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On the Generalization of the Multivariable Popov Criterion for Slope-Restricted Nonlinearities

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Abstract— **The Popov criterion has proved to be a powerful tool for analyzing the absolute stability of Lur'e systems, where a linear time-invariant system is in feedback interconnection with a nonlinear operator. However, its applicability is limited by the stringent requirements that the linear system is strictly proper and the input of the nonlinear operator has a bounded time derivative. In this paper, we relax these requirements for the Popov stability criterion on MIMO systems with non-repeated diagonal slope-restricted nonlinearities by demonstrating the phase containment of Popov multipliers within the well-established O'Shea-Zames-Falb multipliers.**

I. Introduction

In the realm of robust control theory, the Popov criterion has proved to be a pivotal stability criterion for the analysis of absolute stability. The criterion dates back to [1] where Popov proposed conditions for the absolute stability of a class of nonlinear systems with a scalar nonlinear operator within an open sector. It has wide applications across various fields such as large-scale systems [2], power systems [3], [4], and neural networks [5]. In particular, recent studies in [6], [7] have demonstrated that Lur'e type Lyapunov functions associated with the Popov criterion contain the Bregman divergence function in the optimization literature as a special case when the former is applied to the mirror descent method. This highlights the effectiveness of the Popov criterion and implies a broader application of robust control theory to cutting-edge optimization algorithms [8], [9].

The extensive applications of the Popov criterion have stimulated sustained development efforts over decades [10]–[16]. It has been generalized by [10]–[12] to multiple-input-multiple-output (MIMO) systems. It is shown in [13], [14] that Popov criterion can be applied with indefinite multipliers. However, stringent requirements are still imposed: the stable linear time-invariant (LTI) system in the feedback interconnection is strictly proper and the input of the nonlinear operator must have a bounded time derivative. Notably, [14] has proposed a relaxation of these constraints, allowing for a nonzero direct feedthrough term provided the product of the Popov multiplier and the direct feedthrough term is zero. However, such a condition is still restricted, as it simply rules out the feedthrough term for the single-input-single-output (SISO) case. More recently, the work [15], [16] has further extended the SISO Popov criterion to systems with nonzero feedthrough terms through the O'Shea-Zames-Falb (OZF) multipliers.

The relation between Popov and OZF multipliers has been widely explored in the literature [15]–[19]. The Popov multipliers have been treated as the first-order Zames–Falb multipliers having a pole at infinity [17], but such a claim is not rigorous without mathematical proof. It has been proved in [15], [16] that Popov multipliers are phased-contained within OZF multipliers in the SISO case, i.e., given any Popov multiplier for a slope-restricted or monotone and bounded nonlinear operator, there always exists a substitute OZF multiplier certifying the same stability result. Applying this insight to MIMO systems is particularly intriguing, as analyzing dynamical systems usually involves loop transformations, which results in MIMO systems with direct feedthrough terms [7], [20]. While this generalization seems possible, its realization is non-trivial. A critical impediment lies in the intricacy of characterizing the phase for MIMO systems, a concept that is straightforward in the SISO context but complicated and multifaceted for the MIMO case [21]–[23]. Moreover, [19] has discouraged this idea in the MIMO case by providing counterexamples showing that phase substitution is difficult to achieve with a finite-dimensional approximation of OZF multipliers. Nonetheless, we would like to note that [19] considers identical poles and causal OZF multipliers for the MIMO case, which may introduce conservatism in numerical tests.

In this work, we rigorously explore the generalization of the Popov criterion to MIMO systems for slope-restricted nonlinearities, without restrictions on the strict properness of the linear system and boundedness of the input's time derivative to the nonlinear operator. Our investigation includes verification of such feasibility in the MIMO case, addressing the phase containment of the Popov multipliers within OZF multipliers using strong positivity.

II. Preliminaries

A. Notation

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ the space of n-dimensional real vectors, and $\mathbb{R}^{m\times n}$ the space of $m \times n$ real matrices, respectively. Let I_d and 0_d denote the $d \times d$ identity matrix and zero matrix, respectively, with subscripts omitted when the dimensionality is clear from the context. The notation $\mathrm{diag}(\alpha_1, \ldots, \alpha_d)$ denotes a $d \times d$ diagonal matrix with α_i as its i-th diagonal entry. Define $\mathbf{RL}_\infty$ as the set of all proper real rational functions that are bounded on the imaginary axis, and $\mathbf{RH}_\infty$ as the set of all proper

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real rational functions without poles in the closed right-half plane. The set of $m \times n$ matrices with elements in $\mathbf{RL}_\infty$ ($\mathbf{RH}_\infty$) is denoted by $\mathbf{RL}_\infty^{m\times n}$ $\left(\mathbf{RH}_\infty^{m\times n}\right)$. Let $\mathbf{L}_2^m[0,\infty)$ be the Hilbert space of all square-integrable and Lebesgue measurable functions $f:[0,\infty)\to\mathbb{R}^m$ with inner product $\langle f,g\rangle := \int_0^\infty f^T(t)g(t)dt$, for $f,\ g\in\mathbf{L}_2^m[0,\infty)$. $\mathbf{L}_2^m[0,\infty)$ is a subspace of $\mathbf{L}_{2e}^m[0,\infty)$, which includes elements that are integrable on any finite interval. For a function $f:\mathbb{R}\to\mathbb{R}$, its $\mathbf{L}_1$ norm is given by $\|f\|_1 = \int_{-\infty}^{\infty}|f(t)|dt$. Given a matrix $G(j\omega)$, $G^*(j\omega) := G^T(-j\omega)$ represents its conjugate transpose and $(G(j\omega))_{\mathrm{H}} := \frac{1}{2}\left(G(j\omega)+G^*(j\omega)\right)$ denotes its Hermitian part.

B. Absolute stability

Definition 1: A memoryless nonlinearity $\phi:\mathbb{R}\to\mathbb{R}$, with $\phi(0)=0$, is *slope-restricted* in the sector $[\alpha,\beta]$, if $\alpha \le \frac{\phi(x)-\phi(y)}{x-y} \le \beta$, for all $x\ne y$ and $x,y\in\mathbb{R}$. It is said to be *odd* if $\phi(x)=-\phi(-x)$, for all $x\in\mathbb{R}$.
The above terms can be generalized to multivariable $\Phi:\mathbb{R}^n\to\mathbb{R}^n$. In particular, $\Phi(x) := [\phi_1(x_1),\ldots,\phi_n(x_n)]^T$, where x_i denotes the i-th element of $x\in\mathbb{R}^n$, is said to be *slope-restricted* in $[\underline{K},\overline{K}]$, with $\underline{K}=\mathrm{diag}(\underline{k}_1,\ldots,\underline{k}_n)$, $\overline{K}=\mathrm{diag}(\overline{k}_1,\ldots,\overline{k}_n)$, if each ϕ_i is slope-restricted in $[\underline{k}_i,\overline{k}_i]$.

Define the truncation operator P_T which does not change a function on the interval $[0,T]$ and gives the value zero on $(T,\infty]$. The operator Φ is said to be *causal* if $P_T\Phi P_T = P_T\Phi$, for all $T\ge 0$. Consider the interconnection

$$v = Gw + e, \quad w = -\Phi(v) + f \tag{1}$$

where $e,f\in\mathbf{L}_{2e}^n[0,\infty)$, G is a causal LTI system with $G(s)\in\mathbf{RH}_\infty^{n\times n}$ and Φ is a memoryless nonlinear operator. System (1) is called the *Lur'e system*, which is depicted by Figure 1. In this work, Φ is slope-restricted in $[0,K]$, with

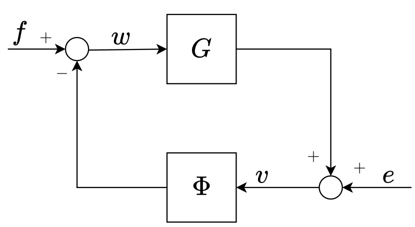

Fig. 1. The Lur'e system, where a linear time-invariant system G is in feedback interconnection with a nonlinear operator Φ.

diagonal matrix $K>0$. The feedback interconnection of G and Φ is *well-posed* if the map $(e,f)\mapsto(v,w)$ defined by (1) has a causal inverse on $\mathbf{L}_{2e}^{2n}[0,\infty)$.

Definition 2: System in Figure 1 is said to be $\mathbf{L}_2$ *stable*, or simply, *stable*, if $(v,w)\in\mathbf{L}_2^{2n}[0,\infty)$ for all exogenous signals $(e,f)\in\mathbf{L}_2^{2n}[0,\infty)$. It is *absolutely stable* if it is stable for all Φ within the class of nonlinear operators.

C. O'Shea-Zames-Falb multipliers and stability theorem

Before introducing the multiplier theorem, let us first define strong positivity based on [24]–[26].

Definition 3: An operator $H:\mathbf{L}_{2e}^n[0,\infty)\to\mathbf{L}_{2e}^n[0,\infty)$ is termed (*strongly*) *positive* if $\langle v,H(v)\rangle \ge \delta\langle v,v\rangle$, $\forall v\in\mathbf{L}_{2e}^m[0,\infty)$, for $\delta=0$ ($\delta>0$).

A (strongly) positive operator H is also (*strongly*) *passive* if it is causal and $H(0)=0$. An LTI system with $G(s)\in\mathbf{RL}_\infty^{n\times n}$ is strongly positive if $(G(j\omega))_{\mathrm{H}} > \delta I$, for some $\delta>0$, $\forall\omega\in\mathbb{R}$. It is easily observed that the slope-restricted Φ is passive.

The multiplier theorem is illustrated through the loop transformation depicted in Figure 2. Loosely speaking, a bounded and causal multiplier $M\in\mathbf{RH}^{n\times n}$ can be designed such that passivity of Φ is preserved under the multiplication of its bounded inverse M^{-1}. Then, if the upper block is strongly passive, the feedback interconnection is stable by the well-known passivity theorem [25]. Additionally, canonical factorization of M allows for relaxation in causality such that $M\in\mathbf{RL}_\infty^{n\times n}$, requiring only that the upper block be strongly positive and the lower block preserve positivity [25], [27]. The O'Shea-Zames-Falb multipliers, introduced in [24],

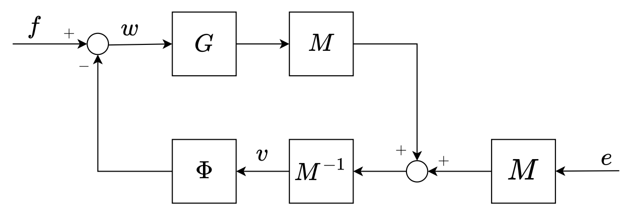

Fig. 2. Diagram of loop transformation with multiplier. The bounded multiplier M is designed such that positivity of Φ is preserved under the multiplication of its bounded inverse M^{-1}. Then, if the upper block is strongly positive, the feedback interconnection is stable.

[28], are currently the largest set of multipliers preserving positivity of slope-restricted and odd nonlinearities [18], [29]. We adopt the following definition of SISO rational OZF multipliers for slope-restricted but not necessarily odd nonlinearities [15], [30].

Definition 4: A SISO rational transfer function of the form $M(s)=1-Z(s)$ is said to be an O'Shea-Zames-Falb (OZF) multiplier if $Z(s)$ satisfies $Z(\infty)=0$, and its inverse Laplace transform satisfies $\|z(t)\|_1<1$. The set of all SISO rational OZF multipliers is denoted by the set $\mathcal{M}\subset\mathbf{RL}_\infty$. The subset of $\mathcal{M}_+\subset\mathcal{M}$ contains all OZF multipliers such that $z(t)\ge 0$, $\forall t\in\mathbb{R}$.
The above definition includes $Z(s)=0$ as a special case, where $M(s)=1$ is a constant gain. We use $\mathcal{M}^{n\times n}$ and $\mathcal{M}_+^{n\times n}$ to denote the sets of all $n\times n$ diagonal matrices with elements from $\mathcal{M}$ and $\mathcal{M}_+$, respectively.

We have the following OZF theorem for non-repeated diagonal nonlinearities, considering rational OZF multipliers.

Theorem 1 ([31]): Consider the feedback interconnection in Figure 1 with $G\in\mathbf{RH}_\infty^{n\times n}$ and Φ is slope-restricted in $[0,K]$, with diagonal matrix $K>0$. Consider a diagonal $M(s)\in\mathcal{M}_+^{n\times n}$. Suppose that the system is well-posed, then the system is absolutely stable if $M(j\omega)\left(G(j\omega)+K^{-1}\right)$ is strongly positive.

Remark 1: MIMO systems with repeated or non-diagonal nonlinearities have been extensively studied [19], [31]–[33].

Nonetheless, we focus specifically on the absolute stability of MIMO systems with non-repeated diagonal nonlinearities in this work to reveal the connection between Popov and OZF multipliers. Moreover, the repeated case can be viewed as adding extra constraints to the characterization of nonlinear operators in the non-repeated case , which can be addressed separately [7].

We consider G to be in the subset $\mathcal{SR}$ of $\mathbf{RH}_\infty^{n\times n}$ defined by

$$\mathcal{SR} = \Big\{\hat{G} \in \mathbf{RH}_\infty^{n\times n} : \hat{G}^{-1} \in \mathbf{RH}_\infty^{n\times n}, \text{ and } \exists \text{ diagonal } P > 0, \text{ s.t. } \Big(P\hat{G}(\infty)\Big)_{\mathrm{H}} > 0\Big\}. \quad (2)$$

To see that this consideration is without loss of generality, let us first consider a linear feedback matrix $\Phi = \tilde{K}$, where $\tilde{K} = \mathrm{diag}(\tilde{k}_1, \ldots, \tilde{k}_n)$, with $\tilde{k}_i \in [0, k_i]$, in Figure 1. Then, a necessary condition for the absolute stability derived from the generalized Nyquist stability criterion [34] is that

$$\det\Big[I + \tilde{K}G(s)\Big] \neq 0, \ \forall \tilde{k}_i \in [0, k_i], \ \forall \mathrm{Re}[s] \geq 0. \quad (3)$$

Define $\hat{G}(s) := G(s) + K^{-1} = K^{-1}(I + KG(s))$, with $K = \mathrm{diag}(k_1, \ldots, k_n)$. Condition (3) guarantees that $\hat{G}$ is invertible, and $\hat{G}^{-1} \in \mathbf{RH}_\infty^{n\times n}$. The loop transformation to obtain $\hat{G}$ is given in Figure 3. The nonlinear operator $\tilde{\Phi}$ is passive. The feedback interconnection is absolutely stable if $\hat{G}$ is strongly passive. The strong passivity can be relaxed by applying the multiplier method, but a necessary condition for the existence of a valid OZF multiplier is that there exists a diagonal and positive definite matrix P such that $\Big(P\hat{G}(\infty)\Big)_{\mathrm{H}} > 0$, as the lower block in Figure 3 preserves passivity with the diagonal P^{-1}.

Remark 2: It can be observed that $\hat{G} \in \mathcal{SR}$ is a necessary condition for absolute stability. It follows that if $\hat{G} = G + K^{-1} \notin \mathcal{SR}$, with $G \in \mathbf{RH}_\infty^{n\times n}$ and $K > 0$, then the feedback interconnection of G and the class of slope-restricted nonlinearities Φ cannot be absolutely stable. This necessity is imposed by the generalized Nyquist stability criterion [34], [35].

D. Phase Containment

The term *phase-contained* is defined in [15] to characterize the equivalence of multipliers for slope-restricted nonlinearities in the SISO case. The concept of phase is not straightforward for the MIMO systems [21]–[23], but we adopt this same terminology as an expansive interpretation through the strong positivity.

Definition 5: Let $\mathcal{M}_{\mathrm{a}}$, $\mathcal{M}_{\mathrm{b}}$ be two sets of multipliers and let $\hat{G} \in \mathcal{SR}$. The class $\mathcal{M}_{\mathrm{a}}$ is said to be phase-contained within the class $\mathcal{M}_{\mathrm{b}}$ if for any multiplier $M_{\mathrm{a}} \in \mathcal{M}_{\mathrm{a}}$ such that

$$\Big(M_{\mathrm{a}}(j\omega)\hat{G}(j\omega)\Big)_{\mathrm{H}} \geq \delta_1 I, \quad \forall\omega \in \mathbb{R}$$

for some constant $\delta_1 > 0$, there exists a multiplier $M_{\mathrm{b}} \in \mathcal{M}_{\mathrm{b}}$ to phase-substitute M_{a}, i.e.,

$$\Big(M_{\mathrm{b}}(j\omega)\hat{G}(j\omega)\Big)_{\mathrm{H}} \geq \delta_2 I, \quad \forall\omega \in \mathbb{R}$$

for some constant $\delta_2 > 0$.

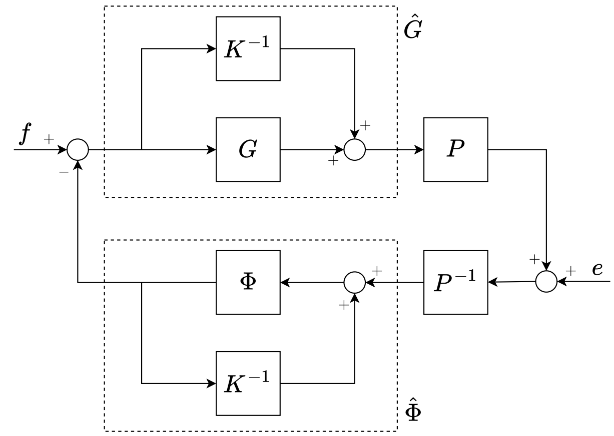

Fig. 3. Diagram of loop transformation. The nonlinear operator $\tilde{\Phi}$ is passive. The feedback interconnection is absolutely stable if $\hat{G}$ is strongly passive. The strong passivity can be relaxed by applying the multiplier method, but a necessary condition for the existence of a valid OZF multiplier is that there exists a diagonal and positive definite matrix P such that $\Big(P\hat{G}(\infty)\Big)_{\mathrm{H}} > 0$, as the lower block preserves passivity with diagonal P^{-1}.

III. Main results

We present the main results in this section. The proofs are included in the appendix.

As the open-loop system G considered in this work may contain the nonzero feedthrough term D, we assume the well-posedness of the negative feedback interconnection of $G(s) = C(sI - A)^{-1}B + D$ and Φ, as defined in Theorem 1. It is worth noting that this assumption is fulfilled if $G(s) + K^{-1}$ is within the set $\mathcal{SR}$ since every element in the set is invertible as indicated by (3).

A. Popov and O'Shea-Zames-Falb multipliers

The class of Popov multipliers for SISO systems is given by

$$M_{\mathrm{p}}(s) = 1 + \lambda s, \quad \lambda \in \mathbb{R}. \quad (4)$$

The first-order OZF multiplier

$$M_{\mathrm{ozf}}(s) = \frac{1 + \lambda s}{1 + \varepsilon s}, \quad \varepsilon \cdot \lambda > 0 \quad (5)$$

is adopted in [15] to show that the class of Popov multipliers (4) are phase-contained within the class of OZF multipliers. The class of first-order OZF multipliers is given below.

Lemma 1 ([15]): Let $M_{\mathrm{ozf}}(s)$ be a first-order transfer function given by (5). Then $M_{\mathrm{ozf}}(s) \in \mathcal{M}_+$ if and only if $\left|1 - \frac{\lambda}{\varepsilon}\right| < \frac{\lambda}{\varepsilon}$.

The above lemma implies that in the MIMO case, the signs of $\varepsilon_1, \ldots, \varepsilon_n$ should coincide with their corresponding $\lambda_1, \ldots, \lambda_n$, in order to construct a valid OZF multiplier. This observation also indicates that a multivariable OZF multiplier should be constructed with multiple poles if one wants to achieve a similar performance as the multivariable Popov multipliers. Moreover, when $\lambda = 0$, the Popov multiplier (4) is reduced to $M_{\mathrm{p}}(s) = 1$, which is a trivial case in SISO systems as no Popov multiplier is required to prove the stability, but this case is not negligible in MIMO systems. To see this, let us look at the multivariable Popov multiplier for MIMO systems [14], given by

$$M_{\mathrm{P}}(s) = Q + s\Lambda \tag{6}$$

where $Q := \mathrm{diag}(q_1, \ldots, q_n)$ is a diagonal positive definite matrix and $\Lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is an indefinite diagonal matrix. If $\lambda_i = 0$ for some $i \in \{1, \ldots, n\}$, $M_{\mathrm{P}}(s)$ is still a Popov multiplier, but $M(s) = \mathrm{diag}\left(\frac{q_1+s\lambda_1}{1+s\varepsilon_1}, \ldots, \frac{q_n+s\lambda_n}{1+s\varepsilon_n}\right)$ fails to be an OZF multiplier due to the i-th element being $M_i(s) = \frac{q_i}{1+s\varepsilon_i}$ for any nonzero ε_i.

Therefore, we adopt the following OZF multiplier

$$M_{\mathrm{OZF}}(s) = \begin{bmatrix} \frac{q_1+s(\lambda_1+p_1\varepsilon_1)}{1+s\varepsilon_1} & & \\ & \ddots & \\ & & \frac{q_n+s(\lambda_n+p_n\varepsilon_n)}{1+s\varepsilon_n} \end{bmatrix}, \tag{7}$$

where $p_i > 0$ is defined in (2), $\varepsilon_i\lambda_i \geq 0$, and $\varepsilon_i = 0$ iff $\lambda_i = 0$.

Lemma 2: $M_{\mathrm{OZF}}(s)$ defined in (7) is a multivariable O'Shea-Zames-Falb multiplier and $M_{\mathrm{OZF}} \in \mathcal{M}_+^{n\times n}$ if and only if $0 < q_i < \frac{2\lambda_i}{\varepsilon_i} + 2p_i$ for all i such that $\lambda_i \neq 0$.

From the above lemma, one has $M_{\mathrm{OZF}}(s) \in \mathcal{M}_+^{n\times n}$ given sufficiently small nonzero ε_i, for all i such that $\lambda_i \neq 0$.

B. Phase containment of Popov multipliers

The following lemma is trivial in the SISO case but is required to verify phase containment of the Popov multipliers in the MIMO case.

Lemma 3: If $M_{\mathrm{P}}(s)\hat{G}(s)$ is strongly positive, then ΛD is a symmetric matrix.

The following lemma on the eigenvalues of matrix product will be used.

Lemma 4: Given matrices $A, B \in \mathbb{R}^{n\times n}$ with A and $B + B^T$ being positive (semi-)definite, then all eigenvalues of AB have positive (nonnegative) real parts.

Next, we show that the class of multivariable Popov multipliers is phase-contained within the class of OZF multipliers.

Lemma 5: Let $\hat{G}(s) = G(s) + K^{-1}$ and suppose that $\hat{G}(s) \in \mathcal{SR}$, then the class of multivariable Popov multipliers is phase-contained in the class of O'Shea-Zames-Falb multipliers.

C. Generalized multivariable Popov criterion

We are now ready to present the generalized multivariable Popov criterion.

Theorem 2 (Generalized multivariable Popov criterion): Consider the feedback interconnection in Figure 1 with $G(s) \in \mathbf{RH}_\infty^{n\times n}$ and Φ is slope-restricted in $[0, K]$, with diagonal matrix $K > 0$. Suppose the system is well-posed, and $G(s) + K^{-1}$ is in the set $\mathcal{SR}$ defined by (2). If there exist constant matrices $Q = \mathrm{diag}(q_1, \ldots, q_n)$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ such that $\left(QK^{-1} + (Q + j\omega\Lambda)G(j\omega)\right)_{\mathrm{H}} \geq \delta I$, $\forall \omega \in \mathbb{R}$, for some constant $\delta > 0$, then the system is stable.

Proof: The proof follows directly from Lemma 5 and Theorem 1. ■

Remark 3: This multivariable Popov criterion is generalized in the sense that the requirements of $D = 0$ or $\Lambda D = 0$, and the input e in Figure 1 having a bounded time derivative are removed. However, it is important to verify in advance the well-posedness of the closed-loop system, and that $G(s) + K^{-1}$ is within the set $\mathcal{SR}$ in (2) before applying the Popov criterion when there is a nonzero D.

IV. Numerical Examples

In this section, we present two numerical examples to illustrate our proposed results.

A. Phase Containment

We adopt Example 4.9 from [19] but with non-repeated diagonal nonlinearities. The system matrices are given by

$$A = \begin{bmatrix} -4 & -3 & 0 \\ 2 & 0 & 0 \\ -1 & -1 & -2 \end{bmatrix},\ B = -\begin{bmatrix} 0 & 4 & 1 & 3 \\ 2 & 0 & 3 & 1 \\ 1 & 0 & 3 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.1 & -0.2 & 1 \\ -1 & -0.3 & 0.1 \\ -0.2 & 0.1 & 1 \\ 0.1 & -0.2 & 0.2 \end{bmatrix}, D = 0.$$

and $\Phi = [\phi_1, \ldots, \phi_n]^T$ with $K = \mathrm{diag}(0.5, 1, 0.5, 5)$. Solving the time-domain conditions derived from Theorem 2 via the KYP lemma [26], we obtain $M_{\mathrm{p}} = Q + j\omega\Lambda$, where $Q = \mathrm{diag}(0.7097, 1.2072, 0.4240, 0.5778)$ and $\Lambda = \mathrm{diag}(0.2392, -0.0829, 0.1419, 0.5108)$. It has been shown in [19] that a finite-dimensional approximation of OZF multipliers cannot achieve the same bound, indicating a failure of the phase containment. However, we believe that this is due to the use of identical poles only, i.e., $\varepsilon_i = \varepsilon > 0$, for all i in (7). To obtain the phase containment, ε_i should have the same sign as its corresponding λ_i, leading to possibly non-causal OZF multipliers. Indeed, in this example, considering $M_{\mathrm{OZF}}(s)$ in (7) but with identical $\varepsilon_1 = \ldots = \varepsilon_4 = \varepsilon > 0$ fails[1] to ensure the inequality $\left(M_{\mathrm{OZF}}(j\omega)\hat{G}(j\omega)\right)_{\mathrm{H}} > \delta I$. even for $\varepsilon = 10^{-9}$, which coincides with the observation in [19]. On the other hand, let $|\varepsilon_1| = \ldots = |\varepsilon_4| = \varepsilon$ and $\varepsilon_i\lambda_i > 0$, it is shown in Figure 4 that strong positivity holds for $\varepsilon = 0.01$, which is relatively small but not infinitesimal.

B. Nonzero direct feedthrough term

Let us consider the same system matrices A, B, C, and nonlinear operator Φ, as the previous example, but with a nonzero direct feedthrough term given by

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \end{bmatrix}.$$

It is important to check that $D + K^{-1}$ has positive eigenvalues. It is readily observed that $\left(D + K^{-1}\right)_{\mathrm{H}}$ is not positive definite. A positive definite matrix P satisfying (2) is given

[1]Note that such a $M_{\mathrm{OZF}}(s)$ may not even be an OZF multiplier by Lemma 1.

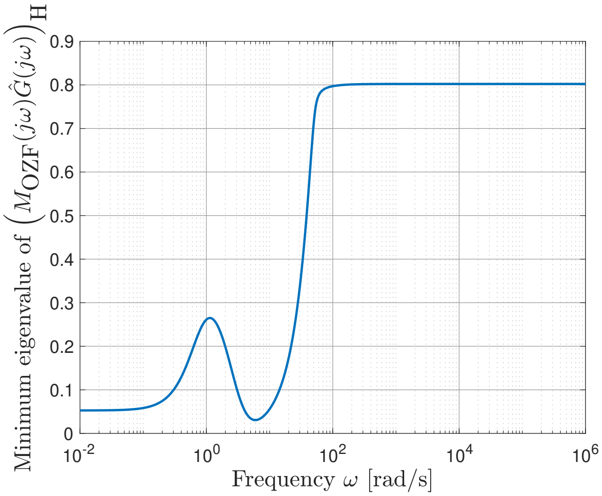

Fig. 4. The minimum eigenvalue of $\left(M_{\mathrm{OZF}}(j\omega)\hat{G}(j\omega)\right)_{\mathrm{H}}$ over frequency ω [rad/s], where $\hat{G}(j\omega) = G(j\omega) + K^{-1}$.

by $P = \mathrm{diag}\,(0.4000, 0.1959, 0.6042, 1.6667)$. Similarly, Theorem 2 gives the Popov multiplier $M_{\mathrm{P}}(j\omega) = Q + j\omega\Lambda$, where $Q = \mathrm{diag}\,(0.8533, 0.2020, 1.6010, 0.8771)$, and $\Lambda = \mathrm{diag}\,(0.1066, 0, 0.4134, 0.1737)$. This Popov multiplier is phase-contained within the set of OZF multipliers by choosing $\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = 0.01$, and $\varepsilon_2 = 0$. It is worth noting that we do not use $M(j\omega) = \frac{1+p_2\varepsilon s}{1+\varepsilon s}$ to approximate the constant in the Popov multiplier $M_{2,\mathrm{P}}(j\omega)$ as it may not represent a first-order OZF multiplier by Lemma 2. More importantly, a constant can already be considered a special case of an OZF multiplier, which exhibits distinct phase properties compared to first-order OZF multipliers.

V. Conclusions

This paper has shown that the class of multivariable Popov multipliers for MIMO systems with slope-restricted nonlinearities is phase-contained within the class of the O'Shea-Zames-Falb multipliers. In other words, any multivariable Popov multiplier can be equivalently replaced by an O'Shea-Zames-Falb multiplier when applied to the absolute stability analysis with slope-restricted nonlinearities. As a result, the requirement that the input has bounded time derivatives has been removed and the feedthrough term D can be nonzero to apply the multivariable Popov stability criterion.

Appendix

Proof of Lemma 2

As $M_{\mathrm{OZF}}(s)$ is diagonal, it suffices to show that each diagonal element $M_{i,\mathrm{OZF}}(s) := \frac{q_i + s(\lambda_i + p_i\varepsilon_i)}{1+s\varepsilon_i}$ is a SISO OZF multiplier. When $\lambda_i = 0$, $\varepsilon_i = 0$ and $M_{i,\mathrm{OZF}}(s)$ degenerates to the constant q_i, which is a special case of OZF multiplier by Definition 4. When $\lambda_i \neq 0$, $M_{i,\mathrm{OZF}}(s)$ can be rewritten as $M_{i,\mathrm{OZF}}(s) = M_{i,\mathrm{OZF}}(\infty) + \widehat{M}_{i,\mathrm{OZF}}(s)$ where $M_{i,\mathrm{OZF}}(\infty) = \frac{\lambda_i + p_i\varepsilon_i}{\varepsilon_i}$ and $\widehat{M}_{i,\mathrm{OZF}}(s) = \frac{q_i - \frac{\lambda_i + p_i\varepsilon_i}{\varepsilon_i}}{1+s\varepsilon_i}$. Applying Lemma 1 using the fact that $\|\widehat{m}_{i,\mathrm{OZF}}(t)\|_1 = |q_i - \frac{\lambda_i + p_i\varepsilon_i}{\varepsilon_i}|$, where $\widehat{m}_{i,\mathrm{OZF}}(t)$ is the inverse Laplace transform of $\widehat{M}_{i,\mathrm{OZF}}(s)$, we have that $M_{i,\mathrm{OZF}}(s)$ is an OZF multiplier if and only if $(\lambda_i + p_i\varepsilon_i)\varepsilon_i > 0$ and $|q_i - p_i - \frac{\lambda_i}{\varepsilon_i}| < \frac{\lambda_i}{\varepsilon_i} + p_i$. The first condition is satisfied since $\lambda_i\varepsilon_i > 0$ and $p_i > 0$, and the latter is equivalent to $0 < q_i < \frac{2\lambda_i}{\varepsilon_i} + 2p_i$.

Proof of Lemma 3

We have $M_{\mathrm{P}}(j\omega)\hat{G}(j\omega) = (Q + j\omega\Lambda)\left(C(j\omega I - A)^{-1}B + D + K^{-1}\right)$. When ω is large, the term $j\omega\Lambda(D+K^{-1})$ dominates other terms, and it follows that its Hermitian part is positive semidefinite, i.e., $\left(j\omega\Lambda(D+K^{-1})\right)_{\mathrm{H}} = \frac{1}{2}j\omega(\Lambda D - D^T\Lambda) \geq 0$ where the equality follows from the fact that ΛK^{-1} is symmetric. Since $j\omega(\Lambda D - D^T\Lambda)$ is a positive semidefinite Hermitian with diagonal elements being zero, it follows that the eigenvalues are solely zero. As the Hermitian matrix is diagonalizable, we have $j\omega(\Lambda D - D^T\Lambda) = 0$, meaning that ΛD is symmetric.

Proof of Lemma 4

Given matrices $S \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$. If ν is an eigenvector of ST, then $T\nu$ is an eigenvector of TS, both associated with the same nonzero eigenvalue μ, as $TS(T\nu) = \mu T\nu$. Thus, ST and TS share the same nonzero eigenvalues. Since A is positive (semi-)definite, it admits a positive (semi-)definite square root $A^{1/2}$. Then, we have $AB = A^{1/2}(A^{1/2}B)$ and thus AB shares the same nonzero eigenvalues with $\left(A^{1/2}B\right)A^{1/2}$, which have positive (non-negative) real parts due to positive (semi-)definiteness of $B + B^T$, as $x^TA^{1/2}BA^{1/2}x = \frac{1}{2}x^TA^{1/2}(B+B^T)A^{1/2}x > (\geq)\ 0$, for any $x \neq 0$.

Proof of Lemma 5

Let $\mathcal{E} := \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$ where $\varepsilon_i = 0$ iff $\lambda_i = 0$, $|\varepsilon_i| = \varepsilon$, for all i such that $\lambda_i \neq 0$, and $\varepsilon > 0$ is a sufficiently small constant. Let $\mathcal{N} := \{1, \ldots n\}$, we denote its subset $\mathcal{N}_{\mathrm{nz}}$ as the set of indices of all nonzero λ_i, and $\mathcal{N}_0 := \mathcal{N} \setminus \mathcal{N}_{\mathrm{nz}}$. Define $E_i \in \mathbb{R}^{n\times n}$ such that its i-th diagonal element is one, and the rest of all other elements are zero. Let us first look at $\left(M_{\mathrm{OZF}}(j\omega)\hat{G}(j\omega)\right)_{\mathrm{H}}$ at high frequencies,

$$
\begin{aligned}
&\lim_{\omega\to\infty}\left(M_{\mathrm{OZF}}(j\omega)\hat{G}(j\omega)\right)_{\mathrm{H}} \\
&= \left(\sum_{i\in\mathcal{N}_{\mathrm{nz}}} E_i\left(\frac{\mathcal{E}\Lambda}{\varepsilon^2} + P\right)\hat{G}(\infty)\right)_{\mathrm{H}} + \left(\sum_{i\in\mathcal{N}_0} E_iQ\hat{G}(\infty)\right)_{\mathrm{H}} \\
&= \left(\sum_{i\in\mathcal{N}_{\mathrm{nz}}} E_i\left(\frac{\mathcal{E}\Lambda}{\varepsilon^2} + P\right)\left(D + K^{-1}\right)\right)_{\mathrm{H}} \\
&\quad + \left(\sum_{i\in\mathcal{N}_0} E_iQ(D+K^{-1})\right)_{\mathrm{H}} \\
&= \sum_{i\in\mathcal{N}_{\mathrm{nz}}} E_i\frac{\mathcal{E}\Lambda}{\varepsilon^2}\left(D+K^{-1}\right) + \left(\sum_{i\in\mathcal{N}_{\mathrm{nz}}} E_iP\left(D+K^{-1}\right)\right)_{\mathrm{H}} \\
&\quad + \left(\sum_{i\in\mathcal{N}_0} E_iQ(D+K^{-1})\right)_{\mathrm{H}} \\
&\geq \left(\sum_{i\in\mathcal{N}_{\mathrm{nz}}} E_iP\left(D+K^{-1}\right)\right)_{\mathrm{H}} + \left(\sum_{i\in\mathcal{N}_0} E_iQ(D+K^{-1})\right)_{\mathrm{H}}
\end{aligned}
$$

where the symmetry in the third equality follows from Lemma 3, and the inequality follows from Lemma 4. Moreover, the first term on the right-hand side is positive semidefinite, and the nonzero diagonal elements of the second term are all positive. Then, it follows from the Schur complement that there exists a diagonal $P > 0$ with sufficiently large elements such that its sum with the right-hand side of the above equation is positive definite, i.e., $\lim_{\omega \to \infty} \left(M_{\mathrm{OZF}}(j\omega)\hat{G}(j\omega) \right)_{\mathrm{H}} \geq \delta_{\mathrm{a}} I$, for some $\delta_{\mathrm{a}} > 0$. As $\left(M_{\mathrm{OZF}}(j\omega)\hat{G}(j\omega) \right)_{\mathrm{H}}$ is continuous on ω, and by the definition of limit at infinity, there exists a sufficiently large ω_0, such that $\left(M_{\mathrm{OZF}}(j\omega)\hat{G}(j\omega) \right)_{\mathrm{H}} \geq (\delta_{\mathrm{a}} - \epsilon) I > 0$, for all $|\omega| \geq \omega_0$, where $\epsilon > 0$. It is worth noting that the minimum value of δ_{a} does not depend on ε but on P and $D + K^{-1}$ instead, so ω_0 can be chosen independently of ε. Next, let us look at the system at lower frequencies,

$$
\begin{aligned}
&\left(M_{\mathrm{OZF}}(j\omega)\hat{G}(j\omega) \right)_{\mathrm{H}} \\
=&\left((I + j\omega\mathcal{E})^{-1} M_{\mathrm{P}}(j\omega)\hat{G}(j\omega) \right)_{\mathrm{H}} \\
=&\left(\left(I - j\omega\mathcal{E} + \omega^2\mathcal{E}^2 (I + j\omega\mathcal{E})^{-1} \right) M_{\mathrm{P}}(j\omega)\hat{G}(j\omega) \right)_{\mathrm{H}} \\
\geq&\delta_1 I + \left(\left(\omega^2\mathcal{E}^2 (I + j\omega\mathcal{E})^{-1} - j\omega\mathcal{E} \right) M_{\mathrm{P}}(j\omega)\hat{G}(j\omega) \right)_{\mathrm{H}}
\end{aligned}
$$

where the inequality follows from strong positivity of $M_{\mathrm{P}}(j\omega)\hat{G}(j\omega)$. Moreover, $\left(\omega^2\mathcal{E}^2 (I + j\omega\mathcal{E})^{-1} - j\omega\mathcal{E} \right) \to 0$ as $\mathcal{E} \to 0$, and it is bounded for $|\omega| \leq \omega_0$, where ω_0 is given previously. Then, for any $|\omega| \leq \omega_0$, there exists a sufficiently small $\varepsilon > 0$, such that $\left(M_{\mathrm{OZF}}(j\omega)\hat{G}(j\omega) \right)_{\mathrm{H}} \geq \delta_{\mathrm{b}} I > 0$, for some constant δ_{b}. In summary, there exists a multivariable OZF multiplier $M_{\mathrm{OZF}}(j\omega)$ in the form of (7) such that $\left(M_{\mathrm{OZF}}(j\omega)\hat{G}(j\omega) \right)_{\mathrm{H}} \geq \delta_2 I$, where $\delta_2 = \min\{\delta_{\mathrm{a}} - \epsilon, \delta_{\mathrm{b}}\}$.

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