathbb{R}^{o_2}$ and $\hat{x}(t) \in \mathbb{R}^n$ are the state vectors, an auxiliary vector, and the estimated states of the observer, respectively. The terms $\mathbb{M}_k$, $\mathbb{X}_k$, $\mathbb{Q}_k$, $\mathbb{T}$, $\mathbb{N}_k$, $\mathbb{Y}_k$, $\mathbb{S}_k$, $\mathbb{L}$ and $\mathbb{P}$ are unknown matrices of appropriate dimension which needs to be calculated such that the estimation error $\tilde{x} = \hat{x} - x$ converges exponentially towards zero.

Let us define the transformation error for the earlier-mentioned observer (12) as $e = \eta - \mathbb{T}x$. Then, from (12) and (11), one can deduce:

$$\begin{aligned} \dot{e} &= \sum_{k=1}^{2^{\bar{\theta}}} \rho_k(\theta(t)) \Big(\mathbb{M}_k e + (\mathbb{M}_k\mathbb{T} + \mathbb{Q}_k C - \mathbb{T}A_k)x \\ &\quad + \mathbb{X}_k \zeta(t) + \mathbb{T}G_k(f(\hat{x}(t)) - f(x))\Big), \end{aligned} \tag{13a}$$

$$\dot{\zeta}(t) = \sum_{i=k}^{2^{\bar{\theta}}} \rho_k(\theta(t)) \Big(\mathbb{N}_k e + \mathbb{Y}_k \zeta + (\mathbb{N}_k\mathbb{T} + \mathbb{S}_k C)x\Big), \tag{13b}$$

$$\tilde{x} = \mathbb{P}e + (\mathbb{P}\mathbb{T} + \mathbb{L}C - \mathbb{I})x. \tag{13c}$$

If the following matrix equalities are satisfied

$$\mathbb{M}_k\mathbb{T} + \mathbb{Q}_k C - \mathbb{T}A_k = 0, \tag{14a}$$

$$\mathbb{N}_k\mathbb{T} + \mathbb{S}_k C = 0, \tag{14b}$$

$$\mathbb{P}\mathbb{T} + \mathbb{L}C = \mathbb{I}, \tag{14c}$$

then, we can reformulate the system (13) into the subsequent form:

$$\dot{e} = \sum_{k=1}^{2^{\bar{\theta}}} \rho_k(\theta(t)) \Big[\mathbb{M}_k e + \mathbb{X}_k \zeta(t) + \mathbb{T}G_k(f(\hat{x}(t)) - f(x))\Big], \tag{15a}$$

$$\dot{\zeta}(t) = \sum_{i=k}^{2^{\bar{\theta}}} \rho_k(\theta(t)) \big(\mathbb{N}_k e + \mathbb{Y}_k \zeta\big), \tag{15b}$$

$$\tilde{x} = \mathbb{P}e. \tag{15c}$$

From [11, Lemma 7], one can express $f(x)-f(\hat{x})$ in the following form:

$$f(x)-f(\hat{x})=\sum_{i,j=1}^{m,\bar{n}} f_{ij}\mathcal{H}_{ij}H_i\tilde{x}, \tag{16}$$

where $\mathcal{H}_{ij}=e_m(i)e_{\bar{n}}^{\top}(j)$ and $f_{ij}\triangleq f_{ij}(\theta_i^{\hat{\theta}_{i,j-1}},\theta_i^{\hat{\theta}_{i,j}})$. The functions $f_{ij}:\mathbb{R}^{\bar{n}}\times\mathbb{R}^{\bar{n}}\to\mathbb{R}$ fulfills $f_{a_{ij}}\le f_{ij}\le f_{b_{ij}}$, where $f_{a_{ij}}$ and $f_{b_{ij}}$ are constants. Without loss of generality, let us assume that $f_{a_{ij}}=0$. Hence,

$$0\le f_{ij}\le f_{b_{ij}}. \tag{17}$$

Through the use of (16), we can rewrite (15) as:

$$\dot{\tilde{e}}=\sum_{k=1}^{2^{\bar{\theta}}}\rho_k(\theta(t))\left(\begin{bmatrix}\mathbb{M}_k+\sum_{i,j=1}^{m,\bar{n}} f_{ij}\mathbb{T}G_k\mathcal{H}_{ij}H_i\mathbb{P} & \mathbb{X}_k\\ \mathbb{N}_k & \mathbb{Y}_k\end{bmatrix}\tilde{e}\right), \tag{18a}$$

$$\tilde{x}=\begin{bmatrix}\mathbb{P} & \mathbb{O}\end{bmatrix}\tilde{e}, \tag{18b}$$

where $\tilde{e}=\begin{bmatrix}e^{\top} & \zeta^{\top}\end{bmatrix}^{\top}$.

B. Parameterization of the observer matrices

In this section, we present the parameterization of the algebraic constraint (14a)-(14c), which aids in mitigating the complexity during the computation of the observer's gain matrices.

Let us consider a full-row rank matrix $\mathbb{E}\in\mathbb{R}^{o_1\times n}$ such that $\begin{bmatrix}\mathbb{E}\\ C\end{bmatrix}$ is a full-column matrix.

Further, the constraint (14c) is expressed in the following manner:

$$\begin{bmatrix}\mathbb{P} & \mathbb{L}\end{bmatrix}\begin{bmatrix}\mathbb{T}\\ C\end{bmatrix}=\mathbb{I}_n. \tag{19}$$

It implies that the rank of $\begin{bmatrix}\mathbb{T}\\ C\end{bmatrix}=n$. Thus, we can introduce two arbitrary matrices $\mathbb{T}\in\mathbb{R}^{o_1\times n}$ and $\mathbb{K}\in\mathbb{R}^{o_1\times p}$ so that we can obtain:

$$\begin{bmatrix}\mathbb{T} & \mathbb{K}\end{bmatrix}\begin{bmatrix}\mathbb{I}_n\\ C\end{bmatrix}=\mathbb{E}. \tag{20}$$

From solution of (20), one can deduce:

$$\mathbb{T}=\mathbb{E}\begin{bmatrix}\mathbb{I}_n\\ C\end{bmatrix}^{\dagger}\begin{bmatrix}\mathbb{I}_n\\ \mathbb{O}\end{bmatrix}, \tag{21a}$$

$$\text{and } \mathbb{K}=\mathbb{E}\begin{bmatrix}\mathbb{I}_n\\ C\end{bmatrix}^{\dagger}\begin{bmatrix}\mathbb{O}\\ \mathbb{I}_p\end{bmatrix}. \tag{21b}$$

Later on, the substitution of (20), i.e., $\mathbb{E}=\mathbb{T}+\mathbb{K}C$, into (14a) leads to

$$\begin{bmatrix}\mathbb{M}_k & \Omega_k\end{bmatrix}\begin{bmatrix}\mathbb{E}\\ C\end{bmatrix}=\mathbb{T}A_k, \tag{22}$$

where

$$\Omega_k=-\mathbb{M}_k\mathbb{K}+\mathbb{Q}_k. \tag{23}$$

By using (21a) along with the solution of (22) and (23), one can obtain:

$$\mathbb{M}_k=\underbrace{\mathbb{T}A_k\begin{bmatrix}\mathbb{E}\\ C\end{bmatrix}^{\dagger}\begin{bmatrix}\mathbb{I}_{o_1}\\ \mathbb{O}\end{bmatrix}}_{\mathbb{M}_{1_k}}-\mathcal{Y}_{2_k}\underbrace{\left(\mathbb{I}_{o_1+p}-\begin{bmatrix}\mathbb{E}\\ C\end{bmatrix}\begin{bmatrix}\mathbb{E}\\ C\end{bmatrix}^{\dagger}\right)\begin{bmatrix}\mathbb{I}_{o_1}\\ \mathbb{O}\end{bmatrix}}_{\mathbb{M}_{3_0}}, \tag{24a}$$

$$\Omega_k=\underbrace{\mathbb{T}A_k\begin{bmatrix}\mathbb{E}\\ C\end{bmatrix}^{\dagger}\begin{bmatrix}\mathbb{O}\\ \mathbb{I}_p\end{bmatrix}}_{\Omega_{1_k}}-\mathcal{Y}_{2_k}\underbrace{\left(\mathbb{I}_{o_1+p}-\begin{bmatrix}\mathbb{E}\\ C\end{bmatrix}\begin{bmatrix}\mathbb{E}\\ C\end{bmatrix}^{\dagger}\right)\begin{bmatrix}\mathbb{O}\\ \mathbb{I}_p\end{bmatrix}}_{\Omega_{3_0}}, \tag{24b}$$

where $\mathcal{Y}_{2_k}$ are arbitrary matrices.

Now, we can rewrite (14b) and (14c) in the following compact form:

$$\begin{bmatrix}\mathbb{N}_k & \mathbb{S}_k\\ \mathbb{P} & \mathbb{L}\end{bmatrix}\begin{bmatrix}\mathbb{T}\\ C\end{bmatrix}=\begin{bmatrix}\mathbb{O}\\ \mathbb{I}_n\end{bmatrix}. \tag{25}$$

Later on, from $\mathbb{T}=\mathbb{E}-\mathbb{K}C$, we achieve:

$$\begin{bmatrix}\mathbb{T}\\ C\end{bmatrix}=\begin{bmatrix}\mathbb{I}_{o_1} & -\mathbb{K}\\ \mathbb{O} & \mathbb{I}_p\end{bmatrix}\begin{bmatrix}\mathbb{E}\\ C\end{bmatrix} \tag{26}$$

Through the utilisation of (26) and (25), we deduce:

$$\mathbb{N}_k=-\mathcal{Y}_{3_k}\mathbb{M}_{3_0}, \tag{27a}$$

$$\mathbb{S}_k=-\mathcal{Y}_{3_k}\underbrace{\left(\mathbb{I}_{o_1+p}-\begin{bmatrix}\mathbb{E}\\ C\end{bmatrix}\begin{bmatrix}\mathbb{E}\\ C\end{bmatrix}^{\dagger}\right)\begin{bmatrix}\mathbb{K}\\ \mathbb{I}_p\end{bmatrix}}_{\mathbb{S}_{3_0}}, \tag{27b}$$

$$\mathbb{P}=\begin{bmatrix}\mathbb{E}\\ C\end{bmatrix}^{\dagger}\begin{bmatrix}\mathbb{I}_{o_1}\\ \mathbb{O}\end{bmatrix}, \tag{27c}$$

$$\mathbb{L}=\begin{bmatrix}\mathbb{E}\\ C\end{bmatrix}^{\dagger}\begin{bmatrix}\mathbb{K}\\ \mathbb{I}_p\end{bmatrix}. \tag{27d}$$

Through the deployment of (24a), (27a) and (27b), the system (18a) is altered as:

$$\dot{\tilde{e}}=\sum_{k=1}^{2^{\bar{\theta}}}\rho_k(\theta(t))\Big((\bar{A}_k-\mathcal{L}_k\mathcal{C})\tilde{e}\Big), \tag{28}$$

where $\bar{A}_k=\begin{bmatrix}(\mathbb{M}_{1_k}+\sum_{i,j=1}^{m,\bar{n}} f_{ij}\mathbb{T}G_k\mathcal{H}_{ij}H_i\mathbb{P} & \mathbb{O}\\ \mathbb{O} & \mathbb{O}\end{bmatrix}$, $\mathcal{L}_k=\begin{bmatrix}\mathcal{Y}_{2_k} & \mathbb{X}_k\\ \mathcal{Y}_{3_k} & \mathbb{Y}_k\end{bmatrix}$ and $\mathcal{C}=\begin{bmatrix}\mathbb{M}_{3_0} & \mathbb{O}\\ \mathbb{O} & -\mathbb{I}_{o_2}\end{bmatrix}$.

From (18b), the convergence of the estimation error (e) towards zero relies on the stability of the error dynamic (28). Thus, the objective of this letter is to determine the observer parameters such that the system (28) is exponentially stable.

In the sequel, the computation of the observer parameters through a new LMI approach is illustrated.

IV. Main Result

In this section, the essential conditions for ensuring the exponential convergence of the system (28) is developed. For the simplicity of the presentation, we have divided this section as follows: First, the Lyapunov-based stability of (28) is studied. Further, two new LMI conditions are derived.

A. Stability analysis

Let us consider the following quadratic Lyapunov function for the error dynamic (28):

$$V(\tilde{e})=\tilde{e}^{\top}P\tilde{e},\ \text{where } P=\begin{bmatrix}P_1 & P_2\\ P_2 & P_2\end{bmatrix}>0. \tag{30}$$

From (28), one can deduce:

$$\dot{V}(\tilde{e})=\tilde{e}^{\top}\left(\sum_{k=1}^{2^{\bar{\theta}}}\rho_k(\theta(t))\big(P(\bar{A}_k-\mathcal{L}_k\mathcal{C})+(\bar{A}_k-\mathcal{L}_k\mathcal{C})^{\top}P\big)\right)\tilde{e}. \tag{31}$$

The system (28) is exponentially stable if the Lyapunov function (30) fulfils

$$\dot{V}(\tilde{e})\le-\sigma V(\tilde{e}),\ \text{where } \sigma>0. \tag{32}$$

$$\mathbb{Z} = \begin{bmatrix} \mathbb{Z}_1 & \mathbb{Z}_{b_2^1} & \dots & \mathbb{Z}_{b_m^1} \\ \star & \mathbb{Z}_2 & \dots & \mathbb{Z}_{b_m^2} \\ & & \ddots & \vdots \\ \star & \star & & \vdots \\ \star & \star & \dots & \mathbb{Z}_m \end{bmatrix}, \text{where } \mathbb{Z}_i = \begin{bmatrix} Z_{i1} & Z_{a_{i2}^1} & \dots & Z_{a_{i\bar{n}}^1} \\ \star & Z_{i2} & \dots & Z_{a_{i\bar{n}}^2} \\ & & \ddots & \vdots \\ \star & \star & & \vdots \\ \star & \star & \dots & Z_{i\bar{n}} \end{bmatrix}, \mathbb{Z}_{b_i^j} = \begin{bmatrix} Z_{b_{i1}^{j1}} & Z_{b_{i2}^{j1}} & \dots & Z_{b_{i\bar{n}}^{j1}} \\ Z_{b_{i1}^{j2}} & Z_{b_{i2}^{j2}} & \dots & Z_{b_{i\bar{n}}^{j2}} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{b_{i1}^{j\bar{n}}} & Z_{b_{i2}^{j\bar{n}}} & \dots & Z_{b_{i\bar{n}}^{j\bar{n}}} \end{bmatrix}, \tag{29}$$

where $Z_{ij}, Z_{a_{ij}^k} \in \mathbf{S}^{\bar{n}} \forall i,k \in \{1,\dots,m\}, \& j \in \{1,\dots,\bar{n}\}$; $Z_{b_{ij}^{kj}} \in \mathbf{S}^{\bar{n}}, \forall i \in \{2,\dots,m\}, k \in \{1,\dots,m-1\}, \& j \in \{1,\dots,\bar{n}\}$ such that $\mathbb{Z} > 0$.

Through the use of (31) and (32), we get

$$P(\bar{\mathcal{A}}_k - \mathcal{L}_k\mathcal{C}) + (\bar{\mathcal{A}}_k - \mathcal{L}_k\mathcal{C})^\top P + \sigma P \leq 0. \tag{33}$$

Furthermore, we can express the inequality (33) in the following manner:

$$\Sigma_k + \mathcal{N}_k \leq 0, \tag{34}$$

where

$$\mathcal{N}_k = \sum_{i,j=1}^{m,\bar{n}} \underbrace{\left(P\begin{bmatrix}\mathbb{T}G_k\mathcal{H}_{ij} \\ \mathbb{O}\end{bmatrix}\right)}_{U_{ij_k}^\top} \underbrace{\left(f_{ij}\overbrace{\begin{bmatrix}H_i\mathbb{P} & \mathbb{O}\end{bmatrix}}^{\mathbb{H}_i}\right)}_{V_{ij}} + V_{ij}^\top U_{ij_k},$$

$\Sigma_k = P\mathcal{A}_k + \mathcal{A}_k^\top P - \mathcal{R}_k^\top \mathcal{C} - \mathcal{C}^\top \mathcal{R}_k + \sigma P$, $\mathcal{A}_k = \begin{bmatrix}\mathbb{M}_{1_k} & \mathbb{O} \\ \mathbb{O} & \mathbb{O}\end{bmatrix}$ and $\mathcal{R}_k^\top = P\mathcal{L}_k$.

In the literature, numerous methodologies have been established to address Lipschitz nonlinearities, denoted as the term $\mathcal{N}_k$, for example, [12], [6], [13], and so on. These approaches are developed from various mathematical tools, such as Schur's lemma [14], Algebraic Riccati Equations [15], and Young inequalities [16], [13]. In this paper, we have used a new technique to address nonlinearities, inspired by the methods introduced in [10].

From [10], one can rewrite the term $\mathcal{N}_k$ as:

$$\mathcal{N}_k = (U_k)^\top(\mathbf{H}\Phi) + \Phi^\top\mathbf{H}^\top(U_k), \tag{35}$$

where

$$U_k = \begin{bmatrix}(U_{k_{11}})^\top & \cdots & (U_{k_{1\bar{n}}})^\top & \cdots & (U_{k_{m\bar{n}}})^\top\end{bmatrix}^\top, \tag{36}$$

$$\mathbf{H} = \text{block-diag}\{\underbrace{\mathbb{H}_1,\dots,\mathbb{H}_1}_{\bar{n} \text{ times}},\dots,\underbrace{\mathbb{H}_m,\dots,\mathbb{H}_m}_{\bar{n} \text{ times}}\}, \tag{37}$$

$$\Phi = \begin{bmatrix}f_{11}\mathbb{I} & \cdots & f_{1\bar{n}}\mathbb{I} & \cdots & f_{m1}\mathbb{I} & \cdots & f_{m\bar{n}}\mathbb{I}\end{bmatrix}^\top. \tag{38}$$

Through the utilisation of (34) and (35), the condition described in (32) is satisfied if

$$\Sigma_k + (U_k)^\top(\mathbf{H}\Phi) + \Phi^\top\mathbf{H}^\top(U_k) \leq 0, \tag{39}$$

B. LMI approach 1: exploiting Lemma 2

In this part, the development of an LMI criterion through the employment of the standard Young's inequality and Lemma 2 is illustrated.

The implementation of the inequality (1) on (35) leads to

$$(U_k)^\top(\mathbf{H}\Phi) + \Phi^\top\mathbf{H}^\top(U_k) \leq (U_k)^\top(\mathbb{Z})^{-1}U_k + \Phi^\top\mathbf{H}^\top\mathbb{Z}\mathbf{H}\Phi, \tag{40}$$

where the considered matrix $\mathbb{Z}$ is illustrated in (29).
Let us consider a new matrix:

$$\Phi_m = \begin{bmatrix}f_{b_{11}}\mathbb{I} & \dots & f_{b_{1\bar{n}}}\mathbb{I} & \dots & f_{b_{m1}}\mathbb{I} & \dots & f_{b_{m\bar{n}}}\mathbb{I}\end{bmatrix}^\top. \tag{41}$$

where $f_{b_{ij}}$ are specified in (17).
Through the deployment of Lemma 2, we get

$$\Phi^\top\mathbf{H}^\top\mathbb{Z}\mathbf{H}\Phi \leq \Phi_m^\top\mathbf{H}^\top\mathbb{Z}\mathbf{H}\Phi_m. \tag{42}$$

From (40) and (42), the inequality (39) holds if

$$\Sigma_k + (U_k)^\top\mathbb{Z}^{-1}(U_k) + \Phi_m^\top\mathbf{H}^\top\mathbb{Z}\mathbf{H}\Phi_m \leq 0. \tag{43}$$

Now, we are ready to state the following theorem.

Theorem 1: If there exist a matrix $\mathbb{Z}$ under the form described by (29) along with the matrices $P_1 > 0 \in \mathbf{S}^{o_1}$, $P_2 > 0 \in \mathbf{S}^{o_2}$, $\mathcal{R}_k \in \mathbb{R}^{(o_1+o_2)\times(o_1+o_2+p)}$ and a positive scalar σ such that the following LMI is feasible:

$$\begin{bmatrix}\Sigma_k & (U_k)^\top & (\mathbb{Z}\mathbf{H}\Phi_m)^\top \\ \star & -\mathbb{Z} & \mathbb{O} \\ \star & \star & -\mathbb{Z}\end{bmatrix} < 0, \ \forall\ k \in \{1,\dots,2^{\bar{\theta}}\}, \tag{44}$$

where Σ_k, U_k, $\mathbf{H}$ and Φ_m are described in (34), (36), (37) and (41), respectively, then, the estimation error $\tilde{x}$ converges exponentially to 0.

The matrix $\mathcal{L}_k$ is computed as $\mathcal{L}_k = P^{-1}\mathcal{R}_k^\top$. Once we determine the matrix $\mathcal{L}_k$, other remaining observer parameters are calculated through the use of the conditions specified in Section III-B. We have provided a detailed algorithm related to this at the end of this segment.

Proof: By implementing Schur's lemma on the inequality (43), one can easily deduce LMI (44). The feasibility of LMI (44) implies that the condition (33) is fulfilled. It infers that the Lyapunov function (30) satisfies the condition specified in (32). From (18b), it is easy to infer that the estimation error $\tilde{x}$ is exponentially stable and converges towards zero. ■

C. LMI approach 2: deploying LPV approach

In the last two decades, several LPV-based LMI techniques have been proposed in the literature for nonlinear observer design problems (see for example [17], [11], [9]). In this section, a new LMI condition is developed by incorporating a well-known LPV approach along with new matrix-multiplier-based LMI techniques.

One can obtain the following inequality by applying (2) on (35):

$$\mathcal{N}_k \leq (U_k + \mathbb{Z}\mathbf{H}\Phi)^\top(2\mathbb{Z})^{-1}(U_k + \mathbb{Z}\mathbf{H}\Phi), \tag{45}$$

The inequality (17) infers that all elements of Φ are bounded and belong to a convex set whose vertices are defined as:

$$\mathcal{F}_{H_m} = \left\{\{\mathcal{F}_{11},\dots,\mathcal{F}_{1\bar{n}},\dots,\mathcal{F}_{m1},\dots,\mathcal{F}_{m\bar{n}}\} : \mathcal{F}_{ij} \in [0, f_{b_{ij}}]\right\}.$$

From the convexity principle, we get

$$\mathcal{N}_k \leq \left[\left((U_k) + \mathbb{Z}\mathbf{H}\Phi\right)^\top(2\mathbb{Z})^{-1}\left((U_k) + \mathbb{Z}\mathbf{H}\Phi\right)\right]_{\Phi\in\mathcal{F}_{H_m}}. \tag{46}$$

From (46), the inequality (39) is true if

$$\Sigma_k + \left[\left((U_k) + \mathbb{Z}\mathbf{H}\Phi\right)^\top(2\mathbb{Z})^{-1}\left((U_k) + \mathbb{Z}\mathbf{H}\Phi\right)\right]_{\Phi\in\mathcal{F}_{H_m}} \leq 0. \tag{47}$$

Theorem 2: Let us consider a matrix $\mathbb{Z}$ as specified in (29) along with $P_1 > 0 \in \mathbf{S}^{o_1}$, $P_2 > 0 \in \mathbf{S}^{o_2}$, $\mathcal{R}_k \in$

$\mathbb{R}^{(o_1+o_2)\times(o_1+o_2+p)}$ and a positive scalar σ. The estimation error $\tilde{x}$ exponentially converges to zero if the following LMI is feasible:

$$\begin{bmatrix} \Sigma_k & \left(U_k + \mathbb{Z}\mathbf{H}\Phi\right)^\top \\ \star & -2\mathbb{Z} \end{bmatrix}_{\Phi\in\mathcal{F}_{H_m}} < 0,\ \forall\, k \in \{1,\dots,2^{\bar{\theta}}\}, \quad (48)$$

where Φ is defined in (38) and other remaining variables are described in Theorem 1. The computation of observer parameters is identical to the one shown in Theorem 1.

Proof: Through the deployment of Schur's lemma on the inequality (47), LMI (48) is obtained. If LMI (48) is feasible, then the condition specified in (33) is satisfied. It infers that the Lyapunov function (30) holds the criterion illustrated in (32). Thus, the system dynamic (28) is exponentially stable. ■

Remark 1: The established LMI (48) is deduced by employing the well-known LPV approach. The authors of [10] have demonstrated the effectiveness of the LPV-based LMI approach, providing a superior optimal solution compared to LMI (44). However, the LPV-based LMI method has a drawback. The LMI presented therein encompasses a large number of LMIs as compared to the one described in Theorem 1. From a computation perspective, solving such a large number of LMIs with any LMI solver can be challenging. Since this parameter calculation is offline, one can deploy this method at the cost of a heavy computation burden.

D. Comment about proposed LMI approach

In order to show the superiority of the established LMI technique, the computation of the number of decision variables inside LMIs (44) and (48) is as follows:

$$\begin{aligned} \mathcal{D} = &\underbrace{2^{\bar{\theta}}\cdot\left(\frac{(m\bar{n})\cdot(m\bar{n}+1)}{2}\right)\cdot\left(\frac{\bar{n}(\bar{n}+1)}{2}\right)}_{\mathcal{D}_2} + \\ &\underbrace{2^{\bar{\theta}}\cdot\left((o_1+o_2)\cdot(o_1+o_2+p)+\frac{o_1(o_1+1)}{2}+\frac{o_2(o_2+1)}{2}\right)}_{\mathcal{D}_1}, \end{aligned} \quad (49)$$

where the terms $\mathcal{D}_1$ and $\mathcal{D}_2$ represent the number of decision variables inside Σ_k and the proposed matrix multiplier $\mathbb{Z}$, respectively.

The number of decision variables inside the approaches proposed in [6] and [7] are illustrated as

$$\bar{\mathcal{D}}_1 = 2^{\bar{\theta}}\cdot\left(np+\frac{n\cdot(n+1)}{2}\right), \quad (50)$$

$$\text{and}\quad \bar{\mathcal{D}}_2 = (1+n_\theta)\cdot np + \frac{n(n+1)}{2} + \frac{(m\bar{n})\cdot\bar{n}(\bar{n}+1)}{2}, \quad (51)$$

respectively. Since $m,\ n,\ p,\ \bar{n},\ o_1,\ o_2 \in \mathbb{R}^+$, we get: $\bar{\mathcal{D}}_1 < \bar{\mathcal{D}}_2 < \mathcal{D}$. Thus, the established dynamic observer method contains more decision variables than the existing approaches. These extra numbers of decision variables add extra degrees of freedom in observer design, increase the accuracy of the estimation and improve the robustness of the estimation.

The detailed procedure for computing the observer parameters is presented in Algorithm 1.

In the sequel, the efficiency of the proposed LMI-based observers over existing approaches is emphasised through a numerical example.

Algorithm 1 Computation of the generalized dynamic observer (12) parameters

Step 1: Consider a full-row rank matrix $\mathbb{E}\in\mathbb{R}^{o_1\times n}$

Step 2: By using (21a), (24a), (27c), determine the matrices $\mathbb{T}$, $\mathbb{T}_2$, $\mathbb{M}_{1_k}$, $\mathbb{M}_{3_0}$ and $\mathbb{P}$

Step 3: Compute the values of $\mathcal{Y}_{2_k}$, $\mathcal{Y}_{3_k}$, $\mathbb{X}_k$ and $\mathbb{Y}_k$ from the solution of the LMI (44) or LMI (48).

Step 4: From (21a), (21b) and (24b), determine the matrices $\mathbb{T}$, $\mathbb{K}$ and Ω_k, respectively.

Step 5: Deduce the remaining matrices $\mathbb{M}_k$, $\mathbb{Q}_k$, $\mathbb{N}_k$, $\mathbb{S}_k$ and $\mathbb{L}$ by utilising (24a), (23), (27a), (27b) and (27d).

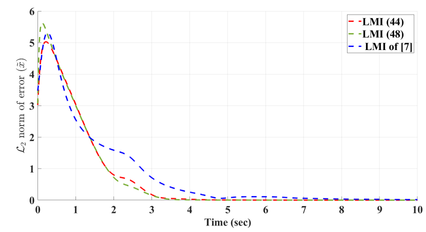

Fig. 1: $\mathcal{L}_2$ norm of estimation error in different cases

V. Numerical Example

This section is dedicated to highlighting the performance of the established LMI-based generalized dynamic observer.

To illustrate the proposed methodology, we have considered the NLPV system in the form of (7) along with the following parameters: $A(\theta)=\begin{bmatrix}0&1&0\\0&0&1\\-\theta&-3&-\theta\end{bmatrix}$, $B(\theta)=\begin{bmatrix}0\\0\\\theta\end{bmatrix}$, $G(\theta)=\begin{bmatrix}\theta&0\\0&0\\0&\theta\end{bmatrix}$ and $C=\begin{bmatrix}1\\0\\0\end{bmatrix}^\top$. The nonlinear function inside system dynamics is given by: $f(x)=\begin{bmatrix}\sin(x_1)\\\sin(x_3)\end{bmatrix}$. The time-varying parameter $\theta(t)$ and input of system $u(t)$ are described as: $\theta(t)=2+\sin(t)$ and $u(t)=3\sin(t)$. The initial values of x for this system is illustrated as $x_0=\begin{bmatrix}2&-2&2\end{bmatrix}^\top$.

For the state estimation of the aforementioned system, the generalized dynamic observer (12) is used. The parameters of the observer are computed by using the **Algorithm 1**. In order to compare the LMI performance, both LMIs are solved by using the YALMIP toolbox, and gain parameters are computed.

Through the use of the observer parameters, the established observer is implemented in MATLAB for both LMIs (44) and (48). Figure 1 depicts the plot of $\mathcal{L}_2$ norm of estimation error achieved in both cases. The convergence of estimation error in all cases is showcased in Figure 1. In addition to this, one can also notice that the convergence rate of $\tilde{x}$ is faster in the case of LMI (48) as compared to other cases.

Further, the Root Mean Square values of estimation errors (RMSE) are calculated in both cases and showcased in Table I. To highlight the observer performance, we have

TABLE I: Comparison of RMSE values of $\tilde{x}$ in several cases for $5 \leq t \leq 10$

	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$
LMI (48)	1.0074×10^{-5}	2.5885×10^{-4}	4.7820×10^{-4}
LMI (44)	1.6223×10^{-5}	5.0291×10^{-4}	0.0013
[7]	0.0186	0.0343	0.0524
[6, Theorem 1]	LMI is infeasible		

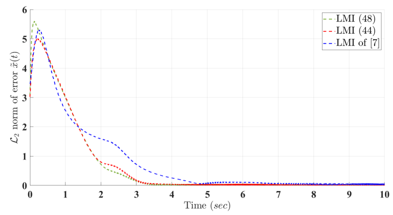

Fig. 2: $\mathcal{L}_2$ norm of $\tilde{x}$ in different cases for the case of systems with parametric uncertainty

compared the obtained RMSE values of $\tilde{x}$ with the same derived from the methods proposed in [7] and [6]. Table I infers that the estimation accuracy is better in the case of LMI (48) as compared to other cases. Thus, the efficiency of the proposed LMI-based observer is emphasized.

Further, the performance of the proposed generalized dynamic observer is evaluated in the presence of parametric uncertainty. Let us consider the system dynamics (7) with the matrix $A_{\text{new}} = A + 0.1 \cdot \mathbb{I}_3 \cdot \alpha(t)$, where $\alpha(t) = \sin(10t)$ depicts uncertainty factor. The dynamic observer (12) with the aforementioned parameters is deployed in MATLAB environment for the state estimation purpose.

Figure 2 represents the graph of estimation error obtained in three aforementioned cases for the earlier-mentioned system. Both figures showcase the exponential convergence of estimation error with the optimal attenuation of uncertainty. It infers that the established observer is robust to the systems with parameter uncertainty. Further, Table II represents RMSE values of estimation error obtained in all cases for the above-mentioned example in the presence of uncertainty. It highlights that the proposed methodology mitigates the impact of uncertainty on the estimated error more effectively than the methods developed in the literature. Thus, through the utilisation of this example, the superiority and robustness of the developed observer are emphasised.

TABLE II: Comparison of RMSE values of $\tilde{x}$ for the case of systems with parametric uncertainty ($5 \leq t \leq 10$)

	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$
LMI (48)	0.0070	0.0170	0.0200
LMI (44)	0.0082	0.0188	0.0209
[7]	0.0234	0.0395	0.0559
[6, Theorem 1]	LMI is infeasible		

VI. Conclusion

In this paper, the challenge of designing a generalized dynamic observer for Nonlinear Parameter-Varying (NLPV) systems is addressed. The issue is resolved through the development of two new LMI conditions. These conditions not only provide the necessary observer parameters but also ensure the exponential stability of the estimation error of the proposed observer. These LMI conditions are formulated by using the polytopic method. By incorporating the reformulated Lipschitz property, Young inequalities and well-known LPV approach, the proposed matrix multiplier-based LMIs are obtained. Notably, the developed LMIs involve a higher number of decision variables compared to existing ones, deliberately incorporating matrix multipliers. This intentional inclusion leads to an enhancement in LMI feasibility. The performance of the observer is verified through the use of a numerical example.

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