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Homogeneity with respect to a part of variables and accelerated stabilization

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***Abstract*—The problem of transforming a locally asymptotically stabilizing time-varying control law to a globally stabilizing one with accelerated finite/fixed-time convergence is studied. The solution is based on an extension of the theory of homogeneous systems to the setting where the symmetry and stability properties only hold with respect to a part of the state variables. The proposed control design advances the kind of approaches first studied in [1], and relies on the implicit Lyapunov function framework. Examples of finite-time and nearly fixed-time stabilization of a nonholonomic integrator are reported.**

I. Introduction

For many systems, the convergence rates of estimation or regulation errors to zero must satisfy the desired specifications. Different approaches have been developed to ensure asymptotic, rated exponential, finite-time or fixed-time convergence properties, see e.g., [2], [3], [4], [5], [6], [7]. Another important issue in nonlinear control and estimation involves enlarging domains of convergence for the tracking/estimation errors, where the local domains are much easier to obtain (using, for example, linearization techniques), but the global ones are more desirable.

An appealing class of nonlinear systems, the local behavior of which is the same as global one (e.g., local attractiveness implies global asymptotic stability), is composed of systems with homogeneous dynamics. Homogenization of the closed-loop systems by a suitable dilation of the control laws, which allows (local) asymptotic stabilizing controls to be transformed to ones with a rated convergence, was studied in [1], where it was shown that exponential convergence rates can be achieved for a class of nonholonomic systems. In [8], [9], [10], [11], [12] such a homogenization approach was employed for finite/fixed-time stabilization of linear autonomous systems. In [1], [9], [10], [11], [12] the implicitly defined Lyapunov functions (see also [13]) are exploited for stability analysis.

In this work, the time-varying feedback stabilization approach of [1] is extended to finite-time and fixed-time stabilization. To this end, the theory of homogeneous systems is extended to the case of partial symmetry, when only a part of the state is dilated, and the variables representing generator dynamics, which inform time-dependence of the control law, are not scaled. These developments partly build on the insights from the previous extension of the theory of homogeneous systems to time-varying models in [14], [15]. In such a settling, stability and convergence only with respect to a part of the state variables are considered in the sense of uniform output stability from [16], [17].

The paper is organized as follows. The background on stability and homogeneity is reviewed in Section II. The control problem statement is introduced in Section III. The properties of partially homogeneous and stable systems are investigated in Section IV. The control design is described in Section V. The results of numerical experiments are reported in Section VI. Some proofs are omitted due to space limitations.

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Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|\cdot\|$ is used for the Euclidean norm on $\mathbb{R}^n$.
- For a (Lebesgue) measurable function $d : \mathbb{R}_+ \to \mathbb{R}^s$ and $[t_0, t_1) \subset \mathbb{R}_+$ define the norm $\|d\|_{[t_0,t_1)} = \text{ess}\sup_{t\in[t_0,t_1)}\|d(t)\|$, then $\|d\|_\infty = \|d\|_{[0,+\infty)}$ and the set of d with the property $\|d\|_\infty < +\infty$ we denote as $\mathcal{L}^s_\infty$ (i.e., this is the set of essentially bounded measurable functions).
- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and it is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}_+$ and $\beta(s, \cdot)$ is decreasing to zero for each fixed $s \in \mathbb{R}_+$.
- For a set $\mathcal{A} \subset \mathbb{R}^n$, we denote its boundary and interior by $\partial\mathcal{A}$ and $\text{int}(\mathcal{A})$, respectively.
- A finite series of integers $1, 2, ..., n$ is denoted by $\overline{1,n}$, and $\{\overline{1,n}\} = \{1, 2, ..., n\}$.

II. Preliminaries

The standard stability notions and their definitions can be found in [4].

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A. Uniform output stability

Consider a non-autonomous differential equation:

$$\dot{x}(t) = f(x(t), d(t)),\ t \geq 0, \qquad (1)$$
$$y(t) = h(x(t)),$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $d(t) \in D \subseteq \mathbb{R}^s$ is the vector of external inputs and $d \in \mathcal{D} \subseteq \mathcal{L}_\infty^s$ ($\mathcal{D}$ is the set of admissible inputs), $y(t) \in \mathbb{R}^p$ is the output; $f : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are locally Lipschitz continuous functions, $f(0, 0) = 0$ and $h(0) = 0$. The unique maximal solution of the system (1) for an initial condition $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{D}$ is denoted by $X(t, x_0, d)$ for those $t \geq 0$ for which the solution exist; then $Y(t, x_0, d) = h(X(t, x_0, d))$.

The system (1) is called forward complete if for any $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{D}$ the solution $X(t, x_0, d)$ exists for all $t \geq 0$.

In the sequel, we assume that the inputs $d(t)$ take values in a compact set $D \subset \mathbb{R}^s$ for almost all instants of time $t \geq 0$, i.e., $\sup_{d\in\mathcal{D}} \|d\|_\infty < +\infty$.

Definition 1. [16] A forward complete system (1) is uniformly output stable (uOS) with respect to the inputs in $\mathcal{D}$ if there exists $\beta \in \mathcal{KL}$ such that

$$\|Y(t, x_0, d)\| \leq \beta(\|x_0\|, t),\ \forall t \geq 0$$

holds for all $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{D}$. It is state-independent uniformly output stable (SIuOS) with respect to the inputs in $\mathcal{D}$ if this estimate can be strengthened to

$$\|Y(t, x_0, d)\| \leq \beta(\|h(x_0)\|, t),\ \forall t \geq 0$$

satisfying for all $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{D}$.

Definition 2. [16] A system (1) is uniformly bounded input bounded state stable (UBIBS) if it is forward complete and there exists $\sigma \in \mathcal{K}$ such that

$$\|X(t, x_0, d)\| \leq \max\{\sigma(\|x_0\|), \sigma(\|d\|_\infty)\},\ \forall t \geq 0$$

for all $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{D}$.

Theorem 1. *[16] A UBIBS system* (1) *is uOS if and only if it admits a smooth Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *satisfying for some* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *and* $\alpha_3 \in \mathcal{KL}$*:*

$$\alpha_1(\|h(x)\|) \leq V(x) \leq \alpha_2(\|x\|),\ \forall x \in \mathbb{R}^n;$$
$$\frac{\partial V(x)}{\partial x} f(x, d) \leq -\alpha_3(V(x), \|x\|),\ \forall x \in \mathbb{R}^n,\ \forall d \in D.$$

Moreover, a forward complete system (1) *is SIuOS if and only if there exists a smooth Lyapunov function* V *with*

$$\alpha_1(\|h(x)\|) \leq V(x) \leq \alpha_2(\|h(x)\|),\ \forall x \in \mathbb{R}^n;$$
$$\frac{\partial V(x)}{\partial x} f(x, d) \leq -\alpha_4(V(x)),\ \forall x \in \mathbb{R}^n,\ \forall d \in D$$

for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *and* $\alpha_4 \in \mathcal{K}$.

The given Lyapunov conditions have been developed in [17] to the case of non-UBIBS systems. In the papers [18], [19] these characterizations of SIuOS property have been extended to the case when f is locally Lipschitz continuous on $\mathbb{R}^n \setminus \{x \in \mathbb{R}^n : h(x) = 0\} \times \mathbb{R}^s$ and continuous everywhere, $h : \mathbb{R}^n \to \mathbb{R}^p$ is continuously differentiable (in that work the case $D = \{0\}$ was considered, but the proof stays valid under mild modifications for any compact set D).

Following [19], the accelerated output convergence rates can be defined as follows.

Definition 3. A forward complete system (1) is uniformly finite-time output stable (uFTOS) if it is uOS and for any $x_0 \in \mathbb{R}^n$ there exists $T_{x_0} \in [0, +\infty)$ such that $Y(t, x_0, d) = 0$ for all $t \geq T_{x_0}$ and $d \in \mathcal{D}$. The function $T_0(x_0) = \inf\{T_{x_0} \geq 0 : Y(t, x_0, d) = 0\ \forall t \geq T_{x_0},\ \forall d \in \mathcal{D}\}$ is called the settling-time function.

A system (1) is called uniformly nearly fixed-time output stable (unFxTOS) if (1) is uOS and for any $\rho > 0$ there exists $T_\rho \in [0, +\infty)$ such that $\|Y(t, x_0, d)\| \leq \rho$ for all $t \geq T_\rho$, all $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{D}$.

The system (1) is uniformly fixed-time output stable (uFxTOS) if it is uFTOS and unFxTOS simultaneously, i.e., $\sup_{x_0\in\mathbb{R}^n} T_0(x_0) < +\infty$.

Note that for unFxTOS systems, it is possible that $\sup_{\rho>0} T_\rho = +\infty$. If $h(x) = x$, then these properties are reduced to the conventional (uniform) finite/fixed-time stability [20], [21]. The respective Lyapunov characterizations of the output stability properties with accelerated convergence can be found in [22], [18], [19].

B. Homogeneity

For any $r_i > 0$, $i = \overline{1, n}$ and $\lambda > 0$, define the vector of weights $\mathbf{r} = [r_1, ..., r_n]$ and the dilation matrix $\Lambda_r(\lambda) = \text{diag}\{\lambda^{r_i}\}_{i=1}^n$; $r_{\min} = \min_{i=\overline{1,n}} r_i$ and $r_{\max} = \max_{i=\overline{1,n}} r_i$.

Definition 4. [23], [7] A function $h : \mathbb{R}^n \to \mathbb{R}$ is called $\mathbf{r}$-homogeneous, if for some $\nu \in \mathbb{R}$ the relation

$$h(\Lambda_r(\lambda)x) = \lambda^\nu h(x)$$

holds for any $x \in \mathbb{R}^n$ and all $\lambda > 0$.

A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is called $\mathbf{r}$-homogeneous, if for some $\nu \geq -r_{\min}$ the relation

$$f(\Lambda_r(\lambda)x) = \lambda^\nu \Lambda_r(\lambda) f(x)$$

holds for any $x \in \mathbb{R}^n$ and all $\lambda > 0$.

In both cases, the constant ν is called the degree of homogeneity.

A dynamical system

$$\dot{x}(t) = f(x(t)),\ t \geq 0,\ x(t) \in \mathbb{R}^n$$

is called $\mathbf{r}$-homogeneous of degree ν if this property is satisfied for f in the sense of Definition 4.

For any $x \in \mathbb{R}^n$ and $\varpi \geq r_{\min}$, a homogeneous norm can be defined as follows

$$\|x\|_r = \left(\sum_{i=1}^n |x_i|^{\varpi/r_i}\right)^{1/\varpi}.$$

For all $x \in \mathbb{R}^n$, its Euclidean norm $\|x\|$ is related with the homogeneous one:

$$\underline{\sigma}_r(\|x\|_r) \leq \|x\| \leq \bar{\sigma}_r(\|x\|_r)$$

for some $\underline{\sigma}_r, \bar{\sigma}_r \in \mathcal{K}_\infty$ [24].

III. Problem statement

Consider the problem of accelerated stabilization (in the sense of convergence rates given in Definition 3) of a controllable system of the form:

$$\dot{y}(t) = X_0(y(t)) + \sum_{i=1}^{m} X_i(y(t))u_i(t),\ t \geq 0, \tag{2}$$

where $y(t) \in \mathbb{R}^p$ is the state of (2) and $u(t) \in \mathbb{R}^m$ is the control to be designed; $X_j : \mathbb{R}^p \to \mathbb{R}^p$ are continuous vector fields for $j = \overline{0, m}$. If $X_0(y) = 0$, then (2) becomes driftless; such models frequently appear in the study of mechanical systems with nonholonomic constraints [1]. Since asymptotic stabilization of systems of this kind by a continuous feedback $u(t) = k(y(t))$ (with a continuous function $k : \mathbb{R}^p \to \mathbb{R}^m$) is impossible [25], time-varying controls $u(t) = k(t, y(t))$ (with continuous $k : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^m$) are frequently used [26], [1], [27]. In such a case, the time dependence can be modeled by introducing auxiliary state variables $z(t) \in \mathbb{R}^s$ in the system dynamics, for which there is no requirement for convergence, and the problem of accelerated output $y(t)$ stabilization can be analyzed for an autonomous extended model with the combined state $x(t) = [y^\top(t)\ z^\top(t)]^\top$. Other situations, when additional dynamics can appear, correspond to the trajectory tracking or disturbance compensation scenarios, then $z(t)$ (or a function of it) becomes the reference or the internal model state; the adaptive control, where $z(t)$ represents the parameter estimates, is another example.

In [1] the theory of homogeneous systems was applied to design a feedback law guaranteeing exponential stability of the closed-loop driftless dynamics at the origin. The method is based on scaling of existing feedback functions, which provide only asymptotic (non-exponential) stabilization. A homogeneous norm was used for scaling, similarly as it has been suggested for linear systems in [8], or using implicit Lyapunov functions in [9], [10], [11], [12]. In this work, utilizing similar ideas and extending the theory of homogeneous systems to the case of partial symmetry and stabilization, finite/fixed-time stabilizing feedbacks are synthesized.

IV. Partial homogeneity

Consider an explicit decomposition of (1) for $d \equiv 0$ into output dynamics and the remaining variables (assume such a representation exists):

$$\dot{y}(t) = F(y(t), z(t)), \qquad \dot{z}(t) = G(y(t), z(t)), \tag{3}$$

where $y(t) \in \mathbb{R}^p$ and $z(t) \in \mathbb{R}^s$ are the components of the state vector $x(t) = [y(t)^\top\ z(t)^\top]^\top \in \mathbb{R}^n$ with $n = p + s$, $F : \mathbb{R}^p \times \mathbb{R}^s \to \mathbb{R}^p$ and $G : \mathbb{R}^p \times \mathbb{R}^s \to \mathbb{R}^s$ are locally Lipschitz continuous functions on $\mathbb{R}^p \setminus \{0\} \times \mathbb{R}^s$ and continuous everywhere, and $F(0, z) = 0$ for all $z \in \mathbb{R}^s$. For any initial conditions $y_0 \in \mathbb{R}^p$ and $z_0 \in \mathbb{R}^s$ denote the respective solutions of (3) by $Y(t, y_0, z_0)$ and $Z(t, y_0, z_0)$ for those $t \geq 0$ for which these solutions exist. For brevity of exposition, assume in the sequel that these solutions are defined for all $t \geq 0$, i.e., (3) is forward complete.

Assumption 1. *There exists a vector of weights* $\mathbf{r} = [r_1, ..., r_n]$ *with* $r_i > 0$, $i = \overline{1, n}$ *such that*

$$F(\Lambda_r(\lambda)y, z) = \lambda^\nu \Lambda_r(\lambda)F(y, z),\ G(\Lambda_r(\lambda)y, z) = \lambda^\nu G(y, z)$$

for all $\lambda > 0$, $y \in \mathbb{R}^p$, $z \in \mathbb{R}^s$ *with some* $\nu \geq -r_{\min}$.

Hence, according to Definition 4, F is $\mathbf{r}$-homogeneous of degree ν considering z as a constant.

Proposition 1. *Under Assumption 1, if* $Y(t, y_0, z_0)$ *and* $Z(t, y_0, z_0)$ *are solutions of* (3) *for some* $y_0 \in \mathbb{R}^p$ *and* $z_0 \in \mathbb{R}^s$*, then* $\Lambda_r(\lambda)Y(\lambda^\nu t, y_0, z_0)$ *and* $Z(\lambda^\nu t, y_0, z_0)$ *are solutions to* (3) *for initial conditions* $\Lambda_r(\lambda)y_0$ *and* z_0 *for any* $\lambda > 0$.

Due to the symmetry of solutions, as in the case of the conventional $\mathbf{r}$-homogeneity [28], under Assumption 1, the local output stability of (3) implies global one:

Lemma 1. *Let there exist* $\rho > 0$ *and* $\beta \in \mathcal{KL}$ *such that* $\|Y(t, y_0, z_0)\| \leq \beta(\|y_0\|, t)$ *for all* $t \geq 0$, $z_0 \in \mathbb{R}^s$ *and all* $y_0 \in \mathbb{R}^p$ *with* $\|y_0\| \leq \rho$*. Then, under Assumption 1, the system* (3) *is SIuOS.*

Following [28], a compact set $\mathcal{A} \subset \mathbb{R}^p$ is called *partially positively invariant* for the system (3) if $y_0 \in \mathcal{A}$ and $z_0 \in \mathbb{R}^s$ implies that $Y(t, y_0, z_0) \in \mathcal{A}$ for all $t \geq 0$. Moreover, $\mathcal{A}$ is called *strictly partially positively invariant* if $Y(t, y_0, z_0) \in \text{int}(\mathcal{A})$ for all $t > 0$.

Lemma 2. *Under Assumption 1, if the system* (3) *has a compact partially positively invariant set* $\mathcal{A} \subset \mathbb{R}^p$ *with* $0 \in$ *int*$(\mathcal{A})$*, then* $\|Y(t, y_0, z_0)\|_r \leq \kappa\|y_0\|_r$ *for all* $t \geq 0$*, all* $y_0 \in \mathbb{R}^p$ *and* $z_0 \in \mathbb{R}^s$ *with some* $\kappa > 0$*. In addition, if* $\mathcal{A}$ *is strictly partially positively invariant for* (3)*, then it is uOS.*

It is a well-known fact that any (locally) asymptotically stable homogeneous system is globally finite-time or nearly fixed-time stable provided that it possesses negative or positive homogeneity degree, respectively [28], [7]. The same convergence characteristics for the output stability can be obtained for (3):

Proposition 2. *Under Assumption 1, if* $\|Z(t, y_0, z_0)\| \leq \sigma(\|z_0\|) + \sigma_0$ *for all* $t \geq 0$*, all* $y_0 \in \mathbb{R}^p$ *and* $z_0 \in \mathbb{R}^s$*, for some* $\sigma \in \mathcal{K}$ *and* $\sigma_0 \geq 0$*, then a uOS system* (3) *is uFTOS or unFxTOS provided that the homogeneity degree* ν *is negative or positive, respectively, while for* $\nu = 0$ *the convergence rate of* $Y(t, y_0, z_0)$ *is exponential.*

Remark 1. The requirement of the above results that the output stability properties are verified for all $z_0 \in \mathbb{R}^s$ can be replaced by $z_0 \in \mathcal{Z}$ for some compact set $\mathcal{Z} \subset \mathbb{R}^s$. In this case, global results on output y stability and convergence are preserved but locally in z, for the initial conditions in $\mathcal{Z}$.

The results above extend [28] to the case of partial homogeneity as defined in Assumption 1.

V. Control design

We now demonstrate how these theoretical findings can be used to accelerate time-varying stabilizing feedback for a class of systems (2) with non-zero homogeneity degree. Our approach extends the one for the case of zero homogeneity degree in [1]. To this end we need the following hypotheses:

Assumption 2. *There exists a vector of weights* $\mathbf{r} = [r_1, ..., r_p]$ *with* $r_i > 0$, $i = \overline{1,p}$ *such that*

$$X_j(\Lambda_r(\lambda)y) = \lambda^{\nu_j}\Lambda_r(\lambda)X_j(y),\ \nu_j \geq -r_{\min},\ j = \overline{0,m}$$

holds for all $\lambda > 0$ *and* $y \in \mathbb{R}^p$.

Note that Assumption 2 does not imply any kind of homogeneity of (2).

Assumption 3. *There is a locally Lipschitz continuous vector field* $g : \mathbb{R}^s \to \mathbb{R}^s$ *such that the system*

$$\dot{z}(t) = g(z(t)),\ t \geq 0, \tag{4}$$

with $z(t) \in \mathbb{R}^s$, *admits a compact invariant set* $\mathcal{Z} \subset \mathbb{R}^s$.

A signal generator system (4) is defined in Assumption 3, which introduces the time dependence into the control laws:

Assumption 4. *There exist locally Lipschitz continuous functions* $U_i : \mathbb{R}^p \times \mathbb{R}^s \to \mathbb{R}$, $i = \overline{1,m}$ *and a smooth* $V : \mathbb{R}^p \times \mathbb{R}^s \to \mathbb{R}_+$ *such that for some* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *and* $\alpha_3 \in \mathcal{K}$,

$$\alpha_1(\|y\|) \leq V(y,z) \leq \alpha_2(\|y\|),$$
$$\frac{\partial V(y,z)}{\partial y} f(y,z) + \frac{\partial V(y,z)}{\partial z} g(z) \leq -\alpha_3(\|y\|)$$

hold for all $y \in \mathcal{X} \subset \mathbb{R}^p$ *(with* $0 \in int(\mathcal{X})$*),* $z \in \mathcal{Z}$, *where*

$$f(y,z) = X_0(y) + \sum_{i=1}^{m} X_i(y)U_i(y,z).$$

This assumption implies that for the system (2), (4) there exist non-autonomous (dependent on z) controls U_i, $i = \overline{1,m}$ that render the closed-loop dynamics SIuOS with respect to the output y at least locally in $\mathcal{X} \times \mathcal{Z}$ with a known Lyapunov function V.

Without loosing generality, assume that the set $\mathcal{V} = \{y \in \mathbb{R}^p, z \in \mathbb{R}^s : V(y,z) \leq 1\}$ belongs to the interior of $\mathcal{X} \times \mathcal{Z}$ (this can be always guaranteed by scaling V). Define a map $\psi : \Omega \to (0, +\infty)$, where $\Omega = \mathbb{R}^p \setminus \{0\} \times \mathcal{Z}$, as a solution of the equation:

$$V(\Lambda_r(\psi(y,z))y, z) = 1. \tag{5}$$

Such a $\psi(y,z)$ always exists since V is assumed to be positive definite and radially unbounded in y, while the transformation $\Lambda_r(\psi(y,z))y$ linearly scales the homogeneous norm of y (i.e., $\|\Lambda_r(\psi(y,z))y\|_r = \psi(y,z)\|y\|_r$) and the homogeneous norm is equivalent to the standard one $\|\cdot\|$. According to the Implicit Function Theorem, ψ is locally uniquely defined provided that

$$\sum_{k=1}^{p} \frac{\partial V(\Lambda_r(\psi(y,z))y, z)}{\partial y_k} r_k \psi(y,z)^{r_k - 1} y_k \neq 0$$

for all $(y,z) \in \Omega$, which, considering the above scaling properties, is equivalent to the condition

$$\frac{\partial V(y,z)}{\partial y} \mathcal{H} y > 0,\ \forall (y,z) \in \partial\mathcal{V} \tag{6}$$

with $\mathcal{H} = \text{diag}\{r_k\}_{k=1}^p$, and obviously $\psi(y,z) = 1$ for $(y,z) \in \partial\mathcal{V}$ (the sign $>$ is chosen without loosing generality since $-\mathcal{H}$ is a negative definite matrix). Under (6) for any $(y,z) \in \Omega$ there is a unique solution $\psi(y,z)$ since this condition implies global monotonicity of $V(\Lambda_r(\psi)y, z)$ in ψ. Using the implicit differentiation rule, we get:

$$\begin{bmatrix} \frac{\partial \psi}{\partial y} & \frac{\partial \psi}{\partial z} \end{bmatrix} = -\psi \frac{\begin{bmatrix} \frac{\partial V(\Lambda_r(\psi)y,z)}{\partial y}\Lambda_r(\psi) & \frac{\partial V(\Lambda_r(\psi)y,z)}{\partial z} \end{bmatrix}}{\frac{\partial V(\Lambda_r(\psi)y,z)}{\partial y}\mathcal{H}\Lambda_r(\psi)y}$$

that is well defined under the restriction (6) for all $(y,z) \in \Omega$. Hence, if (6) is verified, ψ is a continuously differentiable function in Ω, and due to these regularity properties and positive definiteness of V with respect to y:

$$\lim_{\|y\| \to +\infty} \psi(y,z) = 0,\ \lim_{\|y\| \to 0} \psi(y,z) = +\infty.$$

Moreover, by construction

$$\psi^{-1}(\Lambda_r(\lambda)y, z) = \lambda\psi^{-1}(y,z)$$

for all $\lambda > 0$, all $(y,z) \in \mathbb{R}^p \times \mathcal{Z}$, i.e., ψ^{-1} is $\mathbf{r}$-homogeneous of degree 1 for z being constant, and it can be considered as a partially homogeneous norm with respect to y.

In order to use the results of Section IV, and recalling assumptions 2–4, let us define the control law for (2) as

$$u_i(y,z) = \begin{cases} K_i(y,z) & \text{if } y \neq 0, \\ 0 & \text{if } y = 0, \end{cases}\ i = \overline{1,m}, \tag{7}$$
$$K_i(y,z) = \psi^{-\nu_0+\nu_i}(y,z)U_i(\Lambda_r(\psi(y,z))y, z),$$

which is locally Lipschitz continuous in Ω provided that

$$\nu_0 \geq \max_{i=\overline{1,m}} \nu_i \tag{8}$$

and partially homogeneous:

$$K_i(\Lambda_r(\lambda)y, z) = \lambda^{\nu_0-\nu_i} K_i(y,z)$$

for all $\lambda > 0$ and all $(y,z) \in \mathbb{R}^p \times \mathcal{Z}$. Consider also the accelerated generator dynamics given by

$$\dot{z}(t) = G(y(t), z(t)) = \psi^{-\nu_0}(y(t), z(t))g(z(t)), \tag{9}$$

which is also locally Lipschitz continuous in Ω and partially homogeneous:

$$G(\Lambda_r(\lambda)y, z) = \lambda^{\nu_0} G(y, z)$$

for all $\lambda > 0$, $(y, z) \in \mathbb{R}^p \times \mathcal{Z}$.

Remark 2. To calculate the value $\psi(y, z)$, a bisection algorithm from [29] can be applied to (5). Typically, an approximate solution based on a small number of iterations is sufficient in practice.

For the closed-loop dynamics (2), (7) define

$$F(y, z) = X_0(y) + \sum_{i=1}^{m} X_i(y) K_i(y, z),$$

then due to established above symmetry relations and Assumption 2,

$$F(\Lambda_r(\lambda)y, z) = \lambda^{\nu_0} \Lambda_r(\lambda) F(y, z)$$

for all $\lambda > 0$, all $(y, z) \in \mathbb{R}^p \times \mathcal{Z}$. Therefore, Assumption 1 is verified for (2), (7), (9). Furthermore, it is easy to check that $\mathcal{V}$ is a strictly partially positively invariant set for this system under Assumption 4 (since $F(y, z) = f(y, z)$ and $G(y, z) = g(z)$ for $(y, z) \in \partial\mathcal{V}$), and by Lemma 2 the closed-loop system is uOS. In addition, Proposition 2 implies that uFTOS or unFxTOS properties for (2), (7), (9) with initial conditions in $\mathbb{R}^p \times \mathcal{Z}$ for $\nu_0 < 0$ or $\nu_0 > 0$, respectively, hold.

Therefore, the following result has been proven:

Theorem 2. *Let* (6) *and Assumption 2 for* (8) *and assumptions 3, 4 be satisfied. Then the system* (2) *under the dynamic control* (7), (9) *is uOS, and uFTOS for $\nu_0 < 0$ or unFxTOS for $\nu_0 > 0$ in $\mathbb{R}^p \times \mathcal{Z}$ (by considering y as the output).*

In [1] such a theorem was obtained for $\nu_0 = 0$ only (leading to exponential stabilization, see Proposition 2).

Remark 3. If $X_0(y) = 0$, combining uFTOS property for $\psi^{-1}(y, z) < 1$ and unFxTOS property for $\psi^{-1}(y, z) > 1$ by switching the homogeneity degree ν_0 in the control (7), (9) it is possible to get uFxTOS with the settling time uniformly bounded in $\mathbb{R}^p \times \mathcal{Z}$ (while preserving the continuity of the control).

The condition (6) can be relaxed by using the concept of generalized linear homogeneity [30], [7] instead of the weighted one.

It is also possible to extend these results to systems with exogenous inputs and input-to-output stability [16]. Furthermore, Assumption 4 can be relaxed by considering the Lyapunov functions V with semi-definite derivatives [1].

VI. Control of a nonholonomic integrator

For illustration, consider an example of (7) with $p = 3$, $m = 2$ and

$$X_0(y) = 0,\ X_1(y) = [1\ 0\ y_2]^\top,\ X_2(y) = [0\ 1\ 0]^\top,$$

which verifies Assumption 2 for $\mathbf{r} = [1, 1, 2]$, $\nu_1 = \nu_2 = -1$ and $\nu_0 \geq -1$. Take

$$g(z) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} z$$

with $z \in \mathbb{R}^2$ satisfying Assumption 3 for $\mathcal{Z} = \{z \in \mathbb{R}^2 : \|z\| = 1\}$. In [27] (see also [1]) it has been shown that the control law

$$U_1(y, z) = -y_1 + y_3 z_1,\ U_2(y, z) = -y_2 + y_3^2 z_2$$

stabilizes asymptotically this system with the Lyapunov function

$$V(y, z) = \frac{1}{2}[(y_1 - \frac{y_3}{2}(z_1 - z_2))^2 + (y_2 - \frac{y_3^2}{2}(z_1 + z_2))^2 + y_3^2],$$

$$\dot{V} = -(y_1 - \frac{y_3}{2}(z_1 - z_2))^2 - (y_2 - \frac{y_3^2}{2}(z_1 + z_2))^2 + \epsilon(y, z),$$

where $\epsilon(y, z)$ contains high-order terms. Hence, Assumption 4 is also true for $\mathcal{X} = \{y \in \mathbb{R}^3 : \|y\| \leq \varepsilon\}$ with $\varepsilon > 0$ being sufficiently small. In [1] it has been claimed that $\varepsilon > 1$ and (6) holds. Therefore, the conditions of Theorem 2 are verified for $\nu_0 = -0.5$ providing uFTOS or for $\nu_0 = 0.5$ ensuring unFxTOS for the control (7), (9):

$$K_1(y, z) = \psi^{-\nu_0}(y, z)(-y_1 + \psi(y, z) y_3 z_1),$$
$$K_2(y, z) = \psi^{-\nu_0}(y, z)(-y_2 + \psi^3(y, z) y_3^2 z_2),$$
$$G(x, z) = \psi^{-\nu_0}(y, z) \begin{bmatrix} z_2 \\ -z_1 \end{bmatrix},$$

where $\psi = \psi(y, z)$ is the solution of the equation:

$$\psi^2[(y_1 - \psi\frac{y_3}{2}(z_1 - z_2))^2 + (y_2 - \psi^3\frac{y_3^2}{2}(z_1 + z_2))^2 + \psi^2 y_3^2] = 2.$$

The results of MATLAB simulations of the closed-loop system using `ode45` are presented in figures 1 and 2 for $\nu_0 = -0.5$ and $\nu_0 = 0.5$, respectively. The algorithm from [29] was used to calculate ψ with 3 bisection iterations on each time step. On the plots, the state norm $\|y\|_r$ (with $\varpi = 1$) in logarithmic scale, the generator output z_1 and the controls u are shown versus the time t. As it can be observed, the state norm convergence is accelerated. Note that the frequency of oscillations of the generator variables increases over time while the controls are converging to zero. Note also that the original algorithm from [27], for the initial conditions 10 times smaller than the ones in Fig. 1, needs more than 10^3 sec to reduce the state norm below 10^{-1}, and the approach of [1], while faster, only reaches level of 10^{-3} in 10^2 sec. The jumps in the control values are due to the calculation of ψ; the size can be reduced if more bisection steps per time are used.

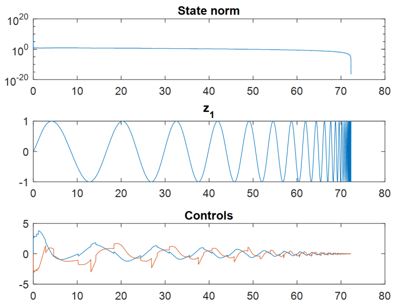

Figure 1. The results of finite-time stabilization: the time histories of $\|y\|_r$, z_1 and u.

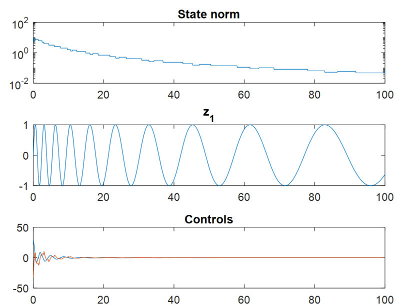

Figure 2. The results of nearly fixed-time stabilization: the time histories of $\|y\|_r$, z_1 and u.

VII. Conclusion

The paper extended several results of the theory of homogeneous systems to the case of systems homogeneous to only a part of the variables. These included scaling of the trajectories, global stability implications from local estimates or from existence of a (strictly) positively invariant set, and finite-time or nearly fixed-time convergence guarantees for negative or positive degrees of homogeneity, respectively. A constructive procedure to generate feedback laws achieving finite/fixed time convergence from a locally asymptotically stabilizing one has been developed, and the results were verified in simulations for a nonholonomic integrator.

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