2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Data-Driven Finite-Time Control for Discrete-Time Linear Time-Invariant Systems

Jinjiang Li, Tao Liu, and Tengfei Liu

Abstract— **This paper studies the data-driven finite-time control (FTC) problem of unknown discrete-time linear time-invariant (LTI) systems with unknown and bounded noise. The proposed FTC aims to guarantee that the state of such a system does not exceed a given bound over a finite time interval under bounded initial conditions. Data-dependent representations are built for the unknown system without and with noise from pre-collected input/state data, based on which a finite-time controller is designed. Sufficient conditions of finite-time stability/boundness of the closed-loop system without/with noise are derived. Compared with model-based FTC methods that strongly depend on some accurate system models, the proposed method is model-free and only relies on pre-collected data. Numerical simulations are performed to illustrate the effectiveness of the proposed scheme.**

I. Introduction

Data-driven control technique bypasses system identification and enables direct learning of controllers from data. Thus, it offers a one-shot design paradigm for control engineers and has gained widespread attention recently.

Among the fruitful literature, Willems' fundamental lemma is one of the most notable works, which states that if the collected input signals of a linear time-invariant (LTI) system are sufficient, i.e., satisfy the persistent excitation (PE) condition, then a nonparametric model for the system can be established by a Hankel matrix constructed from the data [1], [2]. Based on this fundamental lemma, a data-based closed-loop representation for systems with state-feedback control was derived in [3], where the control design problem was transformed into a feasibility problem of data-based linear matrix inequalities (LMIs). To tackle the issue of the large decision variable number in the formulated LMI problem, a noisy data-based quadratic matrix inequality (QMI) method was proposed in [4] to describe all the potential systems and design a state feedback controller. Following the design philosophy in [4], a data-driven controller for linear parameter-varying systems was designed in [5]. A data-driven H_∞ controller for LTI systems with disturbances and measurement noises was developed in [6].

*The work was supported by the National Natural Science Foundation of China through Projects 62173287, 62325303, 62333004 and the Early Career Scheme of the Hong Kong SAR Research Grants Council through Project 27206021.

Jinjiang Li and Tao Liu are with the Department of Electrical and Electronic Engineering, The University of Hong Kong (HKU), Hong Kong SAR, and the HKU Shenzhen Institute of Research and Innovation, Shenzhen, China. (e-mail: jjiangli@eee.hku.hk; taoliu@eee.hku.hk).

Tengfei Liu is with the State Key Laboratory of Synthetical Automation for Process Industries, Northeastern University, Shenyang, China (email: tfliu@mail.neu.edu.cn).

The aforementioned works mainly focus on the steady-state behavior of linear systems but pay no attention to transient behaviors, such as the maximum overshoots or undershoots, which are also very important from a practical perspective. For instance, the roll angle of the ship has to be constrained as a large roll angle can cause discomfort to the crew. Similarly, the speed of a fixed-wing aircraft must be within a specific upper and lower limit [7]. To tackle these problems, two extensively used approaches have been developed. One is the prescribed performance control (PPC) method [8], where most of the existing results are developed for continuous-time systems, and only a few results are established for discrete-time single-input and single-output (SISO) systems [9]. The other one is finite-time control (FTC), which can ensure the state of a discrete-time system within a certain bound in a finite time interval [10]. So far, FTC has been applied to various types of discrete-time systems, whereas PPC has only been applied to SISO systems. In view of the fact that FTC is more universal than PPC in terms of handling discrete-time systems control problems, this paper will use FTC to solve our problem.

FTC is closely related to finite-time stability theory that can be traced back to the 1960s [10]. Since then, many elegant results have emerged for different types of discrete-time systems. Sufficient conditions for achieving FTC of LTI systems via state feedback were proposed in [11], which was further extended to systems with uncertainty in [12]. In [13], an FTC scheme was proposed for systems with conic-type nonlinearities and additive disturbances. In [14], a fuzzy FTC method was developed for nonlinear singularly perturbed systems. However, many existing works on FTC are model-based and strongly rely on an accurate system model, which is usually unavailable in practice.

To solve the abovementioned issues, we propose a data-driven state feedback FTC method for LTI systems. By assuming the availability of noise-free input/state data that satisfies the PE condition, sufficient conditions for finite-time stabilization via state feedback control are established by using the data-based closed-loop representation. Then, the case that the pre-collected data is corrupted by noise is considered, where the data-dependent QMI is used to characterize all possible systems that are compatible with collected data. Sufficient conditions for noisy data-based state feedback controller design are established by utilizing the matrix S-lemma. Unlike conventional FTC methods that depend on the explicit system model, the proposed method directly uses pre-collected data for the controller design.

The rest of this paper is structured as follows. The system to be studied and some necessary preliminaries are stated in Section II. Sufficient conditions for data-driven FTC with noise-free and noisy data are presented in Section III and Section IV, respectively. The effectiveness of the proposed methods is illustrated by numerical examples in Section V. Section VI concludes the paper.

Notation: Let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}^+$, and $\mathbb{N}$ denote the sets of real numbers, integers, positive integers, and natural numbers, respectively. Let $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ be the n-dimensional real vector space, and set of $n\times m$ real matrices, respectively. I_n is the $n\times n$ identity matrix, $0_{n\times m}$ is the $n\times m$ zero matrix with $n,m\in\mathbb{Z}^+$, and $\text{diag}\{M_1,\ldots,M_n\}$ is block diagonal matrix with digaonal entry $M_i\in$, $i=1,\ldots,n$. The symbol * in a symmetric block matrix represents the symmetric part of the matrix. $z_{[k,k+T]}$ denotes the sequence $\{z(k),z(k+1),\ldots,z(k+T)\}$ with $k\in\mathbb{Z},T\in\mathbb{Z}^+$.

II. Problem formulation

Consider a discrete-time LTI system

$$x(k+1)=Ax(k)+Bu(k)+w(k) \tag{1}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$, and $w(k)\in\mathbb{R}^n$ are the state, input, and unknown noise, respectively. $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times m}$ are the system matrices.

Assume that we have pre-collected the system's input/state trajectory $(x_{d,[0,T]},u_{d,[0,T-1]})$, $T\in\mathbb{Z}^+$. Define the input sequence U_-, state sequences X_- and X_+, and unknown noise sequence W_- as follows:

$$\begin{aligned}U_-&=\begin{bmatrix}u_d(0) & u_d(1) & \cdots & u_d(T-1)\end{bmatrix},\\ X_-&=\begin{bmatrix}x_d(0) & x_d(1) & \cdots & x_d(T-1)\end{bmatrix},\\ X_+&=\begin{bmatrix}x_d(1) & x_d(2) & \cdots & x_d(T)\end{bmatrix},\\ W_-&=\begin{bmatrix}w(0) & w(1) & \cdots & w(T-1)\end{bmatrix}.\end{aligned} \tag{2}$$

Without loss of generality, we make the following assumptions for the system (1) and pre-collected data (2).

Assumption 1: The system matrices A and B are unknown, and (A,B) is controllable.

Assumption 2: The noise $w(k)$ is norm-bounded satisfying $w(k)^\top w(k)\leqslant\delta_w^2$ with δ_w being a known positive scalar.

Assumption 3 (PE condition): The input sequency $u_{d,[0,T-1]}$ is persistently exciting of order $n+1$.

Remark 1: These assumptions are extensively used in the literature. For Assumption 1, the controllability of the unknown pair (A,B) can be checked by performing a data-driven Hautus test with data sequences U_-, X_-, and X_+ [15]. For Assumption 2, noises in practice are usually bounded. For Assumption 3, the PE condition of the input signal $u_{d,[0,T-1]}$ can be checked by the rank of the corresponding Hankel matrix, i.e, $u_{d,[0,T-1]}$ is persistently exciting of order $n+1$ if its Hankel matrix has full row rank $m(n+1)$.

Definition 1 (Finite-Time Stability [11]): Given two positive constants δ_x and ϵ satisfying $0<\delta_x<\epsilon$, a positive integer $N\in\mathbb{Z}^+$, and a positive-definite matrix R, the LTI system (1) with $u(k)=0$ and $w(k)=0$ is said to be finite-time stable with respect to (δ_x,ϵ,R,N), if $x^T(0)Rx(0)\leqslant\delta_x^2$ implies $x^T(k)Rx(k)<\epsilon^2,\forall k\in\{1,\ldots,N\}$.

Definition 2 (Finite-Time Boundedness [11]): Given three constants δ_x,δ_w and ϵ satisfying $0<\delta_x<\epsilon$ and $\delta_w\geqslant 0$, a positive integer $N\in\mathbb{Z}^+$, and a positive-definite matrix R, the LTI system (1) with $u(k)=0$ is said to be finite-time bounded with respect to $(\delta_x,\delta_w,\epsilon,R,N)$, if $x^T(0)Rx(0)\leqslant\delta_x^2$ and $w^T(0)w(0)\leqslant\delta_w^2$ imply $x^T(k)Rx(k)<\epsilon^2,\forall k\in\{1,\ldots,N\}$.

Remark 2: The concept of finite-time stability studied in this paper is distinct from those commonly discussed in the literature (e.g., [16]). The former is to ensure that the state of the system remains within predetermined bounds within a finite time interval, whereas the latter focuses on the finite settling time of the system.

This paper aims to design a data-driven based state feedback controller such that the closed-loop system of (1) is finite-time stable if $w(k)=0$ and finite-time bounded if $w(k)$ is bounded. These problems are formulated as follows.

Problem 1: When $w(k)=0$, design a noise-free data-based state feedback controller $u=Kx(k)$ with control gain $K\in\mathbb{R}^{m\times n}$ such that the closed-loop system is finite-time stable with respect to (δ_x,ϵ,R,N). When $w(k)\neq 0$, design a noisy data-based state feedback controller $u=Kx(k)$ such that the closed-loop system is finite-time bounded with respect to $(\delta_x,\delta_w,\epsilon,R,N)$.

The following lemmas will be used to facilitate the data-driven design procedure.

Lemma 1 (Schur complement lemma [17]): Consider two symmetric nonsingular matrices $F_{11}\in\mathbb{R}^{n\times n},F_{22}\in\mathbb{R}^{m\times m}$, and a matrix $F_{12}\in\mathbb{R}^{n\times m}$. The following three statements are equivalent

(i) $\begin{bmatrix}F_{11} & F_{12}\\ * & F_{22}\end{bmatrix}\geqslant 0$;

(ii) $F_{11}>0$ and $F_{22}-F_{12}^\top F_{11}^{-1}F_{12}\geqslant 0$;

(iii) $F_{22}>0$ and $F_{11}-F_{12}F_{22}^{-1}F_{12}^\top\geqslant 0$.

Moreover, the equivalence still holds when the above inequalities are all strict inequalities.

Lemma 2 (Data-based closed-loop representation [3]): Consider the system (1) with $w(k)=0$. Suppose that Assumption 3 holds. Then, the following rank condition is satisfied,

$$\text{rank}\begin{bmatrix}U_-\\ X_-\end{bmatrix}=n+m. \tag{3}$$

Then the system (1) under the state feedback controller $u=Kx(k)$ can be equivalently represented as

$$x(k+1)=(A+BK)x(k)=X_+G_Kx(k), \tag{4}$$

where $G_K\in\mathbb{R}^{T\times n}$ satisfies

$$\begin{bmatrix}K\\ I_n\end{bmatrix}=\begin{bmatrix}U_-\\ X_-\end{bmatrix}G_K. \tag{5}$$

Lemma 3 (Matrix S-lemma [4]): Consider two $(k+n)\times(k+n)-$dimensional symmetric matrices

$$M=\left[\begin{array}{c:c}M_{11} & M_{12}\\ \hdashline * & M_{22}\end{array}\right],H\quad=\left[\begin{array}{c:c}H_{11} & H_{12}\\ \hdashline * & H_{22}\end{array}\right].$$

Suppose that $M_{22}\leqslant 0,H_{22}\leqslant 0$, $\ker H_{22}\subseteq\ker H_{12}$, and there exists some matrix Z satisfying the Slater condition, or equivalently there exists some $Z \in \mathcal{S}$ with

$$\mathcal{S} = \left\{ Z \in \mathbb{R}^{k \times n} | \begin{bmatrix} I_n \\ Z \end{bmatrix}^\top H \begin{bmatrix} I_n \\ Z \end{bmatrix} > 0 \right\}.$$

Then, the following inequality holds

$$\begin{bmatrix} I_n \\ Z \end{bmatrix}^\top M \begin{bmatrix} I_n \\ Z \end{bmatrix} > 0, \forall Z \in \mathcal{S},$$

if and only if there exist two scalars $\alpha \geqslant 0$ and $\beta > 0$ such that

$$M - \alpha H \geqslant \begin{bmatrix} \beta I_n & 0_{n \times k} \\ 0_{k \times n} & 0_{k \times k} \end{bmatrix}.$$

III. Data-driven FTC with noise-free data

In this section, we will present a sufficient condition for finite-time stabilization of the discrete-time LTI system (1) via state feedback control with noise-free data. The main result is stated as follows:

Theorem 1: Consider the system (1) with $w(k) = 0$. Suppose that Assumption 1 and Assumption 3 hold. Given two constants δ_x and ϵ satisfying $0 < \delta_x < \epsilon$ and $x^T(0)Rx(0) \leqslant \delta_x^2$, an integer $N \in \mathbb{Z}^+$, and a matrix $R > 0$, the system under the controller $u = Kx(k)$ with $K = U_- Q \left(X_- Q\right)^{-1}$ is finite-time stable with respect to $(\delta_x, \epsilon, R, N)$ if there exist a scalar $\gamma \geqslant 1$, and a matrix Q such that

$$\begin{bmatrix} \gamma X_- Q & X_+ Q \\ * & X_- Q \end{bmatrix} > 0, \forall \gamma \geqslant 1, \tag{6a}$$

$$\lambda_{\max}\left(\tilde{P}\right)\delta_x^2 < \frac{\epsilon^2 \lambda_{\min}\left(\tilde{P}\right)}{\gamma^N}, \tag{6b}$$

hold with $\tilde{P} = R^{-1/2} X_- Q R^{-1/2}$.

Proof: Select the Lyapunov funciton candidate $V(x(k)) = x^T(k)Px(k)$ with $P > 0$. According to [11], the finite-time stability of the closed-loop system of the system (1) under state feedback controller $u = Kx(k)$ is achieved if $V(x(k+1)) < \gamma V(x(k))$ holds, i.e.,

$$(A + BK)P(A + BK)^\top - \gamma P < 0. \tag{7}$$

With Assumption 3 and Lemma 2, the inequality (7) can be represented as the following data-based inequality

$$X_+ G_K P G_K^\top X_+^\top - \gamma P < 0. \tag{8}$$

Define $Q = G_K P$. Multiplying both sides of (5) by P give $X_- Q = P$ and $U_- Q = KP$. Then, inequality (8) can be rewritten as

$$X_+ Q (X_- Q)^{-1} Q^\top X_+^\top - \gamma X_- Q < 0. \tag{9}$$

Applying Lemma 1 to (6a) gives (9). From $\tilde{P} = R^{-1/2} X_- Q R^{-1/2} = R^{-1/2} P R^{-1/2}$, we have

$$V(x(k)) = x(k)^\top R^{\frac{1}{2}} \tilde{P} R^{\frac{1}{2}} x(k) \geqslant \lambda_{\min}\left(\tilde{P}\right) x^\top(k) R x(k). \tag{10}$$

On the other hand, we have

$$V(x(k)) < \gamma V(x(k-1)) < \gamma^k V(x(0)) \leqslant \gamma^N \lambda_{\max}(\tilde{P}) \delta_x^2. \tag{11}$$

Combing (6b), (10) and (11) gives

$$x^T(k)Rx(k) < \frac{\lambda_{\max}\left(\tilde{P}\right)\delta_x^2 \gamma^N}{\lambda_{\min}\left(\tilde{P}\right)} < \epsilon^2, \forall k \in \{1, \dots, N\}, \tag{12}$$

which implies that the closed-loop system is finite-time stable with respect to $(\delta_x, \epsilon, R, N)$ in the absence of noise. This concludes the proof.

Remark 3: It is worth pointing out that when $\gamma = 1$, the system (1) is both finite-time stable and asymptotically stable. However, finite-time stability and Lyapunov stability are two independent concepts. Lyapunov stability is a qualitative concept focusing on the system behavior within a sufficiently long, in principle infinite, time interval, whereas finite-time stability is a quantitative concept that concerns the boundedness of the state of a system over a finite time interval for given initial conditions. Generally, a finite-time stable system may not be asymptotic stable, and vice versa.

Remark 4: Condition (6b) is a nonlinear condition with matrix eigenvalues. When designing the controller, we can replace (6b) with the following LMIs

$$R < X_- Q < \frac{\epsilon^2}{\gamma^N \delta_x^2} R. \tag{13}$$

It is easy to verify that (13) ensures $1 < \lambda_{\min}(\tilde{P}) < \lambda_{\max}(\tilde{P}) < \epsilon^2/(\gamma^N \delta_x^2)$, which further guarantees (6b). Therefore, as long as the LMI problems (6a) and (13) are feasible, we can find a finite-time controller such that the closed-loop system is finite-time stable.

Remark 5: Compared with the model-based methods in [11], [18], the presented work provides a sufficient condition for achieving finite-time stability based only on pre-collected measured data.

IV. Data-driven FTC with noisy data

This section will present a data-driven approach for finite-time boundedness of the system (1) with bounded and unknown noise.

In the presence of unknown noise $w(k)$, the collected data sequences are noisy data. Substituting (2) into (1) gives

$$W_- = X_+ - AX_- - BU_-. \tag{14}$$

According to Assumption 2, the noise sequence W_- satisfies the following QMI

$$\begin{bmatrix} I_n \\ W_-^\top \end{bmatrix}^\top \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ * & \Phi_{22} \end{bmatrix} \begin{bmatrix} I_n \\ W_-^\top \end{bmatrix} \geqslant 0 \tag{15}$$

with $\Phi_{11} = T\delta_w^2 I_n, \Phi_{12} = 0_{n \times T}$, and $\Phi_{22} = -I_T$. In particular, (15) satisfies the generalized Slater condition [4].

Combing (14) with (15), the system matrices A and B satisfy the following QMI

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top H \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} \geqslant 0 \tag{16}$$

with

$$H=\begin{bmatrix} I_n & X_+ \\ 0_{n\times n} & -X_- \\ 0_{m\times n} & -U_- \end{bmatrix}\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ * & \Phi_{22} \end{bmatrix}\times \begin{bmatrix} I_n & X_+ \\ 0_{n\times n} & -X_- \\ 0_{m\times n} & -U_- \end{bmatrix}^\top. \tag{17}$$

Moreover, the matrix H can be represented as the following partitioned matrix

$$H=\left[\begin{array}{c|c} H_{11} & H_{12} \\ \hline * & H_{22} \end{array}\right] \tag{18}$$

with

$$H_{11}=\Phi_{11}+X_+\Phi_{12}^\top+(\Phi_{12}+X_+\Phi_{22})X_+^\top,$$
$$H_{12}=(\Phi_{12}+X_+\Phi_{22})\begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top,$$
$$H_{22}=\begin{bmatrix} X_- \\ U_- \end{bmatrix}\Phi_{22}\begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top.$$

Due to the existence of the unknown noise $w(k)$, the pair (A,B) that matches the collected data may not be unique. Fortunately, the QMI (16) characterizes a set containing all possible pairs of (A,B) that are compatible with the pre-collected data. Therefore, if we can make all the systems that satisfy the QMI (16) finite-time bounded by designing a finite-time controller, then *Problem 1* is solved. To achieve this objective, the following result is proposed.

Theorem 2: Consider the system (1) under Assumption 1-2. Given the triplet $(\delta_x,\delta_w,\epsilon)$ with $0<\delta_x<\epsilon$ and $x^T(0)Rx(0)\leqslant\delta_x^2$, an integer $N\in\mathbb{Z}^+$, and a matrix $R>0$, the system is finite-time bounded w.r.t. $(\delta_x,\delta_w,\epsilon,R,N)$ under the controller $u=Kx(k)$ with $K=LQ_1^{-1}$ if there exist scalars $\alpha\geqslant 0,\beta>0$, $\gamma\geqslant 1$, a matrix $L\in\mathbb{R}^{m\times n}$, and two positive definite matrices $Q_1\in\mathbb{R}^{n\times n}$, $Q_2\in\mathbb{R}^{n\times n}$ so that the following inequalities hold

$$M_1-\alpha H_1\geqslant 0 \text{ for } \gamma\geq 1, \tag{19a}$$

$$\frac{\delta_x^2}{\lambda_{\min}(\tilde{Q}_1)}+\frac{N\delta_w^2}{\lambda_{\min}(Q_2)}<\frac{\epsilon^2}{\lambda_{\max}(\tilde{Q}_1)} \text{ for } \gamma=1, \tag{19b}$$

$$\frac{\delta_x^2}{\lambda_{\min}(\tilde{Q}_1)}+\frac{(1-\frac{1}{\gamma^N})\delta_w^2}{(1-\frac{1}{\gamma})\lambda_{\min}(Q_2)}<\frac{\epsilon^2}{\lambda_{\max}(\tilde{Q}_1)\gamma^N} \text{ for } \gamma>1, \tag{19c}$$

with $\tilde{Q}_1=R^{\frac{1}{2}}Q_1R^{\frac{1}{2}}$, $H_1=\operatorname{diag}\{H,0_{n\times n}\}$, and

$$M_1=\begin{bmatrix} Q_1-\gamma^{-1}Q_2-\beta I_n & 0_{n\times n} & 0_{n\times m} & 0_{n\times n} \\ * & -\gamma^{-1}Q_1 & -\gamma^{-1}L^\top & 0_{n\times n} \\ * & * & 0_{m\times m} & L \\ * & * & * & \gamma Q_1 \end{bmatrix}.$$

Proof: Select the Lyapunov function candidate $V(x(k))=x^T(k)Px(k)$ with $P=Q_1^{-1}$. According to (19a), we claim the following inequality holds:

$$V(x(k+1))<\gamma V(x(k))+\gamma w(k)^\top P_2 w(k) \tag{20}$$

with $P_2=Q_2^{-1}$.

To prove (20), define $\bar{A}=A+BK$. Then, according to (1) and $u=Kx(k)$, (20) can be rewrited as

$$\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^\top\begin{bmatrix} \gamma P-\bar{A}^\top P\bar{A} & -\bar{A}^\top P \\ * & \gamma P_2-P \end{bmatrix}\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}>0. \tag{21}$$

Applying Lemma 1 to (19a) with $F_{11}=M_2-\alpha H$, $F_{12}=\begin{bmatrix}0_{n\times 2n} & L^\top\end{bmatrix}^\top$, $F_{22}=\gamma Q_1$, and

$$M_2=\begin{bmatrix} Q_1-\gamma^{-1}Q_2-\beta I_n & 0_{n\times n} & 0_{n\times m} \\ * & -\gamma^{-1}Q_1 & -\gamma^{-1}L^\top \\ * & * & 0_{m\times m} \end{bmatrix},$$

yields

$$M-\alpha H\geqslant \operatorname{diag}\{\beta I_n,0_{(n+m)\times(n+m)}\} \tag{22}$$

where M is given as

$$M=\left[\begin{array}{c|c} Q_1-\gamma^{-1}Q_2 & 0_{n\times(n+m)} \\ \hline * & -\begin{bmatrix} I_n \\ K \end{bmatrix}\gamma^{-1}Q_1\begin{bmatrix} I_n \\ K \end{bmatrix}^\top \end{array}\right]. \tag{23}$$

Denote the $(2,2)$ block of the partitioned matrix M as M_{22}. Obviously, M_{22} and H_{22} satisfy $M_{22}\leqslant 0$ and $H_{22}\leqslant 0$. In addition, from the definition of H, one has

$$\ker H_{22}=\ker\left(\begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top\right),$$
$$\ker H_{12}=\ker\left((\Phi_{12}+X_+\Phi_{22})\begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top\right), \tag{24}$$

which implies $\ker H_{22}\subseteq\ker H_{12}$. Then, applying Lemma 3 with $Z=[A,B]^\top$, together with (16), the following inequality is fulfilled

$$\begin{bmatrix} I_n \\ Z \end{bmatrix}^\top M\begin{bmatrix} I_n \\ Z \end{bmatrix}>0, \tag{25}$$

which is equivalent to

$$Q_1-\gamma^{-1}\begin{bmatrix} \bar{A}^\top \\ I_n \end{bmatrix}^\top\operatorname{diag}\{Q_1,Q_2\}\begin{bmatrix} \bar{A}^\top \\ I_n \end{bmatrix}>0. \tag{26}$$

Applying Lemma 1 to (26) with

$$F_{11}=Q_1,\ F_{12}=\begin{bmatrix} \bar{A}^\top \\ I_n \end{bmatrix}^\top,\ F_{22}=\gamma^{-1}\operatorname{diag}\{Q_1,Q_2\},$$

gives

$$\begin{bmatrix} \gamma Q_1^{-1}-\bar{A}^\top Q_1^{-1}\bar{A} & -\bar{A}^\top Q_1^{-1} \\ * & \gamma Q_2^{-1}-Q_1^{-1} \end{bmatrix}>0, \tag{27}$$

which implies (21) that is equivalent to (20).

Next, we will demonstrate that (19b)-(19c) and (20) imply that the system (1) is finite-time bounded with respect to $(\delta_x,\delta_w,\epsilon,R,N)$. Consider the case of $\gamma=1$. From (20) and $\tilde{P}=R^{-1/2}PR^{-1/2}$, one has

$$\begin{aligned} V(x(k)) &< V(x(0))+\sum_{i=0}^{k-1}w(i)^\top P_2 w(i) \\ &\leqslant \lambda_{\max}(\tilde{P})\delta_x^2+\lambda_{\max}(P_2)N\delta_w^2, \end{aligned} \tag{28}$$

and

$$V(x(k)) = x^{\mathrm{T}}(k)Px(k) \geq \lambda_{\min}\left(\tilde{P}\right)x^{\mathrm{T}}(k)Rx(k). \quad (29)$$

Combing (28) with (29) gives

$$x^{\mathrm{T}}(k)Rx(k) < \frac{1}{\lambda_{\min}(\tilde{P})}\left[\lambda_{\max}\left(\tilde{P}\right)\delta_x^2 + \lambda_{\max}\left(P_2\right)N\delta_w^2\right]. \quad (30)$$

Using the fact that $P_i = Q_i^{-1}, i = 1, 2$ and condition (19b), it follows that $x^{\mathrm{T}}(k)Rx(k) < \epsilon^2$.

For case of $\gamma > 1$, according to (20), one has

$$\begin{aligned} V(x(k)) &< \gamma^k V(x(0)) + \sum_{i=1}^{k}\gamma^i w(k-i)^\top P_2 w(k-i) \\ &\leqslant \gamma^N \lambda_{\max}(\tilde{P})\delta_x^2 + \lambda_{\max}(P_2)\delta_w^2 \sum_{i=1}^{N}\gamma^i \\ &\leqslant \gamma^N\left[\lambda_{\max}\left(\tilde{P}\right)\delta_x^2 + \lambda_{\max}\left(P_2\right)\delta_w^2\frac{1-\frac{1}{\gamma^N}}{1-\frac{1}{\gamma}}\right]. \end{aligned} \quad (31)$$

Combing (19c), (29) and (31) give $x^{\mathrm{T}}(k)Rx(k) < \epsilon^2$, for $\gamma > 1$. Hence, we have proved (19a)-(19c) guarantee that system (1) is finite-time bounded with respect to $(\delta_x, \delta_w, \epsilon, R, N)$. This concludes the proof.

Remark 6: Conditions (19b) and (19c) are nonlinear conditions and can be replaced by the following LMIs in the controller design

$$\lambda_1 R^{-1} < Q_1 < R^{-1}, Q_2 < I_n, \quad (32)$$

$$\begin{bmatrix} Q_2 & I_n \\ * & \lambda_2 I_n \end{bmatrix} > 0, \quad (33)$$

$$\begin{bmatrix} \varepsilon^2 - \lambda_2\delta_w^2 N & \delta_x \\ * & \lambda_1 \end{bmatrix} > 0 \text{ for } \gamma = 1, \quad (34)$$

$$\begin{bmatrix} \frac{\varepsilon^2}{\gamma^N} - \lambda_2\delta_w^2\left(\frac{1-\frac{1}{\gamma^N}}{1-\frac{1}{\gamma}}\right) & \delta_x \\ * & \lambda_1 \end{bmatrix} > 0 \text{ for } \gamma > 1, \quad (35)$$

for some positive numbers λ_1 and λ_2. The LMI (33) aims at ensuring that $Q_2 > I_n/\lambda_2$. The LMIs (34)-(35) are obtained by taking (19b)-(19c) and (32)-(33) into consideration and performing Schur complements.

Remark 7: Since (16) and (25) are both QMI involved with A and B, the significance of Theorem 2 lies in finding a feedback control gain K such that (25) holds for all possible (A, B) satisfying (16), which can be realized by utilizing the matrix S-lemma, i.e., Lemma 3.

V. NUMERICAL SIMULATIONS

This section will show the effectiveness of the proposed method via examples. To solve the related LMIs, CVX [19] is adopted, and Mosek [20] is selected as the solver.

A. Noise-free scenario

We consider an unstable plant with (A, B) as follows

$$\begin{aligned} A &= \begin{bmatrix} 0.850 & -0.038 & -0.380 \\ 0.735 & 0.815 & 1.594 \\ -0.664 & 0.697 & -0.064 \end{bmatrix}, \\ B &= \begin{bmatrix} 1.431 & 0.705 \\ 1.620 & -1.129 \\ 0.913 & 0.369 \end{bmatrix}. \end{aligned} \quad (36)$$

The information of (A, B) is only used for simulation and is unknown for controller design.

In the absence of noise, similar to the model-based finite-time controller, we can perform an optimization over ϵ using the following algorithm.

Optimization algorithm I

Initialization: Choose δ_x, R, N and initial value for ϵ. Set $\gamma = 1$.
Procedure
if no solution is found and $\gamma < (\epsilon/\delta_x)^{(2/N)}$, **then** increase γ until a solution is found.
if no solution can be found, **then** increase the initial valueof ϵ.
end if
if the LMIs are feasible, **then** keep decreasing ϵ.
elseif $\gamma < (\epsilon/\delta_x)^{(2/N)}$, **then** increasing γ until a solution is found.
end if
return ϵ
End procedure

In the simulation, an input/state trajectory with length $T = 20$ is pre-collected, whose inputs and initial state are generated randomly from a Gaussian distribution with zero mean and unit variance. Set $R = I_3$, $N = 4$ and the initial value of ϵ as $\epsilon = 10$. In optimization, ϵ is decreased by a step size of 0.1, and γ is increased by a step size of 0.01. We consider δ_x with five different values: $\delta_x = \{0.5, 1, 2, 5, 7.5\}$.

Since the proposed controller is a data-driven realization of the model-based controller proposed in Corollary 2 of [11], we compare the optimization results of the two controllers. The obtained results are shown in the following table:

TABLE I: Optimize search result for ϵ

	Model-based control	Data-driven control
$\delta_x = 0.5$	$\gamma = 1, \epsilon = 0.7$	$\gamma = 1, \epsilon = 0.7$
$\delta_x = 1$	$\gamma = 1, \epsilon = 1.3$	$\gamma = 1, \epsilon = 1.3$
$\delta_x = 2$	$\gamma = 1, \epsilon = 2.5$	$\gamma = 1, \epsilon = 2.5$
$\delta_x = 5$	$\gamma = 1, \epsilon = 6.2$	$\gamma = 1, \epsilon = 6.2$
$\delta_x = 7.5$	$\gamma = 1, \epsilon = 9.2$	$\gamma = 1, \epsilon = 9.2$

As we can see from TABLE I, the noise-free data-based controller reaches the same results as the model-based one. This is not surprising, as pointed out by Willems' fundamental lemma, when the data is rich enough, it is possible to achieve exact identification of LTI systems using data. In this example, we fixed δ_x to find the minimum admissible ϵ that guarantees the desired closed-loop finite-time property. Conversely, we can also fix ϵ to search for the maximum admissible δ_x [11].

B. Noisy scenario

In the presence of unknown noise, the data-based representation (16) may contain different (A, B) pairs, and then the optimization, as in the previous example, may give very conservative results since the search terminates as long as one of the possible systems fails to satisfy the LMIs (19a) and (32)-(35). Therefore, in this subsection, similar to the other existing FTC works, we will directly use the proposed sufficient condition to find a controller for prespecified parameters $(\delta_x, \delta_w, \epsilon, R, N)$. In the simulation, we set $\gamma = 1$ and the noise is randomly generated within the ball $\{w \in \mathbb{R}^n | \|w\|^2 \leq \delta_w^2\}$. In particular, we investigate two noise levels: $\delta_w \in \{0.1, 0.5\}$. For each noise level, we collect 250 input/state trajectories. Meanwhile, we consider the following six cases in TABLE II.

TABLE II: Simulation parameters of the noisy scenario

	δ_x	δ_w	ϵ	R	N
Case 1	1	0.1	1.5	I_3	4
Case 2	1	0.1	2	I_3	4
Case 3	1	0.1	5	I_3	4
Case 4	1	0.5	5	I_3	4
Case 5	1	0.5	10	I_3	4
Case 6	1	0.5	100	I_3	4

For each case, we record the number of times the finite-time controller was found using the corresponding noise level dataset and Theorem 2, as shown in TABLE III.

TABLE III: The number of trajectories on which the controller was found

Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
100	224	250	96	226	242

Based on the TABLE II-III, from Case 1-3, we can see that for noise level $\delta_w = 0.1$, by increasing ϵ, we can find a controller from each pre-collected trajectory such that the closed-loop system is finite-time bounded. This is mainly because the increase of ϵ relaxes condition (19b), which makes LMIs more feasible. For Case 4-6, we successfully find controllers from 96, 226, and 242 trajectories, respectively. The proposed method fails to find the controller in some trajectories. This is due to the fact that as the noise level increases, the (A, B) pairs characterized by the QMI also enlarged, and thus it is much harder to find a controller that simultaneously guarantees all possible systems with the same finite-time boundedness.

VI. Conclusion

This work has proposed a data-driven FTC method for discrete-time LTI systems. For the noise-free case, sufficient conditions of the system with the proposed FTC method have been obtained by using a data-based closed-loop representation. In the presence of bounded unknown noise, sufficient conditions for the finite-time boundedness of the closed-loop system have been derived by using QMI and matrix S-lemma. The proposed control method enables control engineers to bypass system identification and use only measured data to design a finite-time controller.

References

[1] J. C. Willems, P. Rapisarda, I. Markovsky, and B. L. D. Moor, "A note on persistency of excitation," *Systems & Control Letters*, vol. 54, pp. 325-329, 2005.

[2] I. Markovsky, E. Prieto-Araujo, and F. Dörfler, "On the persistency of excitation," *Automatica*, vol. 147, p. 110657, 2022.

[3] C. De Persis and P. Tesi, "Formulas for data-driven control: Stabilization, optimality, and robustness," *IEEE Trans. Automatic Control*, vol. 65, pp. 909-924, 2020.

[4] H. J. van Waarde, M. K. Camlibel, and M. Mesbahi, "From noisy data to feedback controllers: Nonconservative design via a matrix S-lemma," *IEEE Trans. Automatic Control*, vol. 67, pp. 162-175, 2022.

[5] J. Miller and M. Sznaier, "Data-driven gain scheduling control of linear parameter-varying systems using quadratic matrix inequalities," *IEEE Control Systems Letters*, vol. 7, pp. 835-840, 2023.

[6] K. Hu and T. Liu, "Data-driven H_∞ control for unknown linear time-invariant systems with bounded disturbances," in *Proc. of the IEEE Conference on Decision and Control*, Cancún, Mexico, 2022, pp. 1423-1428.

[7] M. Ariola, M. Mattei, E. D. Amato, I. Notaro, and G. Tartaglione, "Model predictive control for a swarm of fixed wing USVs," in *Congress of the International Council of the Aeronautical Sciences*, Daejeon, Korea, 2016, pp. 25-30.

[8] C. P. Bechlioulis and G. A. Rovithakis, "Robust adaptive control of feedback linearizable MIMO nonlinear systems with prescribed performance," *IEEE Trans. Automatic Control*, vol. 53, pp. 2090-2099, 2008.

[9] D. Liu and G. H. Yang, "Prescribed performance model-free adaptive integral sliding mode control for discrete-time nonlinear systems," *IEEE Trans. Neural Networks and Learning Systems*, vol. 30, pp. 2222-2230, 2019.

[10] P. Dorato, "Short-time stability in linear time-varying systems," in *Proc. of Southeastern Symposium on System Theory*, 1961, pp. 83-87.

[11] F. Amato and M. Ariola, "Finite-time control of discrete-time linear systems," *IEEE Trans. Automatic Control*, vol. 50, pp. 724-729, 2005.

[12] L. Zhu, Y. Shen, and C. Li, "Finite-time control of discrete-time systems with time-varying exogenous disturbance," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, pp. 361-370, 2009.

[13] M. N. Elbsat and E. E. Yaz, "Robust and resilient finite-time bounded control of discrete time uncertain nonlinear systems," *Automatica*, vol. 49, pp. 2292-2296, 2013.

[14] F. Li, W. X. Zheng, and S. Xu, "Finite-time fuzzy control for nonlinear singularly perturbed systems with input constraints," *IEEE Trans. Fuzzy Systems*, vol. 30, no. 6, pp. 2129-2134, 2022.

[15] H. J. van Waarde, J. Eising, H. L. Trentelman, and M. K. Camlibel, "Data informativity: a new perspective on data-driven analysis and control," *IEEE Trans. Automatic Control*, vol. 65, pp. 4753-4768, 2020.

[16] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal on Control and optimization*, vol. 38, pp. 751-766, 2000.

[17] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Philadephia, PA: Society for Industrial and Applied Mathematics, 1994.

[18] F. Amato, M. Carbone, M. Ariola, and C. Cosentino, "Finite-time stability of discrete-time systems," in *Proc. of the American Control Conference*, Boston, USA, 2004, pp. 1440-1444.

[19] M. Grant and S. Boyd, "CVX: Matlab software for disciplined convex programming, version 2.1," Mar. 2014. [Online]. Available: http://cvxr.com/cvx

[20] M. ApS, "MOSEK optimization toolbox for MATLAB," Jul. 2023. [Online]. Available: https://docs.mosek.com