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Moment matching for second-order systems with pole-zero placement

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***Abstract*— In this paper, a structure-preserving model reduction problem for second-order dynamical systems of high dimension using time-domain moment matching with pole-zero placement is studied. The moments of a second-order system are defined based on the solutions of linear matrix equations. Families of second-order reduced models, parameterized in a set of matrix degrees of freedom, that match the moments of a given second-order system at selected interpolation points are computed. We then provide formulae for the set of matrix parameters such that the reduced order approximation has a set of prescribed poles and zeros. The theory is illustrated on a damped vibratory system (e.g., a chain of mechanical oscillators) of degree n, governed by a second-order dynamical model.**

I. Introduction

Second-order systems are used to model physical systems, including mechanical and electrical systems, see e.g., [1], [2], [3], [4]. In real applications, the model description of a second-order system is of high dimension, and is computationally costly, hence hindering simulation and control. Therefore, model order reduction is needed to preserve the second-order topology. The moment matching techniques provide efficient tools for model reduction, see [5], [6], [7], [8] for an extensive overview for first-order systems. Using Krylov projection matrices, reduced models are constructed to match the original system at selected complex points. Extensions to second-order systems are found in, e.g., [9], [10], [11], where second-order Krylov subspaces are introduced to preserve the second-order structure, see also the survey [12]. A time-domain approach to moment matching has been presented in [8], with moments characterized by the unique solution of the Sylvester equation, based on [6]. The approach has been further developed in e.g., [13], [14], [15], [16] for port-Hamiltonian systems, two-sided moment matching, and simultaneous pole-zero-higher order moments enforcement, respectively. In [17], a time-domain approach based on a signal generator perspective has been taken yielding a notion of moment matching for nonlinear second-order dynamical systems. Necessary and sufficient conditions have been given for a system of second-order equations to achieve moment matching. A family of systems of second-order equations achieving moment matching has been constructed choosing the free parameters of a parameterization of all systems achieving moment matching.

In this paper, using the recent results in the preprint [18], on time-domain moment matching of linear second-order systems, we compute the second-order approximation meeting pole-zero constraints. Representing the moments of the second-order system at a set of interpolation points through the unique solution of a linear matrix equation, a family of reduced models in the second-order form is constructed, matching the prescribed moments, parameterized in a set of several matrix degrees of freedom. We then derive results by finding the unique set of free matrix parameters, and the resulting model, such that a prescribed set of poles and/or zeros are imposed on the second-order approximation. The theory is illustrated using a damped vibratory system [19], [20], with its limitations highlighted in the undamped case. The approach herein is different from [17], where the notion of moments is presented as the time-domain response at a reference signal. We use the frequency domain notion of moments of a complex transfer function and its relation to linear matrix equations [6]. We then compute free parameters to obtain reduced second-order models that achieve moment matching with prescribed poles and zeros.

The paper is organized as follows. In Section II, we present preliminary results regarding time-domain moment matching for second-order linear systems [18]. We also briefly overview the notion of zeros of a system from the state-space perspective, as defined in [21], and formulate the model reduction problem. In Section III, we address the constraints on the family of second-order approximations achieving moment matching, such that a set of prescribed poles and/or zeros are imposed, computing the unique model meeting the constraints. Section IV illustrates the theory on a practical example. The paper ends with concluding remarks.

Notation: $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ are the sets of natural, real, and complex numbers, respectively. For matrix $A \in \mathbb{R}^{n\times m}$, A^T is the transpose of A. $\sigma(A)$ is the set of eigenvalues of A. $I_n \in \mathbb{N}^{n\times n}$ is the identity matrix. $A \succ 0$ means that A is positive definite. A set of complex numbers is symmetric if for any element, its conjugate is in the same set, including multiplicities. $a : b = \{a, a + 1, \dots, b\}$, for $a < b$ integers.

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II. Preliminaries

In this section, we briefly overview recent results on single input-single output, and time-domain moment matching for second-order systems, as in the preprint [18]. Consider a linear time-invariant second-order system described by

$$\mathbf{\Sigma}:\begin{cases} M\ddot{x}(t)+D\dot{x}(t)+Kx(t) &= Bu(t),\\ C_1\dot{x}(t)+C_0x(t) &= y(t),\end{cases} \tag{1}$$

where $M \succ 0, D \succ 0, K \succ 0 \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^n$ and $C_0, C_1 \in \mathbb{R}^{1\times n}$. The transfer function of $\mathbf{\Sigma}$ is

$$W(s) = (C_1 s + C_0)(Ms^2 + Ds + K)^{-1}B. \tag{2}$$

A. Moments of second-order systems

In this section, we characterize the moments of the second-order system $\mathbf{\Sigma}$ in (1) at a set of interpolation points, different from the poles of $\mathbf{\Sigma}$, defined as follows:

$$\Omega := \{s \in \mathbb{C} \mid \det(Ms^2 + Ds + K) = 0\}. \tag{3}$$

Note that $|\Omega| = 2n$. We present the notion of moments of a second-order transfer function (2), as in [5], [8].

Definition 1. Let $s_\star \in \mathbb{C}$ be such that $s_\star \notin \Omega$. The 0-moment of $W(s)$ at $s_\star$ is the complex matrix

$$\eta_0(s_\star) = W(s_\star) = (C_1 s_\star + C_0)(Ms_\star^2 + Ds_\star + K)^{-1}B,$$

and the k-moment at $s_\star$, $k \in \mathbb{N}$, is defined by

$$\eta_k(s_\star) = \frac{(-1)^k}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}s^k}W(s)\right]_{s=s_\star},\ k \geq 1. \tag{4}$$

The next result presents a matrix representation of the moments at a set of symmetric points (including multiplicities) $\eta_k(s_i)$, $i = 1:\nu$, where $\{s_i \in \mathbb{C} \mid i = 1:\nu\} = \sigma(S)$, with $S \in \mathbb{R}^{\nu\times\nu}$.

Theorem 1. *[18] Consider the second-order system* (1) *with transfer function $W(s)$ and the non-derogatory*[1] *matrices $S \in \mathbb{R}^{\nu\times\nu}$ and $Q \in \mathbb{R}^{\nu\times\nu}$. Let $L \in \mathbb{R}^{1\times\nu}$ and $R \in \mathbb{R}^{\nu}$ be such that the pair (L, S) is observable, and the pair (Q, R) is controllable, respectively. Then, the following statements hold.*

1) *If $\sigma(S)\cap\Omega = \emptyset$, there is a one-to-one relation between the moments of $W(s)$ at $\sigma(S)$ and the matrix $C_0\Pi + C_1\Pi S$, where $\Pi \in \mathbb{R}^{n\times\nu}$ is the unique solution of the linear matrix equation*

$$M\Pi S^2 + D\Pi S + K\Pi = BL. \tag{5}$$

2) *If $\sigma(Q)\cap\Omega = \emptyset$, there is a one-to-one relation between the moments of $W(s)$ at $\sigma(Q)$ and the matrix ΥB, where $\Upsilon \in \mathbb{R}^{\nu\times n}$ is the unique solution of the linear matrix equation*

$$Q^2\Upsilon M + Q\Upsilon D + \Upsilon K = RC_0 + QRC_1. \tag{6}$$

[1] A matrix is non-derogatory if its minimal polynomial coincides with its characteristic polynomial.

Remark 1. Matrices Π and Υ can be extracted from the solutions of Sylvester equations of order $2n \times 2\nu$. Indeed, according to [18, Theorem 1], there exists a matrix $\widetilde{\Pi} = \begin{bmatrix}\Pi^T & \Pi_1^T\end{bmatrix}^T$ satisfying the solution of the Sylvester equation

$$\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D\end{bmatrix}\begin{bmatrix}\Pi\\ \Pi_1\end{bmatrix} + \begin{bmatrix}0\\ M^{-1}B\end{bmatrix}L = \begin{bmatrix}\Pi\\ \Pi_1\end{bmatrix}S, \tag{7}$$

if and only if $\Pi_1 = \Pi S$ and Π is the unique solution of (5). Furthermore, there exists $\widetilde{\Upsilon} = [\Upsilon_1\ \Upsilon M]$ satisfying the solution of the Sylevester equation

$$\widetilde{\Upsilon}\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D\end{bmatrix} + R\begin{bmatrix}C_0 & C_1\end{bmatrix} = Q\widetilde{\Upsilon}, \tag{8}$$

if and only if $\Upsilon_1 = Q\Upsilon M + \Upsilon D - RC_1$ and Υ is the unique solution of (6). Hence, Krylov projections can be employed for computing Π and Υ, using first-order approaches as in [9], [22].

B. Moment matching-based second-order reduced systems

Using the characterization of moments in Theorem 1, we now define the families of second-order reduced models achieving moment matching at the given interpolation points. The following results give necessary and sufficient conditions for $\widehat{\mathbf{\Sigma}}$ to achieve moment matching.

Proposition 1. *[18] Consider the reduced model*

$$\widehat{\mathbf{\Sigma}}:\begin{cases} F_2\ddot{\xi}(t)+F_1\dot{\xi}(t)+F_0\xi(t) &= Gu(t),\\ H_1\dot{\xi}(t)+H_0\xi(t) &= \eta(t),\end{cases} \tag{9}$$

with $\xi(t), \dot{\xi}(t) \in \mathbb{R}^{\nu}$, $F_i \in \mathbb{R}^{\nu\times\nu}$, for $i = 0, 1, 2$, and $G \in \mathbb{R}^{\nu}$, $H_1, H_0 \in \mathbb{R}^{1\times\nu}$. Denote the following symmetric set

$$\widehat{\Omega} := \{s \in \mathbb{C} : \det(s^2F_2 + sF_1 + F_0) = 0\}. \tag{10}$$

Let $S \in \mathbb{R}^{\nu\times\nu}$, $L \in \mathbb{R}^{1\times\nu}$ be such that the pair (L, S) is observable. Assume that $\sigma(S)\cap\Omega = \emptyset$ and $\sigma(S)\cap\widehat{\Omega} = \emptyset$. Then, the reduced system $\widehat{\mathbf{\Sigma}}$ matches the moments of $\mathbf{\Sigma}$ at $\sigma(S)$ if and only if $C_0\Pi + C_1\Pi S = H_0P + H_1PS$, where $P \in \mathbb{R}^{\nu\times\nu}$ is the unique solution of the linear matrix equation $F_2PS^2 + F_1PS + F_0P = GL$.

Selecting $P = I_\nu$, we then obtain the family of second-order reduced models

$$\widehat{\mathbf{\Sigma}}_{\mathcal{G}}:\begin{cases} F_2\ddot{\xi}+F_1\dot{\xi}+(GL-F_2S^2-F_1S)\xi &= Gu,\\ H_1\dot{\xi}+(C_0\Pi+C_1\Pi S-H_1S)\xi &= \eta,\end{cases} \tag{11}$$

parameterized in the matrix set $\mathcal{G} = \{F_1, F_2, G, H_1\}$, that matches the moments of $\mathbf{\Sigma}$ at $\sigma(S)$.

C. Zeros of second-order systems

In this section, we recall a notion of zeros for dynamical systems as defined in [21, Chapter 8], see also more general arguments in [23, Chapter 3]. Consider a system $\widehat{\mathbf{\Sigma}}$ defined by (9). We determine conditions such that for the input $u(t) = u_0\,\mathrm{e}^{st}$ and the state evolution $\xi(t) = \xi_0\,\mathrm{e}^{st}$, with $u_0, \xi_0 \neq 0$, the resulting output satisfies $\eta(t) = 0$, for all t.

Substituting u and ξ in $\widehat{\boldsymbol{\Sigma}}$ yields

$$\mathrm{e}^{st}(s^2F_2\xi_0 + sF_1\xi_0 + F_0\xi_0 - Gu_0) = 0,$$
$$\mathrm{e}^{st}(H_1 s + H_0) = 0,$$

written equivalently, in matrix form, as

$$\begin{bmatrix} s^2F_2 + sF_1 + F_0 & -G \\ H_1 s + H_0 & 0 \end{bmatrix} \begin{bmatrix} \xi_o \\ u_0 \end{bmatrix} = 0.$$

Hence, $s = z$ is a zero of $\widehat{\boldsymbol{\Sigma}}$ if

$$\operatorname{rank} \begin{bmatrix} z^2F_2 + zF_1 + F_0 & -G \\ H_1 z + H_0 & 0 \end{bmatrix} < \nu + 1,$$

or, equivalently,

$$\det \begin{bmatrix} z^2F_2 + zF_1 + F_0 & -G \\ H_1 z + H_0 & 0 \end{bmatrix} = 0. \tag{12}$$

D. Problem formulation

We now formulate a moment matching-based model reduction problem for second-order systems.

Problem 1. Consider the second-order system $\boldsymbol{\Sigma}$ as in (1) with the transfer function W and the family of ν order models $\widehat{\boldsymbol{\Sigma}}_{\mathcal{G}}$, as in (11), matching ν moments of W at $\sigma(S)$. Let $\{\lambda_1, \dots, \lambda_\ell\}$ and $\{z_1, \dots, z_k\}$ be two symmetric sets, such that $\{\lambda_1, \dots, \lambda_\ell\} \cap \Omega = \emptyset$, $\{\lambda_1, \dots, \lambda_\ell\} \cap \sigma(S) = \emptyset$ and $\{z_1, \dots, z_k\} \cap \sigma(S) = \emptyset$. Find the parameter matrices of appropriate dimensions, in the set $\mathcal{G}$ such that

i) $\widehat{\boldsymbol{\Sigma}}_{\mathcal{G}}$ has ℓ poles at λ_i $i = 1:\ell$,
ii) $\widehat{\boldsymbol{\Sigma}}_{\mathcal{G}}$ has k zeros at z_j, $j = 1:k$.

III. Pole-zero placement for second-order reduced systems

In this section, we derive linear constraints on the matrix degrees of freedom in the set $\mathcal{G}$, resulting in the linear system yielding the solution to Problem 1.

A. Second-order pole placement

In this section, we overview the results on second-order time-domain moment matching with pole placement, as in [18]. Considering $\boldsymbol{\Sigma}$ in (1) and the family of approximations $\widehat{\boldsymbol{\Sigma}}_{\mathcal{G}}$ as in (11) that matches the moments of $\boldsymbol{\Sigma}$ at $\sigma(S)$ with $S \in \mathbb{R}^{\nu\times\nu}$, the parameter matrices F_1, F_2, G, and H_1 such that $\widehat{\boldsymbol{\Sigma}}_{\mathcal{G}}$ has the poles at prescribed locations $\lambda_1, \lambda_2, ..., \lambda_\ell$, where $\ell \le \nu$, and $\lambda_i \notin \sigma(S) \cup \Omega$ with Ω defined in (3), are computed. Define $Q_{\mathbf{P}} \in \mathbb{R}^{\ell\times\ell}$ such that $\sigma(Q_{\mathbf{P}}) = \{\lambda_1, \lambda_2, ..., \lambda_\ell\}$. Since $\sigma(Q_{\mathbf{P}}) \cap \Omega = \emptyset$, the matrix equation

$$Q_{\mathbf{P}}^2\Upsilon_{\mathbf{P}}M + Q_{\mathbf{P}}\Upsilon_{\mathbf{P}}D + \Upsilon_{\mathbf{P}}K = R_{\mathbf{P}}C_{\mathbf{P}0} + Q_{\mathbf{P}}R_{\mathbf{P}}C_{\mathbf{P}1}. \tag{13}$$

has the unique solution $\Upsilon_{\mathbf{P}} \in \mathbb{R}^{\ell\times n}$, where $R_{\mathbf{P}} \in \mathbb{R}^{\ell\times 1}$ is any matrix such that the pair $(Q_{\mathbf{P}}, R_{\mathbf{P}})$ is controllable, and $C_{\mathbf{P}0}, C_{\mathbf{P}1} \in \mathbb{R}^{1\times n}$ such that $C_{\mathbf{P}0}\Pi = C_{\mathbf{P}1}\Pi = 0$, i.e. $C_{\mathbf{P}0}^T \in \ker(\Pi^T)$ and $C_{\mathbf{P}1}^T \in \ker(\Pi^T)$ with Π the unique solution of (5). The next result provides sufficient conditions given by linear constraints on the free parameters of the reduced model $\widehat{\boldsymbol{\Sigma}}_{\mathcal{G}}$ such that the approximating model $\widehat{\boldsymbol{\Sigma}}_{\mathcal{G}}$ has poles at $\sigma(Q_{\mathbf{P}})$.

Theorem 2. *Consider $\widehat{\boldsymbol{\Sigma}}_{\mathcal{G}}$ in (11) as a reduced model that matches the moments of the system (1) at $\sigma(S)$. Let Π and $\Upsilon_{\mathbf{P}} \in \mathbb{R}^{\ell\times n}$ be the unique solutions of the second-order Sylvester equations in (5) and (13), respectively. Assume that $\operatorname{rank}(\Upsilon_{\mathbf{P}}\Pi) = \ell$. If the following constraints hold*

$$\begin{aligned} \Upsilon_{\mathbf{P}}\Pi F_2 &= \Upsilon_{\mathbf{P}}M\Pi, \\ \Upsilon_{\mathbf{P}}\Pi F_1 &= \Upsilon_{\mathbf{P}}D\Pi, \\ \Upsilon_{\mathbf{P}}\Pi G &= \Upsilon_{\mathbf{P}}B, \end{aligned} \tag{14}$$

then $\sigma(Q_{\mathbf{P}}) = \{\lambda_1, \lambda_2, ..., \lambda_\ell\} \subseteq \widehat{\Omega}$ with $\widehat{\Omega}$ in (10) the set of poles of the reduced model $\widehat{\boldsymbol{\Sigma}}_{\mathcal{G}}$.

Remark 2. Theorem 2 yields the sufficient conditions (14) on the set $\mathcal{G}$ such that $\ell \le \nu$ of the poles of (11) are fixed, when the pair (L, S) is observable and the pair $(Q_{\mathbf{P}}, R_{\mathbf{P}})$ is controllable. Furthermore, if $\ell = \nu$ and $\Upsilon_{\mathbf{P}}\Pi$ is assumed invertible, then $\widehat{\Omega} = \sigma(Q_{\mathbf{P}})$, if and only if

$$F_2 = (\Upsilon_{\mathbf{P}}\Pi)^{-1}\Upsilon_{\mathbf{P}}M\Pi, \quad F_1 = (\Upsilon_{\mathbf{P}}\Pi)^{-1}\Upsilon_{\mathbf{P}}D\Pi,$$
$$G = (\Upsilon_{\mathbf{P}}\Pi)^{-1}\Upsilon_{\mathbf{P}}B.$$

B. Second-order zero placement

Consider a second-order system (1) with the transfer function (2) and the family of ν order models $\widehat{\boldsymbol{\Sigma}}_{\mathcal{G}}$ that match ν moments of (1), for all matrices in $\mathcal{G}$. Let $\{z_1, ..., z_k\} \subset \mathbb{C}$ be a symmetric set of zeros of $W(s)$ as in (2), with $k < \nu$ and $\{z_1, \dots, z_k\} \cap \sigma(S) = \emptyset$. In the sequel, we compute the set $\mathcal{G}$ such that $z_1, ..., z_k$ are zeros of (11). In practice, the zeros of the plant are rendering control design difficult.
By (12), we are looking for matrices in the set $\mathcal{G}$, such that

$$\det \begin{bmatrix} F_2z_i^2 + F_1z_i + (GL - F_2S^2 - F_1S) & -G \\ H_1z_i + (C_0\Pi + C_1\Pi S - H_1S) & 0 \end{bmatrix} = 0,\ i = 1:k.$$

Now let $Q_{\mathbf{Z}} \in \mathbb{R}^{k\times k}$, with $\sigma(Q_{\mathbf{Z}}) = \{z_1, \dots, z_k\}$ and $R_{\mathbf{Z}} \in \mathbb{R}^k$ be such that $(Q_{\mathbf{Z}}, R_{\mathbf{Z}})$ is controllable. Let $\Upsilon_{\mathbf{Z}} \in \mathbb{R}^{k\times n}$ be the unique solution of the Sylvester equation

$$Q_{\mathbf{Z}}^2\Upsilon_{\mathbf{Z}}M + Q_{\mathbf{Z}}\Upsilon_{\mathbf{Z}}D + \Upsilon_{\mathbf{Z}}K = R_{\mathbf{Z}}C_0 + Q_{\mathbf{Z}}R_{\mathbf{Z}}C_1. \tag{15}$$

with $\operatorname{rank}\Upsilon_{\mathbf{Z}} = k$. The moments of W at z_i are given by $\Upsilon_{\mathbf{Z}}B$ [18]. Since $W(z_i) = 0$, then $\Upsilon_{\mathbf{Z}}B = 0$. The next result imposes linear constraints on $\mathcal{G}$ such that the reduced model $\widehat{\boldsymbol{\Sigma}}_{\mathcal{G}}$ has k zeros of $W(s)$ at $\{z_1, \dots, z_k\}$.

Theorem 3. *Consider $\widehat{\boldsymbol{\Sigma}}_{\mathcal{G}}$ in (11) a system of order ν matching the moments of (1) at $\sigma(S)$. Consider the matrix $Q_{\mathbf{Z}} \in \mathbb{R}^{k\times k}$ with $\sigma(Q_{\mathbf{Z}}) = \{z_1, \dots, z_k\}$, a symmetric set and let $R_{\mathbf{Z}} \in \mathbb{R}^k$ be such that the pair $(Q_{\mathbf{Z}}, R_{\mathbf{Z}})$ is controllable. Let Π and $\Upsilon_{\mathbf{Z}} \in \mathbb{R}^{k\times n}$ be the unique solutions of the matrix equations in (5) and (15), respectively. Assume that $\operatorname{rank}(\Upsilon_{\mathbf{Z}}\Pi) = k$. If the following constraints hold*

$$H_1 = C_1\Pi, \tag{16a}$$
$$\Upsilon_{\mathbf{Z}}\Pi F_2 = -\Upsilon_{\mathbf{Z}}M\Pi, \tag{16b}$$
$$\Upsilon_{\mathbf{Z}}\Pi F_1 = -\Upsilon_{\mathbf{Z}}D\Pi, \tag{16c}$$
$$\Upsilon_{\mathbf{Z}}\Pi G = 0, \tag{16d}$$

then $\{z_1, \dots, z_k\} = \sigma(Q_{\mathbf{Z}})$ are zeros of the system $\widehat{\boldsymbol{\Sigma}}_{\mathcal{G}}$.

C. A solution to Problem 1

Let $\widehat{\mathbf{\Sigma}}_{\mathcal{G}}$, as in (11), define a family of ν order models that match ν moments of (1) at $\{s_1, \ldots, s_\nu\}$, parameterized in the set of matrices $\mathcal{G} = \{F_1, F_2, G, H_1\}$ of appropriate dimensions. Let $\{\lambda_1, \ldots, \lambda_\ell\}$ and $\{z_1, \ldots, z_k\}$ be symmetric sets (including multiplicities), such that $\{\lambda_1, \ldots, \lambda_\ell\} \cap \Omega = \emptyset$, $\{\lambda_1, \ldots, \lambda_\ell\} \cap \sigma(S) = \emptyset$ and $\{z_1, \ldots, z_k\} \cap \sigma(S) = \emptyset$, $\ell + k \leq \nu$. To write the solution of Problem 1 as a linear system, we collect the constraints (14) and (16) yielding the system of matrix equations in the unknowns F_1, F_2, G, H_1.

Corollary 1. *Let $\widehat{\mathbf{\Sigma}}_{\mathcal{G}}$, as in* (11)*, define a family of ν order models that match ν moments of* (1) *at $\{s_1, \ldots, s_\nu\}$. Let Π be the solution of the matrix equation* (5)*, $\Upsilon_{\mathbf{P}}$ be the solution of the matrix equation* (13) *and $\Upsilon_{\mathbf{Z}}$ the solution of the matrix equation* (15)*. Denote by $\Upsilon = \begin{bmatrix} \Upsilon_{\mathbf{P}}^T & \Upsilon_{\mathbf{Z}}^T \end{bmatrix}^T \in \mathbb{R}^{\nu \times n}$. Furthermore, let $\{\lambda_1, \ldots, \lambda_\ell\}$ and $\{z_1, \ldots, z_k\}$ be symmetric sets (including multiplicities), such that $\{\lambda_1, \ldots, \lambda_\ell\} \cap \Omega = \emptyset$, $\{\lambda_1, \ldots, \lambda_\ell\} \cap \sigma(S) = \emptyset$ and $\{z_1, \ldots, z_k\} \cap \sigma(S) = \emptyset$, $\ell + k \leq \nu$. Assuming* (16a) *holds and if*

$$\Upsilon \Pi F_2 = \begin{bmatrix} (\Upsilon_{\mathbf{P}} M \Pi)^T & -(\Upsilon_{\mathbf{Z}} M \Pi)^T \end{bmatrix}^T, \tag{17a}$$

$$\Upsilon \Pi F_1 = \begin{bmatrix} (\Upsilon_{\mathbf{P}} D \Pi)^T & -(\Upsilon_{\mathbf{Z}} D \Pi)^T \end{bmatrix}^T, \tag{17b}$$

$$\Upsilon \Pi G = \begin{bmatrix} (\Upsilon_{\mathbf{P}} B)^T & 0 \end{bmatrix}^T, \tag{17c}$$

then $\{\lambda_1, \ldots, \lambda_\ell\}$ are poles and $\{z_1, \ldots, z_k\}$ are zeros of $\widehat{\mathbf{\Sigma}}_{\mathcal{G}}$ as in (11)*, respectively.*

Corollary 1 yields the sufficient conditions (17) on $\mathcal{G}$ such that $\ell + k \leq \nu$ of the poles and zeros of (11) are fixed, when S, L are arbitrary matrices such that the pair (L, S) is observable. In practice, for the reduced order modeling of physical systems, additional structure on the dynamics of the reduced order model is imposed. Hence, note that in Corollary 1, if $\ell + k < \nu$, then the sufficient conditions expressed through the linear systems (17) have an infinite number of solutions, respectively. Furthermore, if $\ell + k = \nu$ and $\Upsilon \Pi$ is invertible, then

$$F_2 = (\Upsilon \Pi)^{-1} \begin{bmatrix} \Upsilon_{\mathbf{P}} M \Pi \\ -\Upsilon_{\mathbf{Z}} M \Pi \end{bmatrix}, \; F_1 = (\Upsilon \Pi)^{-1} \begin{bmatrix} \Upsilon_{\mathbf{P}} D \Pi \\ -\Upsilon_{\mathbf{Z}} D \Pi \end{bmatrix}, \tag{18a}$$

$$G = (\Upsilon \Pi)^{-1} \begin{bmatrix} \Upsilon_{\mathbf{P}} B \\ 0 \end{bmatrix}, \tag{18b}$$

provides the unique model (11) having the poles $\lambda_i, i = 1 : \ell$ and the zeros $z_j, j = 1 : k$.

Algorithm 1. Consider a second-order system described by (1) and choose $\nu < n \in \mathbb{N}$.

1) Consider the symmetric sets $\{s_i \in \mathbb{C} \setminus \sigma(A) \mid i = 1 : \nu\}$, $\{\lambda_1, \ldots, \lambda_\ell\} \subset \mathbb{C}$, $s_j \neq \lambda_j, j = 1 : \ell$ and $\{z_1, \ldots, z_k\} \subset \mathbb{C}$, $\ell + k \leq \nu$.
2) Let $S \in \mathbb{R}^{\nu \times \nu}$, such that $\sigma(S) = \{s_i \mid i = 1 : \nu\}$. Pick $L \in \mathbb{R}^{1 \times \nu}$ such that (L, S) is observable.
3) Compute $\Pi \in \mathbb{R}^{n \times \nu}$, the solution of (5).
4) Compute $\Upsilon_{\mathbf{P}}$ and $\Upsilon_{\mathbf{Z}}$, the solutions of (13) and (15), respectively.
5) Compute $\mathcal{G}$, using (18) and substitute in $\widehat{\mathbf{\Sigma}}_{\mathcal{G}}$ as in (11).

IV. Second-order moment matching model reduction for vibratory systems

In this section, we illustrate the theory on two vibratory systems (e.g., a chain of mechanical oscillators) of degree n governed by the second-order dynamical model (1), as in e.g., [20] (see also [24]).

A. Damped vibrating system

We first apply the results to the damped vibratory case described by the following relations [20]

$$\begin{aligned} M &= \operatorname{diag}(m_1, \ldots, m_n) \succ 0, \\ D &= \begin{bmatrix} c_1 & -c_1 & & & \\ -c_1 & c_1 + c_2 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -c_{n-1} \\ & & & -c_{n-1} & c_{n-1} + c_n \end{bmatrix} = D^T, \\ K &= \begin{bmatrix} k_1 & -k_1 & & & \\ -k_1 & k_1 + k_2 & -k_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -k_{n-1} \\ & & & -k_{n-1} & k_{n-1} + k_n \end{bmatrix} = K^T, \\ B^T &= C_0 = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}, \; C_1 = 0, \end{aligned} \tag{21}$$

with masses m_i, stiffness constants k_i and damping factors c_i, $i = 1 : n$. The input u is a force applied to mass m_1, whereas the output y measures the displacement of mass m_1. We select $n = 10$, with the masses, the stiffness and damping constants computed as in [20], [19],

$$\begin{aligned} m_n &= 1, \\ m_{n-i} &= 2 \frac{(n+i-1)!(n-i-1)!}{(n-1)!^2}, i = 1 : n-1, \\ c_i &= 1, i = 1 : n, \\ K_{ii} &= i, i = 1 : n, \\ K_{i,i+1} &= -\frac{1}{2} \sqrt{i(2n-i-1)}, i = 1 : n-2, \\ K_{n-1,n} &= -\frac{1}{\sqrt{2}} \sqrt{n(n-1)}. \end{aligned} \tag{22}$$

We approximate this system with a second-order system $\widehat{\mathbf{\Sigma}}_{\mathcal{G}}$ as in (11) of order $\nu = 6$. Hence, selecting the interpolation points $\{0, 0, \pm j, \pm 5j\}$, we define

$$\begin{aligned} S &= \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} \right), \\ L &= \begin{bmatrix} 1 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} \end{bmatrix}, \end{aligned}$$

such that the pair (L, S) is observable. Solving the Sylvester equation (7), yields the unique solution (19) of the matrix equation (5), with $\operatorname{rank} \Pi = 6$. A family of second-order models of order $\nu = 6$ matching the moments $C_0 \Pi = [0.284 \;\; -0.010 \; 11.765 \; 0.0019 \; 0.015 - 0.119]$ of (21), with given data in (22), is described by (11) parameterized in

$$\Pi = \begin{bmatrix} 1.08e-06 & -3.08e-08 & 0.0002 & 5.40e-07 & -1.5e-09 & 3.75e-09 \\ 2.17e-05 & -6.14e-07 & 0.004 & 9.72e-06 & 4.48e-08 & -1.13e-07 \\ 0.0002 & -5.87e-06 & 0.04 & 8.2e-05 & -6.59e-07 & 1.66e-06 \\ 0.001 & -3.58e-05 & 0.2 & 0.0004 & 6.32e-06 & -1.59e-05 \\ 0.005 & -0.0002 & 0.74 & 0.001 & -4.45e-05 & 0.0001 \\ 0.02 & -0.0005 & 2.0733 & 0.003 & 0.0002 & -0.0006 \\ 0.05 & -0.002 & 4.49 & 0.002 & -0.001 & 0.003 \\ 0.10 & -0.003 & 7.7 & -0.004 & 0.004 & -0.01 \\ 0.19 & -0.007 & 10.59 & -0.01 & -0.009 & 0.04 \\ 0.28 & -0.01 & 11.77 & 0.002 & 0.02 & -0.12 \end{bmatrix}, \quad (19)$$

$$\Upsilon\Pi = \begin{bmatrix} -8.11e-05 & 2.82e-06 & -0.004 & 2.54e-06 & -7.13e-07 & 9.46e-06 \\ -0.002 & 5.94e-05 & -0.07 & 3.11e-05 & -4.43e-05 & 0.0004 \\ 0.0001 & -4.42e-06 & 0.008 & -8.45e-06 & -2.86e-06 & 1.36e-05 \\ 0.004 & -0.0002 & 0.15 & -2.58e-05 & 0.0002 & -0.002 \\ 0.0006 & -1.99e-05 & 0.032 & -4.36e-05 & -1.96e-05 & 9.25e-05 \\ -0.04 & 0.002 & -2.85 & 0.002 & 0.0008 & -0.004 \end{bmatrix} \quad (20)$$

F_1, F_2, G. We now consider $\ell = 4$ and $k = \nu - \ell = 2$ and $\{\lambda_{1,2}, \lambda_{3,4}\} = \{-0.009 \pm 1.9j, -0.01 \pm 2.4j\}$ and $z_{1,2} = -0.004 \pm 2j$. We construct the matrices

$$Q_{\mathbf{P}} = \text{diag}\left(\begin{bmatrix} -0.009 & 1.9 \\ -1.9 & -0.009 \end{bmatrix}, \begin{bmatrix} -0.01 & 2.4 \\ -2.4 & -0.01 \end{bmatrix}\right),$$
$$Q_{\mathbf{Z}} = \begin{bmatrix} -0.004 & 2 \\ -2 & -0.004 \end{bmatrix}, \ R_{\mathbf{P}} = [0 \ \sqrt{2} \ 0 \ \sqrt{2}]^T, \ R_{\mathbf{Z}} = [0 \ \sqrt{2}]^T,$$

such that $(Q_{\mathbf{P}}, R_{\mathbf{P}})$ is controllable and $(Q_{\mathbf{Z}}.R_{\mathbf{Z}})$ is controllable. Solving the matrix equations (13) and (15), respectively, yield the solutions $\Upsilon_{\mathbf{P}}$ and $\Upsilon_{\mathbf{Z}}$, respectively. Furthermore, $\text{rank}\,\Upsilon_{\mathbf{P}} = 4$ and $\text{rank}\,\Upsilon_{\mathbf{Z}} = 2$. We now build $\Upsilon = [\Upsilon_{\mathbf{P}}^T \Upsilon_{\mathbf{Z}}^T]^T$. The computations yield $\Upsilon\Pi$ as in (20), invertible. Hence, applying Corollary 1, the second-order model (11) that matches the moments of (21) at $\sigma(S)$ and has four poles in $\lambda_i, i = 1:4$ and two zeros in $z_j, j = 1:2$ is given by F_1, F_2, G as in (18),

$$F_1 = \begin{bmatrix} -691.02 & 26.495 & -12096 & 87.959 & 92.037 & -396.39 \\ -16310 & 625.11 & -288270 & 2098.3 & 2197 & -9516.1 \\ 2.1082 & -0.080938 & 35.731 & -0.25902 & -0.27044 & 1.1422 \\ -207.54 & 7.9763 & -3410.9 & 24.33 & 25.328 & -104.13 \\ -573.75 & 22.221 & -7575.8 & 53.414 & 54.504 & -187.47 \\ -128.75 & 4.9742 & -1830 & 12.896 & 13.247 & -48.679 \end{bmatrix},$$
$$F_2 = \begin{bmatrix} 66133 & -2081.3 & 5.2163e+06 & -1561.7 & 433.17 & -1637.7 \\ 1558300 & -49029 & 1.2304e+08 & -36457 & 10351 & -39278 \\ -202.82 & 6.3892 & -15943 & 4.9372 & -1.2684 & 4.7352 \\ 20114 & -634.18 & 1.5771e+06 & -493.15 & 118 & -433.15 \\ 56723 & -1798 & 4.3537e+06 & -1690.4 & 248.38 & -808.87 \\ 12631 & -399.74 & 9.7581e+05 & -356.28 & 60.872 & -207.03 \end{bmatrix},$$
$$G = \begin{bmatrix} -4.3361 & -165.44 & -0.010469 & 4.2793 & -33.762 & -9.9479 \end{bmatrix}^T.$$

Note that the second-order structure is preserved, even if the matrices F_1, F_2 are not necessarily "mass" and "damping" matrices. The transfer function of the second-order reduced approximation is a complex rational function of order 12, with the poles $\{-0.11 \pm 4.226j, -0.059 \pm 3.024j, -0.01 \pm 2.4j, -0.009 \pm 1.9j, -0.008 \pm 1.753j, -0.011 \pm j\}$ and the zeros $\{0.073 \pm 3.657j, -0.021 \pm 2.684j, -0.004 \pm 2j, -0.007 \pm 1.959j, -0.003 \pm 1.419j\}$. Note that the approximation has the imposed poles and zeros.
Figure 1 presents the step response of the original tenth order vibrating system, closely matched by the step response of the sixth order approximation. Figure 2 shows that the magnitude plot of the approximation closely follows the magnitude plot of the original system. Finally, note that the resulting approximant is stable and minimum phase similar to the original system.

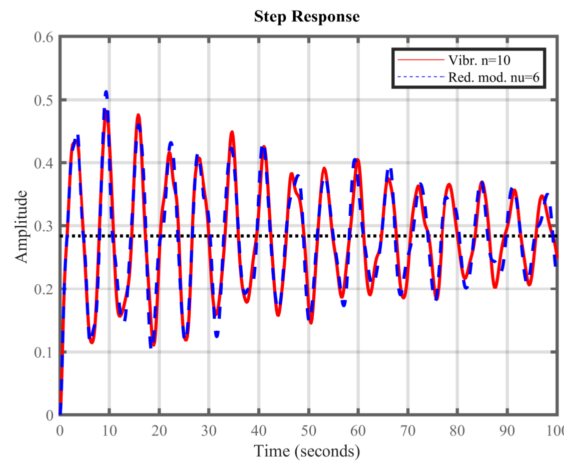

Fig. 1. Step response of the vibrating system (21) with (22), $n = 10$ (solid line) v. the second-order approximation of order $\nu = 6$ matching six moments at $\sigma(S)$ and having $\ell = 4$ poles and $k = 2$ zeros imposed

B. The undamped case

We consider an undamped vibratory system of degree n governed by the second-order dynamical model (1), as in [24], [19] with $D = 0$ in (21). In this case, the transfer function of (21) is $W(s)$ as in (2) such that

$$W(j\omega) = \frac{\det(\widehat{K} - \omega^2 \widehat{M})}{\det(K - \omega^2 M)}, \quad (23)$$

where $\widehat{K}$ and $\widehat{M}$ are obtained by removing the last line and the last column in K and M, respectively. The poles of (23) are $\{p_i, \overline{p_i}\} = \{\pm j\omega_i\}, i = 1:n$. The zeros of (23) are $\{z_j, \overline{z_j}\} = \{\pm j\mu_j\}, j = 1:n{-}1$. For this undamped second-order vibrating systems, equations (5) and (6) become

$$M\Pi S^2 + K\Pi = BL, \quad (24)$$
$$Q_{\mathbf{P}}^2 \Upsilon_{\mathbf{P}} M + \Upsilon_{\mathbf{P}} K = RC_0,$$

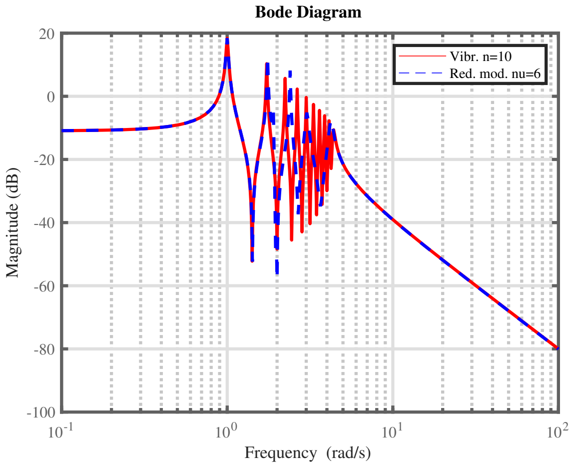

Fig. 2. Magnitude plot of the vibrating system (21) with (22), $n = 10$ (solid line) v. the second-order approximation of order $\nu = 6$ matching six moments at $\sigma(S)$ and having $\ell = 4$ poles and $k = 2$ zeros imposed

equivalent to

$$\Pi S^2 + M^{-1}K\Pi = M^{-1}BL,$$
$$Q_{\mathbf{P}}^2\Upsilon_{\mathbf{P}} + \Upsilon_{\mathbf{P}}KM^{-1} = RC_0M^{-1},$$

provided that M is invertible. Note that one does not have to match moments at complex-conjugated frequencies as, up to a coordinate transformation $S^2 = \operatorname{diag}(-\varpi_i^2, -\varpi_i^2)$ is derogatory and in this case, the solution Π in (24) would have $\operatorname{rank}\Pi < \nu$. For approximating vibrating systems the ℓ imposed poles and the k imposed zeros have to be in complex-conjugated pairs, respectively. Then, e.g.,

$$Q_{\mathbf{P}} = \operatorname{diag}(Q_i),\ Q_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix},\ i = 1 : \frac{\ell}{2}.$$

Therefore $Q_{\mathbf{P}}^2 = \operatorname{diag}(-\omega_i^2, -\omega_i^2)$, derogatory, implying the solution $\Upsilon_{\mathbf{P}}$ in (24) satisfies $\operatorname{rank}\Upsilon_{\mathbf{P}} < \ell$. Hence, the matrix $\Upsilon\Pi$ in Corollary 1 is such that $\operatorname{rank}\Upsilon\Pi < \nu$ and the results in (18) cannot be applied to this particular case of second-order systems.

V. Conclusions

In this paper, we have addressed a time-domain moment matching with pole-zero placement, structure-preserving model reduction problem for second-order systems. The definition of moments of second-order systems, based on the solutions of linear matrix equations, has been employed, leading to families of second-order reduced models. The models are parameterized in a set of matrices and all match the moments of a given second-order system at selected interpolation points. We have provided formulae for the set of matrices such that the reduced order approximation has a set of prescribed poles/zeros. As an illustrative numerical example, a second-order, damped vibratory system, described as a chain of damped mechanical oscillators, has been presented. The limitations of the method have been exposed in the undamped case, where derogatory matrices appear.

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