Using Circulation to Mitigate Spurious Equilibria in Control Barrier Function

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Abstract— **Control Barrier Functions and Quadratic Programming are increasingly used for designing controllers that consider critical safety constraints. However, like Artificial Potential Fields, they can suffer from the stable spurious equilibrium point problem, which can result in the controller failing to reach the goal. To address this issue, we propose introducing circulation inequalities as a constraint. These inequalities force the system to explicitly circulate the obstacle region in configuration space, thus avoiding undesirable equilibria. We conduct a theoretical analysis of the proposed framework and demonstrate its efficacy through simulation studies. By mitigating spurious equilibria, our approach enhances the reliability of CBF-based controllers, making them more suitable for real-world applications.**

I. Introduction

Control barrier functions [1] (**CBF**s) have attracted much interest in recent years in the field of automatic control, due to the fact that they provide a convenient incorporation of safety constraints as linear inequalities in the control input [2]–[13]. These inequalities can be used into a Quadratic Programming (**QP**) optimization problem, that is formulated based on two objectives: making the resulting control action as close as possible to a *nominal controller*, that solves the task (e.g., navigating towards a goal location) if we disregard all the constraints; ensuring that all the safety constraints are satisfied. Solving this QP optimization problem provides a control input that is able to maintain safety while driving the robot to complete the task.

In the QP formulation, the constraints are linear in the control input provided that we assume an affine dynamics for the system. This linearity appears since we use *local information* of the problem, namely the gradients of the distance and Lyapunov functions. This property of the approach induces *spurious equilibrium points* into the system, and so the closed loop system can become stuck in such a point. Even worse, these spurious equilibrium points can even be *stable* [14]. This problem also appears in other approaches that use local information to navigate. A quintessential example is using *artificial potential functions* (**APF**) [15], in which a weighted sum of gradients of attractive and

repulsive potentials are employed for navigation. The QP-CBF formulation is qualitatively similar to the APF. In fact, the Karush-Kuhn-Tucker conditions (**KKT**), the fact that many of the CBF inequalities are generated using gradients of distance functions and the fact that the nominal controller is usually the negative gradient of a potential function, imply that the CBF-QP controller can be seen as a sum of attractive and repulsive vectors with a choice of weights that comes from the optimization procedure.

Considering this local equilibrium issue, there are works that aim to modify the traditional QP-CBFs/APF approaches to circumvent it. For example, [14] introduces a new CBF inequality into the original CBF formulation that can, in the case that there is a single convex obstacle, remove the equilibrium points in the boundary of the obstacle. In the proposed approach in this paper, the spurious equilibrium point problem is mitigated by introducing an inequality, which we call *a circulation inequality*. This inequality forces the configuration to circulate the obstacle when near it. This is motivated by the observation that QP-CBF approaches often fail in large/complex obstacles, specially when the nominal controller is defined in terms of the gradient of a simple potential function as, for example, the squared distance between the configuration and the goal in the configuration space. This is because the controller seeks to decrease this potential function, and for large obstacles this may be impossible to do in all situations: sometimes the configuration must move around (i.e., circulate) the obstacle, resulting in temporary increase of this function so as to enable eventual reaching of the goal at a later time.

The idea of inducing circulation of obstacles appears in other works as well. Early approaches for circulating obstacles appear in the *bug algorithm* [16]. In order to avoid obstacles, [17] proposes the addition of a circulation term, called "gyroscopic force", and [18], [19] propose the addition of a circulation term based on magnetic fields. Furthermore, in [20], the authors propose a modification of the classic attractive/repulsive potential fields approach so as to improve its "efficiency" while retaining the Lyapunov stability analysis from traditional potential fields. The approach is motivated by the observation that if the attractive and repulsive vectors conflict with each other - i.e., if the cosine of the angle between them is less than 0 - then, within a certain range of angles, the repulsive vector can be replaced by a different vector that is orthogonal to the attractive vector without violating the Lyapunov inequalities for the closed loop system. This modification of the attractive/repulsive potential fields approach allows the robot to move faster towards the goal since the impact of the repulsive potential is

reduced in the system. This modification of making (in some circumstances) the repulsive and attractive fields orthogonal can be seen as a way to generate a circulation behavior in the controller. However, although this technique can remove some spurious equilibrium points in some situations, it cannot completely solve the problem. The "modification of the repulsive/attractive vectors" philosophy is also employed in this work, since the proposed optimization formulation is conceptually similar to APF approaches. The difference is that the modification is introduced in this paper by adding a new constraint into the optimization problem. This proposed strategy allows us to remove the spurious equilibrium points in some circumstances, as shown in Figure 1.

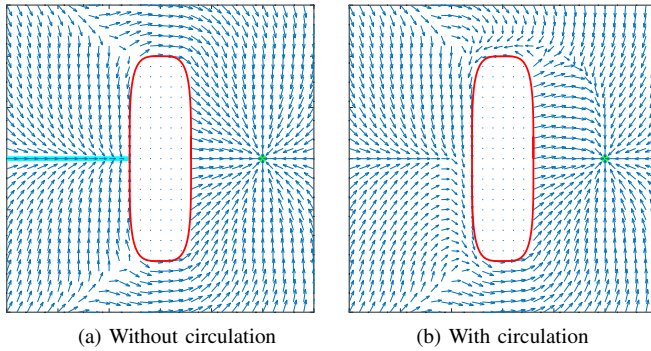(a) Without circulation   (b) With circulation

Fig. 1: Vector field generated by traditional CBF-QP formulations (2) (left) and our formulation (3) (right), with a nominal controller that points towards the goal. Note that in the original formulation, there is a set of points (at the left of the obstacle) such that if we start from this set, we converge to spurious equilibria near the boundary of the obstacle instead of converging towards the target.

The problem considered in [21], which is to track a path while avoiding obstacles, is similar to the problem addressed in this paper. To implement obstacle avoidance, a "circulation mode", based on the controller proposed in [22], is initiated when the robot is near the obstacle until it can return to navigating towards the target. This switching between circulation and goal-directed navigation modes is done by monitoring the sign of a variable and can produce discontinuities in the vector field. Our proposal is similar in the sense that we also enforce circulation when near an obstacle, but the enforcing of this behavior is achieved by imposing a circulation constraint, which gradually increases in its effect as the robot approaches the obstacle, instead of switching the vector fields in a discontinuous fashion. Since our formulation is based on a strictly convex optimization problem with continuous data, our proposed construction can be shown to have continuity properties using classical theorems on QPs. Finally, our previous work [23] also proposes a circulation-like inequality for CBFs, but considering only the planar case. So, our approach generalizes this result to arbitrary dimensions and present theoretical guarantees.

In this paper, we extend the QP-CBF formulations with a circulation constraint, providing certain theoretical results (as characterization of the equilibrium points, feasibility, and

continuity). The major challenge is related to how to *define* circulation in higher dimensions. In order to implement the constraint, we need to specify precisely *how* to circulate the obstacles. Clockwise or counter-clockwise can be used in planar situations, but when we consider higher dimensions the situation is more complex, in which topology may forbid us to defining circulation in a continuous way.

Paper organization: The notations are summarized in Section II. The proposed CCBF-QP (Circulation Control Barrier Function Quadratic Program) approach and its properties are discussed in Section III. Practical aspects in application of the proposed approach are discussed in Section IV. Simulation studies of the approach are presented in Section V. Finally, concluding remarks are summarized in Section VI.

II. MATHEMATICAL NOTATION

We denote by $\mathbb{R}^+$ ($\mathbb{R}^-$) the set of nonnegative (nonpositive) reals. All vectors are column vectors unless stated otherwise, and for a vector v, v^T denotes its transpose. If $f : \mathbb{R}^n \mapsto \mathbb{R}$, ∇f is its gradient, written as a column vector. For a closed set $\mathcal{C}$, we denote its boundary by $\partial \mathcal{C}$ and $Int(\mathcal{C}) \triangleq \mathcal{C} - \partial \mathcal{C}$ its interior. We denote the identity matrix of appropriate order by I. A square matrix is *skew-symmetric* if $\Omega^T = -\Omega$ and *orthonormal* if $\Omega^T \Omega = I$.

III. THE CCBF-QP FORMULATION AND ITS PROPERTIES

A. The CCBF-QP formulation

Our objective is to define a controller that reaches a goal in the configuration space while avoiding obstacles. For this, we will present assumptions and definitions that will allow us to describe formally the problem and our solution.

Assumption 1. *We assume the following:*

1) *The configuration space $\mathcal{Q}$ is an even-dimensional Euclidean space, that is, $\mathcal{Q} \subset \mathbb{R}^n$, n even. Note that $\mathcal{Q}$ can include forbidden configurations.*

2) *A single n-dimensional obstacle $\mathcal{C} \subset \mathbb{R}^n$ is considered in the configuration space modeled as a closed set. Furthermore, a function $D : \mathbb{R}^n \mapsto \mathbb{R}^+$ is assumed that computes a metric of "closeness" from the point $q \in \mathcal{Q}$ and $\mathcal{C}$. D is not required to be the Euclidean distance between q and $\mathcal{C}$, but*

$$D(q) > 0 \;\Rightarrow q \notin \mathcal{C}. \tag{1}$$

Furthermore, D is assumed to be differentiable (i.e., $\nabla D(q)$ is continuous) everywhere in $\mathbb{R}^n - Int(\mathcal{C})$ and $\nabla D(q) \neq 0$ almost everywhere. Define the set $\mathcal{W} \triangleq \{q \in \mathbb{R}^n \mid q \in \mathbb{R}^n - Int(\mathcal{C}) \;,\; \nabla D(q) \neq 0\}$. For $q \in \mathcal{W}$, we also define the normal vector $N(q) \triangleq \nabla D(q)/\|\nabla D(q)\|$.

3) *A skew-symmetric orthonormal matrix is defined such that $\Omega \in \mathbb{R}^{n \times n}$. If $q \in \mathcal{W}$, we define the tangent vector as $T(q) \triangleq \Omega N(q)$.*

4) *A target configuration $q_g \in \mathcal{W}$ is to be achieved, with a nominal controller $u_d : \mathbb{R}^n \mapsto \mathbb{R}^n$ achieving this task, as the negative gradient of a positive definite differentiable function $V(q)$ such that V and ∇V vanish only at $q =$*

q_g. Furthermore, it is assumed that $\|\nabla V(q)\|$ is upper bounded by all q, i.e., $\max_{q \in \mathbb{R}^n} \|\nabla V(q)\|$ is finite.

5) *The set of configuration-dependent constraints to our control input can be written as $\mathcal{U}(q) \triangleq \{\mu \mid A(q)\mu \geq b(q)\}$, in which $A : \mathbb{R}^n \mapsto \mathbb{R}^{l \times n}$ and $b : \mathbb{R}^n \mapsto \mathbb{R}^l$ are continuous functions of q. Furthermore, $\forall q$, let the set $\mathcal{U}(q)$ contain a ball of radius $r > 0$ centered at the origin, i.e., if $\|\mu\| \leq r$, then $A\mu \geq b$.*

6) *A continuous function $\alpha : \mathbb{R}^+ \mapsto \mathbb{R}^-$ exists such that $\alpha(0) = 0$ and $\alpha(D) < 0$ for $D < 0$ (e.g., $\alpha(D) = -\eta D$ for a constant $\eta > 0$).*

7) *A continuous function $\beta : \mathbb{R}^+ \mapsto \mathbb{R}$ exists with the following properties: β is decreasing, $\beta(0) > 0$, and $\beta(D) \to -\infty$ as $D \to \infty$ (e.g., $\beta(D) = a - bD$ for positive constants a, b). Furthermore, $\beta(D(q_g)) < 0$ and $\beta(0) < r$.*

8) *The system's dynamics is $\dot{q} = u$.*

The most striking assumption is related to the even-dimensional assumption in Assumption 1-1. It turns out that even-dimensional spaces are more amenable for defining the circulation of obstacles than odd-dimensional spaces. This is a topological issue: the *hairy ball theorem* [24] establishes that there is no way to define a continuous and non-vanishing circulation field (i.e., tangent to the surface) on the surfaces of odd-dimensional balls. Thus, since N is in the surface of a n-dimensional ball (thanks to $\|N\| = 1$ and $N \in \mathbb{R}^n$), creating a continuous non-vanishing function that maps every possible N to a vector orthogonal to it would be tantamount to generating a continuous non-vanishing field at the surface of the n-sphere, which is forbidden by the hairy ball theorem when n is odd. As a consequence of this result, there is no such matrix Ω as defined in Assumption 1-3 if n is odd[1], because $F(N) = \Omega N$ would define a continuous and non-vanishing tangential field in the surface of an odd-dimensional sphere. In Subsection IV-C, we discuss a workaround for the case when n is odd.

To reach the target q_g while avoiding the obstacle, a common approach is to formulate a *minimally invasive QP using CBFs* and compute the control input u as:

$$u(q) = \quad \arg\min_{\mu} \ \|\mu - u_d(q)\|^2$$
$$\text{such that} \quad N(q)^T \mu \geq \alpha(D(q))$$
$$A(q)\mu \geq b(q) \qquad (2)$$

for $q \in \mathcal{W}$ (we leave $u(q)$ undefined if $q \notin \mathcal{W}$). The QP tries to minimally modify the nominal controller $u_d(q)$ while imposing the CBF constraint $\dot{D} = \nabla D(q)^T \mu \geq \|\nabla D(q)\| \alpha(D(q)) \geq 0$ that guarantees that the system will not enter the forbidden set $\mathcal{C}$.

Unfortunately, the formulation in (2) can have stable equilibrium points other than the only desired one $q = q_g$. The result of (2) is essentially a sum of potential fields, which

is known for having spurious equilibrium points, dependent on the repulsive potential $N(q)$. To give an intuition about it, we note that if we disregard the constraints $A(q)u \geq b(q)$, (2) can be solved analytically:

$$u(q) = -\nabla V(q) + N(q)\lambda(q)$$

in which $\lambda(q) = \max\left(0, N(q)^T \nabla V(q) + \alpha(D(q))\right)$, the dual variable for the normal constraint, is a nonnegative configuration-dependent weight for the repulsive potential.

These spurious equilibrium points appear near the surface of the obstacle when the controller cannot force the configuration to move around the obstacle to avoid it. Therefore, in this paper, we augment (2) with an inequality that forces the configuration to circulate the obstacle when in its neighborhood by including the inequality:

$$T(q)^T u \geq \beta(D(q))$$

(in which β was defined in Assumption 1 - 7) that forces the following behavior: when near obstacles (that is, when $\beta > 0$), we have a positive projection into the tangent vector of the obstacle, which means that we should circulate it. If D is large enough, β becomes very negative and thus the constraint is trivialized. This is specially true for $q = q_g$: we do not want to force any circulation when the goal is achieved and this is why Assumption 1 - 7 has $\beta(D(q_g)) < 0$. Nevertheless, after the inclusion of this inequality into (2), we have the **CCBF-QP** (Circulation Control Barrier Function Quadratic Program), including the validity of all terms in Assumption 1:

$$u(q) = \quad \arg\min_{\mu} \ \|\mu - u_d(q)\|^2$$
$$\text{such that} \quad N(q)^T \mu \geq \alpha(D(q))$$
$$T(q)^T \mu \geq \beta(D(q))$$
$$A(q)\mu \geq b(q). \qquad (3)$$

Figure 1 makes the comparison in the computed vector fields between (2) and (3), disregarding the constraints $A(q)\mu \geq b(q)$, for a smooth rectangle-like obstacle and goal $p_{goal} = [1 \ 0]^T$ (green dot in the two pictures). We can see that without the circulation term, there is a spurious line of attractive points (in cyan), with a very large attractive basin to the left of the obstacle. If we include the circulation terms, there are no spurious attractive points anymore.

Next, we establish some properties for this formulation.

B. Feasibility of CCBF-QP

One important question is whether the CCBF-QP formulation always admits a feasible solution. The following result answers this question positively.

Theorem 1. *Let K be such that $\beta(0) < K < r$ (such K always exists, since $\beta(0) < r$, see Assumption 1-7). Then, if $q \in \mathcal{W}$, the CCBF-QP formulation always has a feasible solution: $\mu_f \triangleq K T(q)$. Furthermore, this particular solution lies in the interior of the feasible set, achieving all inequalities without equality.*

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[1]Another approach without invoking the hairy ball theorem is that if n is odd and Ω is skew-symmetric, $\det(\Omega^T) = \det(-\Omega)$ implies $\det(\Omega) = (-1)^n \det(\Omega)$ and thus $\det(\Omega) = 0$. Hence, odd-dimensional skew-symmetric matrices are non-invertible and cannot be orthonormal.

Proof: We shall verify all of the three constraints in (3), and emphasize that the inequalities are achieved without equality.

- For $N(q)^T \mu_f \geq \alpha(D(q))$, we note that $N(q)^T T(q) = 0$ and $\alpha(D(q)) < 0$ if $q \notin \mathcal{C}$.
- For $T(q)^T \mu_f \geq \beta(D(q))$, we note that $T(q)^T T(q) = \|T(q)\|^2 = 1$ and that $K > \beta(D(q))$ since β being decreasing (Assumption 1-7) implies that the maximum value for β occurs at $D = 0$ and thus $K > \beta(0) \geq \beta(D(q))$.
- For the constraint $A\mu \geq b$, we use the fact that $\|\mu\| \leq r$ is a sufficient condition for this inequality to hold (Assumption 1-5). Now $\|KT(q)\| = K < r$. The inequality being strict means that μ_f lies strictly inside the set $A\mu \geq b$, i.e., $A\mu_f > b$.

This implies that CCBF-QP allows as a feasible action for the system to move tangent to its current contour surface of $D(q)$ (in the direction specified by $T(q)$).

C. Continuity of CCBF-QP

The continuity of $u(q)$ is established in the sequel.

Theorem 2. *If $u(q)$ comes from a CCBF-QP, it is a continuous function of q for any $q \in \mathcal{W}$.*

Proof: This is mainly a consequence of a result [25] that states that the single minimizer of a strictly convex quadratic program varies continuously with the parameters. See the extended version [26] for the proof.

D. CCBF-QP Equilibrium Points

We will characterize the subset of $\mathcal{W}$ in which the output of CCBF-QP is the null vector, that is, the *equilibrium points* of the dynamical system $\dot{q} = u(q)$.

Theorem 3. *Considering the set $q \in \mathcal{W}$, in the CCBF-QP formulation, $u(q) = 0$ occurs only in the two mutually exclusive conditions:*

1) *$q = q_g$;*
2) *$\beta(D(q)) = 0$ and $-u_d(q)$ is positive parallel to $T(q)$, that is, there exists a positive scalar λ_T for which $u_d + \lambda_T T = 0$.*

Proof: This comes by analyzing the KKT conditions. See the extended version [26] for the proof.

IV. PRACTICAL ASPECTS

A. The function D

The scalar function D must encompass all the system constraints, codified as an obstacle $\mathcal{C}$ in the configuration space. However, in practice, it is easier to describe these constraints using m scalar functions as $F_i(q) \geq 0$, that is, $\mathcal{C} = \{q \in \mathbb{R}^n \mid F_i(q) \geq 0 \ i = 1, 2, .., m\}$. In this case, one could use as function D the minimum of these F_i, since clearly $D > 0$ implies that $q \notin \mathcal{C}$. However, even if F_i is everywhere differentiable, the function D defined in this way can have many non-differentiable points. More specifically, whenever a q exists in which the minimum is achieved for at least two different functions F_i, D is non-differentiable

unless the respective gradients ∇F_i are equal at this q, which is a very strong condition. Being non-differentiable brings several practical problems because the controller will be discontinuous at that point.

This problem can be removed by using an approximate version of the minimum function. The *softmin* function [27] is a good candidate, defined (with an averaging term) as $\min^h(g_1, g_2, ..., g_m) \triangleq -h \ln\left(\frac{1}{m} \sum_k \exp(-g_k/h)\right)$. This function is always differentiable in the arguments g_i and approximates the minimum when h is positive and close to 0. However, it can be shown that $\min^h(g_1, g_2, ..., g_m) \geq \min(g_1, g_2, ..., g_m)$, and thus if $D(q) = \min_i^h F_i(q)$, $D > 0$ is chosen, this may not imply $F_i(q) > 0$ for all i. In this case, a margin $\delta > 0$ is introduced followed by a choice of $D(q) \triangleq \min_i^h F_i(q) - \delta$.

B. Generating matrices Ω

In order to implement the algorithm, skew-symmetric orthonormal matrices, i.e, $\Omega \in \mathbb{R}^{n \times n}$ such that $\Omega^T = -\Omega = \Omega^{-1}$ are needed, which exist only in even-dimensional spaces. A different choice of Ω implies a different way to circulate the obstacle.

If $n = 2$, there are only two such matrices:

$$\Omega_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \ , \ \Omega_2 = -\Omega_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (4)$$

corresponding to counter-clockwise and clockwise rotations, respectively. But for $n > 2$ (and even), infinite such matrices exist. A simple way to create these matrices is to divide the set $\{1, 2, ..., n\}$ into $n/2$ pairs (i_k, j_k), $k = 1, 2, ..., n/2$. We then start from the $n \times n$ identity matrix, and multiply each i_k^{th} row by -1. We then swap the i_k^{th} row with the j_k^{th} row, and we can check by inspection that the resulting matrix is skew-symmetric and orthonormal.

We can create $n!/(n/2)!$ such matrices, but for even numbers $n > 2$, there are other ways. One simple way is as follows: it is easy to show that if $\Omega \in \mathbb{R}^{n \times n}$ is skew-symmetric orthonormal and $Q \in \mathbb{R}^{n \times n}$ is an orthonormal matrix, then $\hat{\Omega} = Q\Omega Q^T$ is also a skew-symmetric orthonormal matrix. We can use this fact to generate new skew-symmetric orthogonal matrices by constructing Ω by the simple pairing index procedure described above, constructing an orthonormal matrix Q and then calculating $\hat{\Omega} = Q\Omega Q^T$. To generate orthonormal matrices in $\mathbb{R}^{n \times n}$, we use the fact that if A is skew-symmetric and I is the $n \times n$ identity matrix, $Q = (I - A)^{-1}(I + A)$ is orthonormal. We can generate a skew-symmetric matrix by using any matrix $B \in \mathbb{R}^n$ and computing $A = B - B^T$.

C. Odd dimension extension

In Assumption 1-1, we consider even n. If n is odd, there is no matrix $\Omega \in \mathbb{R}^{n \times n}$ that is skew-symmetric and orthonormal. One alternative for handling the odd-dimensional case is to drop one of these properties. On the one hand, being skew-symmetric is important for guaranteeing that $T = \Omega N$ is orthogonal to N (for any N), which guarantees that T indeed induces circulations. On the other hand, being
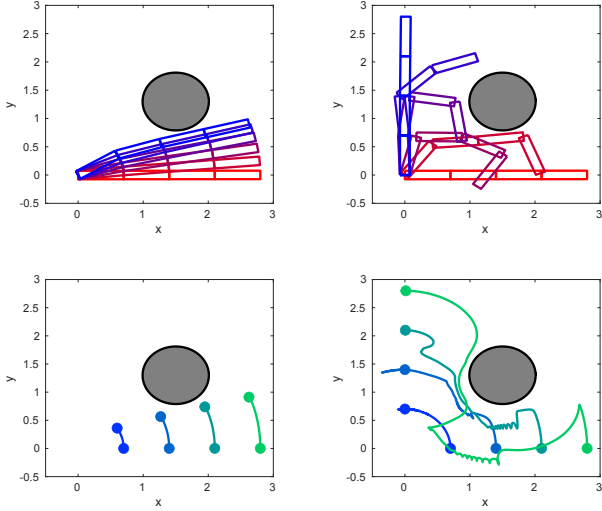
Fig. 2: Manipulator trajectory snapshots without (with) the circulation constraint at left (right) column.

Fig. 3: Snapshots of the trajectory of the manipulator in the *configuration space*.

orthonormal, is not as important as Ω being invertible, which is also incompatible with skew-symmetry when n is odd. Invertibility is important because it guarantees that $T = \Omega N$ only vanishes when N vanishes, i.e., when $q \notin \mathcal{W}$. If T vanishes when $\beta > 0$, the circulation constraint $T^T\mu \geq \beta$ cannot be satisfied and the QP is infeasible, rendering the controller undefined even in some points inside $\mathcal{W}$.

The constraint in Ω for n odd can be relaxed assuming that Ω is skew-symmetric and orthonormal only in an $n - 1$ dimensional subspace of $\mathbb{R}^n$, being singular in the complementary 1-dimensional space. Such matrices can be created using a similar construction as the one described in Subsection IV-B. However, for n odd, the index i that has no pair must have its respective row in the identity matrix set to zero.

V. SIMULATION STUDIES

In this simulation, we have a planar 4DOF revolute manipulator and rectangle-shaped links of size 0.70m×0.15m. The configuration $q \in \mathbb{R}^4$ corresponds to the four joint positions. The arm goes from an initial horizontal configuration to a vertical configuration while avoiding a circular obstacle with radius 0.5m, centered at x=1.5m and $y = 1.3$m.

We introduce functions F_i to model the constraints and obstacles in configuration space, as described in Subsection IV-A. The first constraint is the collision avoidance between the links and the obstacle (i.e., circle). We consider functions $F_i(q)$, $i = 1, 2, 3, 4$, that compute the squared distance (measured in meters square) between each rectangular link and the circle. We also consider a joint limit of $\pm 360^o$ for the first joint and $\pm 120^o$ for all the other joints. These constraints also generate obstacles in the configuration space, which are modeled using functions $F_i(q) = 0.02(q_i - q_{\min,i})$ and $F_i(q) = 0.02(q_{\max,i} - q_i)$ (q_i measured in degrees) for minimum and maximum joint limits, respectively. We thus
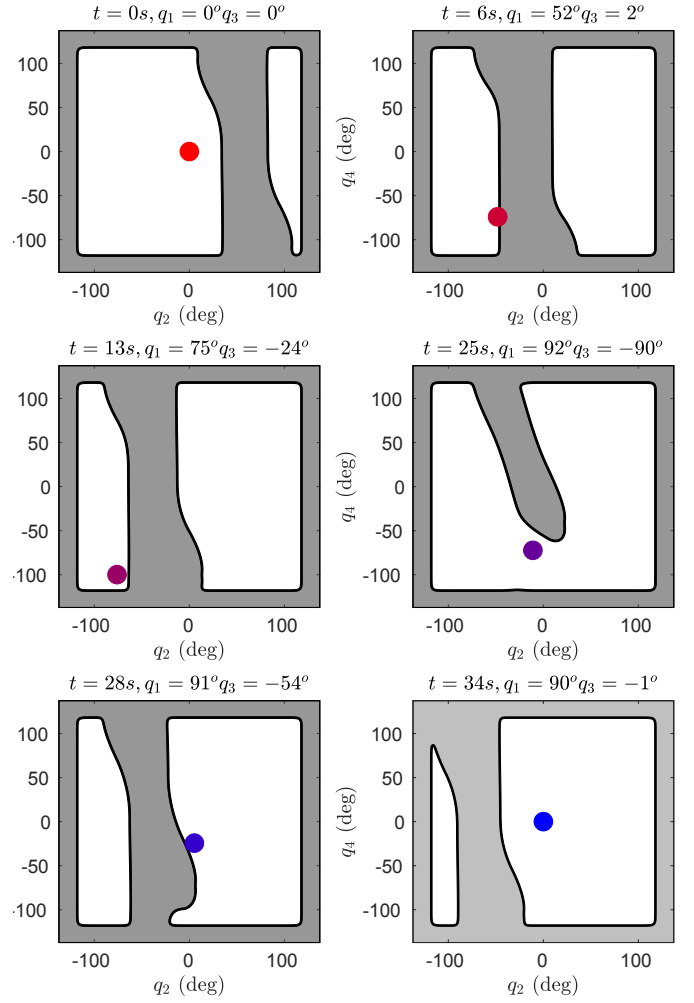
have another 8 functions F_i. For the softmin, we used $h = 0.04$ and a margin of $\delta = 0.15$. We used $\alpha(D) = -0.5D$, $\beta(D) = 0.6 - 2.25D$, and $V(q) = \sum_{i=1}^{4} 0.4|q_i - q_{g,i}|^{1.5}$. We also use the constraints of the form $A\mu \geq b$ to impose a maximum absolute value of 1 to each control input. The employed Ω matrix is

$$\Omega = \begin{bmatrix} 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & 0 \end{bmatrix}. \quad (5)$$

Figure 2-(Top-Right) shows the path made by the manipulator in six different timesteps, Figure 2-(Bottom-Right) shows the path made by the junction points of the links/tip of the manipulator, and Figure 4 shows the evolution of the distance function, which is always greater than 0. It is important to note that without the circulation constraint, the system converges to a spurious equilibrium point when the manipulator touches the circle from below. The snapshots for the simulation without the circulation constraint can be seen
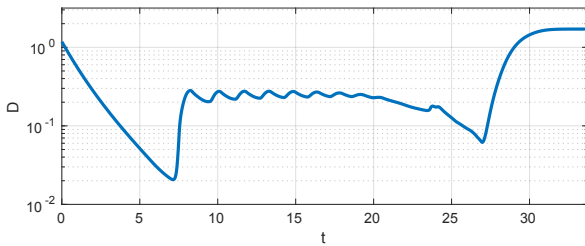
Fig. 4: Evolution of the distance function D.

in the two figures at the Left column of Figure 2. It is also important to mention that, although the obstacle in the 2D space is very simple (a circle), the resulting obstacle $\mathcal{C}$ in the configuration space is very complex. To illustrate this claim, Figure 3 shows snapshots of the obstacle in the *configuration space* during the same six timesteps as Figure 2. Since the configuration space is 4-dimensional, we show 2-dimensional slices of this space by keeping the same configurations q_1, q_3 but changing q_2 and q_4. The obstacle region is in gray.

VI. CONCLUSION

An optimization framework was proposed for controlling a system with integrator dynamics to a goal in the configuration space while considering obstacle avoidance in addition to other constraints. The novelty of the framework is the introduction of a circulation inequality that forces the system to circulate an obstacle - in the configuration space - when sufficiently near it. The motivation for the approach is that often CBF-QP formulations are prone to reach spurious equilibrium points instead of the desired goal. The introduced inequality allows us to extend the range of situations in which the approach is successful with a negligible increase in the complexity of the formulation. We showed several formal results for the controller including continuity, feasibility, and classification of the equilibrium points. Furthermore, the applicability of the approach was demonstrated using simulation studies, in which we highlight that the original CBF formulation alone is not capable of solving the task.

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