

Stability and Performance Analysis of Discrete-Time ReLU Recurrent Neural Networks

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Abstract—This paper presents sufficient conditions for the stability and ℓ_2 -gain performance of recurrent neural networks (RNNs) with ReLU activation functions. These conditions are derived by combining Lyapunov/dissipativity theory with Quadratic Constraints (QCs) satisfied by repeated ReLUs. We write a general class of QCs for repeated ReLUs using known properties for the scalar ReLU. Our stability and performance condition uses these QCs along with a “lifted” representation for the ReLU RNN. We show that the positive homogeneity property satisfied by a scalar ReLU does not expand the class of QCs for the repeated ReLU. We present examples to demonstrate the stability / performance condition and study the effect of the lifting horizon.

I. INTRODUCTION

This paper considers the analysis of recurrent neural networks (RNNs) with ReLU activation functions. These are modeled by the interconnection of a discrete-time, linear time-invariant (LTI) system in feedback with a repeated ReLU. The goal is to derive sufficient conditions to prove stability and performance (as measured by the induced ℓ_2 gain) for this RNN. This work is motivated by the increasing interest in RNNs for inner-loop feedback control. Such RNNs can potentially improve performance over more standard LTI controllers. However, the analysis of closed-loop stability and performance is challenging due to the nonlinear activation functions in the RNN. The sufficient conditions in this paper are one step to address this issue.

Our technical approach combines multiple ingredients in the existing literature. First, we note that the scalar ReLU has slope restricted to $[0, 1]$. Hence the repeated ReLU satisfies a known quadratic constraint (QC) involving doubly hyperdominant matrices [1]–[3]. Second, we present QCs that are specific to repeated ReLUs building on work in [4]–[6]. Next, we write the ReLU RNN using a “lifted” representation over an N -step horizon [7]. This lifting allows us to construct more general QCs that hold for the ReLU across multiple time steps. Finally, we combine Lyapunov and dissipativity theory [8]–[11] with these QCs to obtain sufficient conditions for stability and performance of the lifted ReLU RNN. Stability and performance of the original (unlifted) ReLU RNN follows from this analysis.

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This paper adds to the broader literature on integral quadratic constraint (IQC) conditions [12]–[15]. Our stability/performance condition is similar to the discrete-time, IQC formulations in [16]–[23]. The most closely related conditions are in [17], [24], both of which use lifting and static QCs (rather than dynamic IQCs). There is also growing literature on using these techniques to analyze NNs and RNNs [4], [5], [25]–[27]. We specifically build on the results for ReLU RNNs in [4]–[6]. In particular, [4] derives QCs for ReLU and uses Lyapunov theory to prove stability for continuous-time ReLU RNNs. We build on this work in discrete-time and also prove ℓ_2 bounds. The work in [5] also addresses discrete-time ReLU RNNs, similar to our paper, but uses small-gain arguments. In contrast, we use QCs and Lyapunov/dissipativity theory leading to related, but different conditions.

Our contributions to this existing literature are as follows. First, we write a general class of QCs for repeated ReLU using known existing properties (Lemma 4). We show that the positive homogeneity property satisfied by scalar ReLUs, i.e. $\phi(\beta v) = \beta v$ for all $\beta \geq 0$, does not expand the class of QCs for repeated ReLUs (discussed in Section V-A). Second, we develop a stability condition combining QCs and a dissipation inequality (Section V-B). We use the discrete-time lifting, originally proposed in [7], that maintains the state dimension. This is different from the liftings used previously in [17], [24] for which the lifted state grows with the lifting horizon. Finally, we present examples to illustrate the effects of the ReLU QCs and lifting time horizon (Section VI). We show that the ReLU QCs can significantly reduce the conservatism as compared to QCs developed for the more general class of slope-restricted nonlinearities (although the performance is problem dependent).

II. NOTATION

This section briefly reviews basic notation regarding vectors, matrices, and signals. Let \mathbb{R}^n and $\mathbb{R}^{n \times m}$ denote the sets of real $n \times 1$ vectors and $n \times m$ matrices, respectively. Moreover, $\mathbb{R}_{\geq 0}^n$ and $\mathbb{R}_{\geq 0}^{n \times m}$ denote vectors and matrices of the given dimensions with non-negative entries. Let D^n denote the set of $n \times n$ diagonal matrices and $D_{\geq 0}^n$ the subset of diagonal matrices with non-negative entries. Finally, $M \in \mathbb{R}^{n \times n}$ is a *Metzler matrix* if the off-diagonal entries are non-negative, i.e. $M_{ij} \geq 0$ for $i \neq j$. A matrix $M \in \mathbb{R}^{n \times n}$ is *doubly hyperdominant* if the off-diagonal elements are non-positive, and both the row sums and column sums are non-negative.

Next, let \mathbb{N} denote the set of non-negative integers. Let $v: \mathbb{N} \rightarrow \mathbb{R}^n$ and $w: \mathbb{N} \rightarrow \mathbb{R}^n$ be real, vector-valued sequences.

Define the inner product $\langle v, w \rangle := \sum_{k=0}^{\infty} v(k)^\top w(k)$. The set ℓ_2 is an inner product space with sequences v that satisfy $\langle v, v \rangle < \infty$. The corresponding norm is $\|v\|_2 := \sqrt{\langle v, v \rangle}$.

III. PROBLEM STATEMENT

Consider the interconnection shown on the left of Figure 1 with a static nonlinearity Φ wrapped in feedback around the top channels of a nominal system G . This interconnection is denoted as $F_U(G, \Phi)$. The nominal part G is a discrete-time, linear time-invariant (LTI) system described by the following state-space model:

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 w(k) + B_2 d(k) \\ v(k) &= C_1 x(k) + D_{11} w(k) + D_{12} d(k) \\ e(k) &= C_2 x(k) + D_{21} w(k) + D_{22} d(k), \end{aligned} \quad (1)$$

where $x \in \mathbb{R}^{n_x}$ is the state. The inputs are $w \in \mathbb{R}^{n_w}$ and $d \in \mathbb{R}^{n_d}$ while $v \in \mathbb{R}^{n_v}$ and $e \in \mathbb{R}^{n_e}$ are outputs. The static nonlinearity $\Phi: \mathbb{R}^{n_v} \rightarrow \mathbb{R}_{\geq 0}^{n_w}$ maps v to w elementwise by $w_i = \phi(v_i)$ for $i = 1, \dots, n_w$ where $\phi: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ is the ReLU function. The ReLU, shown on the right of Figure 1, is:

$$\phi(v) = \begin{cases} 0 & \text{if } v < 0 \\ v & \text{if } v \geq 0 \end{cases}. \quad (2)$$

The repeated ReLU is the function $\Phi: \mathbb{R}^{n_v} \rightarrow \mathbb{R}_{\geq 0}^{n_w}$ defined by $\Phi(v) = (\phi(v_1), \phi(v_2), \dots, \phi(v_{n_v}))$ and ϕ is the scalar ReLU. The interconnection $F_U(G, \Phi)$ is known as a linear fractional transformation (LFT) in the robust control literature [28]. The interconnection has its roots in the Lur'e decomposition used in the absolute stability problem [11].

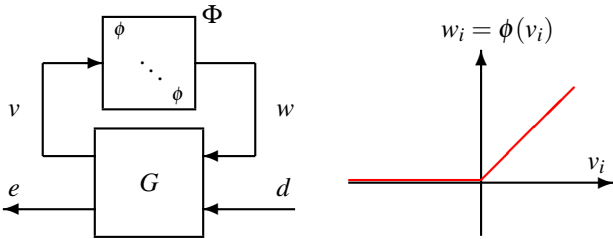


Fig. 1: Left: Interconnection $F_U(G, \Phi)$ of a nominal discrete-time LTI system G and repeated ReLU Φ . Right: Graph of scalar ReLU ϕ .

This feedback interconnection involves an implicit equation if $D_{11} \neq 0$. Specifically, the second equation in (1) combined with $w(k) = \Phi(v(k))$ yields:

$$v(k) = C_1 x(k) + D_{11} \Phi(v(k)) + D_{12} d(k). \quad (3)$$

This equation is *well-posed* if there exists a unique solution $v(k)$ for all values of $x(k)$ and $d(k)$. Well-posedness of this equation implies that the dynamic system $F_U(G, \Phi)$ is well-posed in the following sense: for all initial conditions $x(0) \in \mathbb{R}^{n_x}$ and inputs $d \in \ell_2$ there exists unique solutions x, v, w and $e \in \ell_2$ to the system $F_U(G, \Phi)$. There are simple sufficient conditions for well-posedness of (3), e.g. Lemma 1 in [4] (which relies on results in [29], [30]). Thus, we'll assume well-posedness for simplicity in the remainder of the paper.

A well-posed interconnection $F_U(G, \Phi)$ is *internally stable* if $x(k) \rightarrow 0$ from any initial condition $x(0)$ with $d(k) = 0$ for $k \in \mathbb{N}$. In other words, $F_U(G, \Phi)$ is internally stable if $x = 0$ is a globally asymptotically stable equilibrium point with no external input. A well-posed interconnection $F_U(G, \Phi)$ has *finite induced- ℓ_2 gain* if there exists $\gamma < \infty$ such that the output e generated by any $d \in \ell_2$ with $x(0) = 0$ satisfies $\|e\|_2 \leq \gamma \|d\|_2$. This bound is denoted as $\|F_U(G, \Phi)\|_{2 \rightarrow 2} \leq \gamma$. The goal of this paper is to derive sufficient conditions, using the properties of the ReLU, that verify $F_U(G, \Phi)$ is internally stable, has finite induced ℓ_2 gain, and can be used to compute an upper bound on γ .

IV. PRELIMINARY RESULTS

This section presents preliminary results from the literature on quadratic constraints (QCs) for repeated ReLU.

A. Quadratic Constraints for Slope-Restricted Functions

First, consider a scalar function $\phi: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ that satisfies $\phi(0) = 0$ and has slope restricted to $[0, 1]$, i.e.:

$$0 \leq \frac{\phi(v_2) - \phi(v_1)}{v_2 - v_1} \leq 1, \quad \forall v_1, v_2 \in \mathbb{R}, v_1 \neq v_2.$$

By putting $v_1 = 0$ in the inequality above, we get the $[0, 1]$ sector bound condition:

$$0 \leq \frac{\phi(v)}{v} \leq 1, \quad \forall v \in \mathbb{R}, v \neq 0$$

This subsection presents QCs that hold for any activation function ϕ that satisfies these conditions. This includes, but is not limited to, the ReLU.

The graph of ϕ is restricted to the $[0, 1]$ sector, i.e. its graph lies between lines that pass through the origin with slope 0 and 1. Thus the following QC holds $\forall v \in \mathbb{R}$ and $w = \phi(v)$:

$$w(v - w) \geq 0. \quad (4)$$

Moreover, any two points on the graph of ϕ are connected by a line with slope in $[0, 1]$. Hence the following QC holds for all $v_1, v_2 \in \mathbb{R}$, $w_1 = \phi(v_1)$, and $w_2 = \phi(v_2)$:

$$(w_2 - w_1)(v_2 - v_1 - (w_2 - w_1)) \geq 0. \quad (5)$$

Next, consider a repeated nonlinearity $\Phi: \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}^m$ that maps v to w elementwise $w_k = \phi(v_k)$ for $k = 1, \dots, m$. We can derive a QC for Φ based on scaled combinations of the sector and slope constraints for the data $\{(v_k, w_k)\}_{k=1}^m$. Specifically, for any non-negative constants λ_k and γ_{kj} :

$$\begin{aligned} \lambda_k w_k (v_k - w_k) &\geq 0 \quad \text{for } k = 1, \dots, m \\ \gamma_{kj} (w_k - w_j) (v_k - v_j - (w_k - w_j)) &\geq 0. \quad \text{for } k, j = 1, \dots, m. \end{aligned}$$

Summing over all these constraints gives an inequality of the form $w^\top Q_0 (v - w) \geq 0$ where

$$(Q_0)_{kj} = \begin{cases} \lambda_k + \sum_{k \neq j} (\gamma_{kj} + \gamma_{jk}) & \text{if } k = j \\ -(\gamma_{kj} + \gamma_{jk}) & \text{else} \end{cases}. \quad (6)$$

Q_0 is a symmetric, doubly hyperdominant matrix. Lemma 1, stated next, provides a more general QC for this case in that it allows for Q_0 to be non-symmetric.

Lemma 1: Let $\Phi: \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}^m$ be a repeated nonlinearity with $\Phi(0) = 0$ and slope restricted (elementwise) to $[0, 1]$. If $Q_0 \in \mathbb{R}^{m \times m}$ is doubly hyperdominant then the following QC holds $\forall v \in \mathbb{R}^m$ and $w = \Phi(v)$:

$$\begin{bmatrix} v \\ w \end{bmatrix}^\top \begin{bmatrix} 0 & Q_0^\top \\ Q_0 & -(Q_0 + Q_0^\top) \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \geq 0. \quad (7)$$

Proof: This follows from the results in Section 3.5 of [2]. Specifically, for any pair v and $w = \Phi(v)$, define $\tilde{v} := v - w \in \mathbb{R}^m$. Equation 5 implies that $\{(\tilde{v}_k, w_k)\}_{k=1}^m$ is similarly ordered data, i.e. $\tilde{v}_k > \tilde{v}_j$ implies $w_k \geq w_j$. Moreover, it follows from (4) that the data is unbiased, i.e. $w_k \tilde{v}_k \geq 0$ for $k = 1, \dots, m$. Theorem 3.8 in [2] implies that if Q_0 is doubly hyperdominant then $w^\top Q_0 \tilde{v} \geq 0$. Thus $w^\top Q_0 (v - w) + (v - w)^\top Q_0^\top w \geq 0$. ■

The proof of Theorem 3.8 in [2] relies on a cyclic rearrangement inequality satisfied by similarly ordered, unbiased data. In fact, the doubly hyperdominance condition is the largest class of QCs that holds for slope-restricted functions whose graph passes through the origin. A precise statement of this fact is given in Theorem 1 of [3].¹

B. Quadratic Constraints for ReLU

The previous subsection presented QCs for functions with slope restricted to $[0, 1]$. This section derives several additional QCs that are specific to the repeated ReLU. This builds on prior work in [4], [5]. The starting point for these QCs are the following properties for the scalar ReLU that have been used previously in the literature [4], [5], [31], [32]:

- 1) *Positivity:* The scalar ReLU is non-negative for all inputs: $\phi(v) \geq 0 \forall v \in \mathbb{R}$.
- 2) *Positive Complement:* The scalar ReLU satisfies $\phi(v) \geq v \forall v \in \mathbb{R}$.
- 3) *Complementarity:* In addition to the $[0, 1]$ sector, the ReLU satisfies the stronger complementarity property: its graph is identically on the line of slope 0 (when $v \leq 0$) or the line of slope 1 (when $v \geq 0$). Thus, $\phi(v)(v - \phi(v)) = 0 \forall v \in \mathbb{R}$.
- 4) *Positive Homogeneity:* The scalar ReLU is homogeneous for all non-negative constants: $\phi(\beta v) = \beta \phi(v) \forall v \in \mathbb{R}$ and $\forall \beta \in \mathbb{R}_{\geq 0}$.

Properties 1-3 can be combined to construct QCs for the repeated ReLU. Lemmas 2 and 3 below are variations of results in [4].

Lemma 2: Let $\Phi: \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}^m$ be a repeated ReLU. If $Q_1 \in D^m$ then the following QC holds $\forall v \in \mathbb{R}^m$ and $w = \Phi(v)$:

$$\begin{bmatrix} v \\ w \end{bmatrix}^\top \begin{bmatrix} 0 & Q_1 \\ Q_1 & -2Q_1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = 0. \quad (8)$$

Proof: Q_1 is diagonal by assumption so that:

$$\begin{bmatrix} v \\ w \end{bmatrix}^\top \begin{bmatrix} 0 & Q_1 \\ Q_1 & -2Q_1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = 2 \sum_{k=1}^m (Q_1)_{kk} w_k (v_k - w_k).$$

¹Theorem 1 in [3] is stated for repeated monotone nonlinearities. A similar fact holds for nonlinearities with slope restricted to $[0, 1]$ by a transformation of the input-output data.

This is equal to zero based on the complementarity property for each $\{(v_k, w_k)\}_{k=1}^m$. Note that the diagonal entries of Q_1 are not necessarily non-negative. ■

Lemma 3: Let $\Phi: \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}^m$ be a repeated ReLU. If $Q_2 = Q_2^\top, Q_3 = Q_3^\top, Q_4 \in \mathbb{R}_{\geq 0}^{m \times m}$ then the following QC holds $\forall v \in \mathbb{R}^m$ and $w = \Phi(v)$:

$$\begin{bmatrix} v \\ w \end{bmatrix}^\top \begin{bmatrix} Q_2 & -(Q_2 + Q_4^\top) \\ -(Q_2 + Q_4) & Q_2 + Q_3 + Q_4 + Q_4^\top \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \geq 0. \quad (9)$$

Proof: The positivity and positive complement properties are linear constraints on $\{(v_k, w_k)\}_{k=1}^m$. These can be combined to form QCs on pairs of points. The QCs below hold for any non-negative $(Q_2)_{kj}, (Q_3)_{kj}$, and $(Q_4)_{kj}$:

$$(Q_2)_{kj} (w_k - v_k)(w_j - v_j) \geq 0, \quad (10)$$

$$(Q_3)_{kj} w_k w_j \geq 0, \quad (11)$$

$$(Q_4)_{kj} w_k (w_j - v_j) \geq 0. \quad (12)$$

Equations 10 and 11 are equivalent to $(w - v)^\top Q_2 (w - v) \geq 0$ and $w^\top Q_3 w \geq 0$, respectively. These QCs are quadratic forms and hence we can assume $Q_2 = Q_2^\top$ and $Q_3 = Q_3^\top$ without loss of generality. Equation 12 is equivalent to $w^\top Q_4 (w - v) + (w - v)^\top Q_4^\top w \geq 0$. ■

V. MAIN RESULTS

The section presents the main results: sufficient conditions for stability and performance of the ReLU RNNs. These sufficient conditions are based on dissipativity theory and quadratic constraints (QCs) for the repeated ReLU. We also combine existing QCs for repeated ReLU into a single general form for use in our stability condition.

A. Combined Set of QCs for Repeated ReLU

The next result combines all the QCs in the Lemmas 1-3. This is a general QC based on the $[0, 1]$ sector / slope restrictions combined with the known properties for scalar ReLU.

Lemma 4: Let $\Phi: \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}^m$ be a repeated ReLU. Let $Q_2 = Q_2^\top, Q_3 = Q_3^\top \in \mathbb{R}_{\geq 0}^{m \times m}$ and $\tilde{Q} \in \mathbb{R}^{m \times m}$ be given with \tilde{Q} a Metzler matrix. Then the following QC holds $\forall v \in \mathbb{R}^m$ and $w = \Phi(v)$:

$$\begin{bmatrix} v \\ w \end{bmatrix}^\top \begin{bmatrix} Q_2 & -\tilde{Q}^\top - Q_2 \\ -\tilde{Q} - Q_2 & Q_2 + Q_3 + \tilde{Q} + \tilde{Q}^\top \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \geq 0. \quad (13)$$

Proof: Gather all the QCs given in Lemmas 1-3. Define $\tilde{Q} = Q_4 - Q_0 - Q_1$ where $Q_0 \in \mathbb{R}^{m \times m}$ is doubly hyperdominant, $Q_1 \in D^m$ and $Q_4 \in \mathbb{R}_{\geq 0}^{m \times m}$. The diagonal entries of \tilde{Q} can be anything while the off-diagonal entries must be non-negative. Hence \tilde{Q} is a Metzler matrix. ■

Finally, the next lemma exploits the positive homogeneity property of the ReLU. Related results are given in Section 4.3 of [5].

Lemma 5: Let $\Phi : \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}^m$ be a repeated ReLU. Assume Φ satisfies the QC defined by $M = M^\top \in \mathbb{R}^{2m \times 2m}$, i.e. $\forall v \in \mathbb{R}^m$ and $w = \Phi(v)$:

$$\begin{bmatrix} v \\ w \end{bmatrix}^\top \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\top & M_{22} \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \geq 0. \quad (14)$$

If $\Lambda \in D_{\geq 0}^m$ then Φ also satisfies the QC defined by $\begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} M \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \in \mathbb{R}^{2m \times 2m}$, i.e. $\forall \bar{v} \in \mathbb{R}^m$ and $\bar{w} = \Phi(\bar{v})$:

$$\begin{bmatrix} \bar{v} \\ \bar{w} \end{bmatrix}^\top \begin{bmatrix} \Lambda^\top M_{11} \Lambda & \Lambda^\top M_{12} \Lambda \\ (\Lambda^\top M_{12} \Lambda)^\top & \Lambda^\top M_{22} \Lambda \end{bmatrix} \begin{bmatrix} \bar{v} \\ \bar{w} \end{bmatrix} \geq 0 \quad (15)$$

Proof: Take any $\bar{v} \in \mathbb{R}^m$ and $\bar{w} = \Phi(\bar{v})$. Let Λ is a diagonal matrix with non-negative entries. Apply positive homogeneity of the scalar ReLU, element-wise, to conclude that $\Lambda \bar{w} = \Phi(\Lambda \bar{v})$. Next, define $v = \Lambda \bar{v}$ and $w = \Lambda \bar{w}$ so that $w = \Phi(v)$. The QC in (14) holds, by assumption, for (v, w) . Substitute $v = \Lambda \bar{v}$ and $w = \Lambda \bar{w}$ to show the QC in (15) also holds for (\bar{v}, \bar{w}) . ■

The positive homogeneity property seems to generalize the class of QCs for repeated ReLU by allowing for an additional scaling $\Lambda \in D_{\geq 0}^m$. However, this additional freedom does not reduce the conservatism of stability tests that incorporate QCs based on Lemma 4. Specifically, if $Q_2, Q_3 \in \mathbb{R}_{\geq 0}^{m \times m}$ then $\Lambda Q_2 \Lambda, \Lambda Q_3 \Lambda \in \mathbb{R}_{\geq 0}^{m \times m}$ for all $\Lambda \in D_{\geq 0}^m$. Similarly, if $\tilde{Q} \in \mathbb{R}^{m \times m}$ is a Metzler matrix then $\Lambda \tilde{Q} \Lambda$ is also a Metzler matrix for all $\Lambda \in D_{\geq 0}^m$. Thus positive homogeneity does not provide additional freedom when combined with the most general QC given in Lemma 4, i.e. we get the same class of QCs. However, the positive homogeneity scaling Λ can provide an additional degree of freedom when combined with more restricted subsets of QCs. For example, suppose we use the restricted class of QCs based on slope restrictions (Lemma 1). In this case, the positive homogeneity property can provide a benefit because if $Q_0 \in \mathbb{R}^{m \times m}$ is doubly hyperdominant and $\Lambda \in D_{\geq 0}^m$ then $\Lambda Q_0 \Lambda$ is not necessarily doubly hyperdominant. In this case, the scaling introduced by positivity homogeneity (Lemma 5) will generalize beyond the initial (restricted) class of slope-restricted QCs.

B. Conditions for Stability and Performance

This section presents a sufficient condition for the feedback connection $F_U(G, \Phi)$ to be stable and have bounded induced ℓ_2 -gain. The condition uses a lifted plant, defined below, combined with Lyapunov/dissipativity theory and the QCs derived in the previous subsection.

The first step is to create a lifted representation for $F_U(G, \Phi)$ where G is the LTI system (1) and Φ is the repeated ReLU. Define the stacked vectors for both the input/output signals of G as:

$$\begin{aligned} V_N(k) &= [v^\top(k) \quad v^\top(k-1) \quad \cdots \quad v^\top(k-N+1)]^\top, \\ W_N(k) &= [w^\top(k) \quad w^\top(k-1) \quad \cdots \quad w^\top(k-N+1)]^\top, \\ D_N(k) &= [d^\top(k) \quad d^\top(k-1) \quad \cdots \quad d^\top(k-N+1)]^\top, \\ E_N(k) &= [e^\top(k) \quad e^\top(k-1) \quad \cdots \quad e^\top(k-N+1)]^\top. \end{aligned}$$

These vectors stack the signals over an N step horizon from $k-N+1$ to k .

The nominal plant dynamics can be lifted [7] to evolve the state N steps from $x(k-N+1)$ to $x(k+1)$. The lifted nominal plant, denoted G_N , has the form:

$$\begin{aligned} x(k+1) &= A^N x(k-N+1) + B_{1,N} W_N(k) + B_{2,N} D_N(k) \\ V_N(k) &= C_{1,N} x(k-N+1) + D_{11,N} W_N(k) + D_{12,N} D_N(k) \\ E_N(k) &= C_{2,N} x(k-N+1) + D_{21,N} W_N(k) + D_{22,N} D_N(k). \end{aligned} \quad (16)$$

The state matrices ($B_{1,N}$, $B_{2,N}$, etc.) can be constructed for the specified horizon N from the state-matrices of the original nominal plant. The lifted plant given here has the same state dimension as the original plant but the input/output dimensions are stacked, e.g. $W_N \in \mathbb{R}^{n_w N}$ and $B_{1,N} \in \mathbb{R}^{n_x \times n_w N}$. A related, but different, lifting is used in [17], [18] where the state-dimension grows with the horizon N .

The repeated ReLU can also be lifted. Define $\Phi_N : \mathbb{R}^{n_v N} \rightarrow \mathbb{R}^{n_w N}$ by applying the scalar ReLU elementwise to the input. In summary, the lifted system $F_U(G_N, \Phi_N)$ maps $D_N(k)$ to $E_N(k)$ based on the interconnection of G_N in (16) and $W_N(k) = \Phi_N(V_N(k))$.

We next state the stability and performance condition for the lifted system. Define a linear matrix inequality (LMI) with the lifted system of G_N :

$$\begin{aligned} LMI(P, M, \gamma^2) &:= \\ &\begin{bmatrix} (A^N)^\top P (A^N) - P & (A^N)^\top P B_{1,N} & (A^N)^\top P B_{2,N} \\ B_{1,N}^\top P (A^N) & B_{1,N}^\top P B_{1,N} & B_{1,N}^\top P B_{2,N} \\ B_{2,N}^\top P (A^N) & B_{2,N}^\top P B_{1,N} & B_{2,N}^\top P B_{2,N} - \gamma^2 I \end{bmatrix} \\ &+ \begin{bmatrix} C_{2,N}^\top \\ D_{21,N}^\top \\ D_{22,N}^\top \end{bmatrix} \begin{bmatrix} C_{2,N}^\top \\ D_{21,N}^\top \\ D_{22,N}^\top \end{bmatrix}^\top + \begin{bmatrix} C_{1,N}^\top & 0 \\ D_{11,N}^\top & I \\ D_{12,N}^\top & 0 \end{bmatrix} M \begin{bmatrix} C_{1,N}^\top & 0 \\ D_{11,N}^\top & I \\ D_{12,N}^\top & 0 \end{bmatrix}^\top \end{aligned} \quad (17)$$

The next theorem provides a stability and performance condition for the ReLU RNN formulated with this LMI. The proof uses a QC for the lifted ReLU and a standard Lyapunov / dissipativity argument.

Theorem 1: Consider the ReLU RNN $F_U(G, \Phi)$ with the LTI system G defined in (1). Assume this interconnection is well-posed. Let G_N and $\Phi_N : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be the lifted system and lifted ReLU for some $N \in \mathbb{N}$ with dimension $m := n_v N$.

Let $Q_2 = Q_2^\top$, $Q_3 = Q_3^\top \in \mathbb{R}_{\geq 0}^{m \times m}$ and $\tilde{Q} \in \mathbb{R}^{m \times m}$ be given with \tilde{Q} a Metzler matrix. Define $M = M^\top \in \mathbb{R}^{2m \times 2m}$ as follows:

$$M := \begin{bmatrix} Q_2 & -\tilde{Q}^\top - Q_2 \\ -\tilde{Q} - Q_2 & Q_2 + Q_3 + \tilde{Q} + \tilde{Q}^\top \end{bmatrix}, \quad (18)$$

Then the lifted ReLU Φ_N satisfies the QC defined by M , i.e. $\begin{bmatrix} v \\ w \end{bmatrix}^\top M \begin{bmatrix} v \\ w \end{bmatrix} \forall v \in \mathbb{R}^m$ and $w = \Phi_N(v)$.

Moreover, if there exists a positive semidefinite matrix $P = P^\top \in \mathbb{R}^{n_x \times n_x}$ and scalar $\gamma \geq 0$ such that $LMI(P, M, \gamma^2) < 0$, then the ReLU RNN $F_U(G, \Phi)$ is internally stable and has $\|F_U(G, \Phi)\|_{2 \rightarrow 2} < \gamma$.

Proof: This theorem is a standard dissipation result [8]–[11], [21]. A proof is given for completeness.

It follows directly from Lemma 4 that Φ_N satisfies the QC defined by M . We'll first use this QC and the LMI condition to show the lifted system $F_U(G_N, \Phi_N)$ is stable and with ℓ_2 gain bounded by γ .

If the LMI is strictly feasible, it remains feasible under small perturbations: $LMI(P + \varepsilon I, M, \gamma^2) + \varepsilon I < 0$ for some sufficiently small $\varepsilon > 0$. Moreover, well-posedness of $F_U(G_N, \Phi_N)$ follows from the assumption that $F_U(G, \Phi)$ is well-posed. Hence the lifted system $F_U(G_N, \Phi_N)$ has a unique causal solution (x, W_N, V_N, E_N) for any given initial condition $x_N(0)$ and input $D_N \in \ell_2^{d_N}$.

Define a storage function by $V(x) := x^\top (P + \varepsilon I)x$. Left and right multiply the perturbed LMI by $[x(k)^\top, W_N(k)^\top, D_N(k)^\top]$ and its transpose. The result, applying the lifted dynamics (16), gives the following condition:

$$V(x(k+1)) - V(x(k)) - \gamma^2 D_N(k)^\top D_N(k) + E_N(k)^\top E_N(k) + \begin{bmatrix} V_N(k) \\ W_N(k) \end{bmatrix}^\top M \begin{bmatrix} V_N(k) \\ W_N(k) \end{bmatrix} + \varepsilon \begin{bmatrix} x(k) \\ W_N(k) \\ D_N(k) \end{bmatrix}^\top \begin{bmatrix} x(k) \\ W_N(k) \\ D_N(k) \end{bmatrix} \leq 0$$

The term involving M is non-negative as the lifted ReLU satisfies the QC defined by M . Thus the dissipation inequality simplifies to:

$$V(x(k+1)) - V(x(k)) + E_N(k)^\top E_N(k) \leq (\gamma^2 - \varepsilon) D_N(k)^\top D_N(k) - \varepsilon x(k)^\top x(k) \quad (19)$$

Note that the inequality still holds even though the term $-\varepsilon W_N^\top(k) W_N(k) \leq 0$ has been dropped.

First, note that internal stability of the lifted system $F_U(G_N, \Phi_N)$ follows from the inequality (19). Specifically, if $D_N(k) = 0$ for all k then (19) simplifies to the following Lyapunov inequality:

$$V(x(k+1)) - V(x(k)) \leq -\varepsilon x(k)^\top x(k)$$

Hence V is a Lyapunov function and the lifted system is globally asymptotically stable (Theorem 27 in Section 5.9 of [33]).

Next, note that the ℓ_2 -gain bound for the lifted system $F_U(G_N, \Phi_N)$ also follows from (19). Specifically, assume $x(0) = 0$ and $D_N \in \ell_2$. Summing (19) from $k = 0$ to $k = T - 1$ and using $V(x(0)) = 0$ yields:

$$V(x(T)) + \sum_{k=0}^{T-1} E_N(k)^\top E_N(k) \leq \sum_{k=0}^{T-1} (\gamma^2 - \varepsilon) D_N(k)^\top D_N(k)$$

Note that $V(x(T)) \geq 0$ because P is positive semidefinite. Moreover, the right side is upper bounded by $(\gamma^2 - \varepsilon) \|D_N\|_2^2$ for all $T \in \mathbb{N}$. This implies that $E_N \in \ell_2$ and $\|E_N\|_2 < \gamma \|D_N\|_2$.

In summary, the lifted system $F_U(G_N, \Phi_N)$ is internally stable and satisfies $\|F_U(G_N, \Phi_N)\|_{2 \rightarrow 2} < \gamma$. The lifting is an isomorphism and preserves signal norms: $\|d\|_2 = \|D_N\|_2$ and $\|e\|_2 = \|E_N\|_2$. It follows that the original system also satisfies the gain bound $\|F_U(G, \Phi)\|_{2 \rightarrow 2} < \gamma$. Moreover, the state of the lifted system corresponds to the evolving of the state of the original system forward by N steps. Simple bounding arguments can be used to show that if the lifted

state converges to the origin then so does the original state. Hence internal stability of the lifted system also implies internal stability of the original system. ■

VI. NUMERICAL EXAMPLES

A. Stability Analysis

A variety of examples are given in [19] to study the use of discrete-time Zames-Falb multipliers for analyzing stability of Lurье systems. We will use Example 6 in Table 1 of [19] as a benchmark to illustrate our ReLU RNN stability and performance condition.

Consider the Lurье system shown in Figure 2 where $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a nonlinear function, α is a non-negative scaling, and $G_{11}(z) = \frac{2z+0.92}{z^2-0.5z}$ is a discrete-time system. The results in [19] focus on the class of nonlinearities that have slope restricted to $[0, 1]$. The goal is to find the stability margin, i.e. the largest value of $\alpha \geq 0$ for which this Lurье system is stable for all nonlinearities in this class. Their reported results are shown in Table I. The Circle and Tsytkin criteria both can be used to prove stability for α up to 0.6510. The Zames-Falb condition in [19] proves stability for α up to 1.087.² In fact, the Zames-Falb condition achieves the largest possible stability margin for the class of $[0, 1]$ slope restricted nonlinearities. This follows because the Lurье system is unstable for $\phi(v) = v$ and $\alpha = 1.087$. The destabilizing value $\alpha = 1.087$ is called the Nyquist gain.

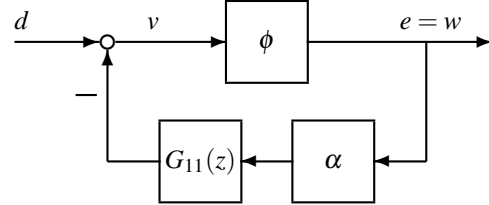


Fig. 2: Lurье System

Test type	Circle	Tsytkin	Zames-Falb	Nyquist
max α guarantee	0.6510	0.6510	1.0870	1.0870

TABLE I: Stability margins: Results from [19] for the Lurье system and nonlinearities with slope restricted to $[0, 1]$.

Next, we used the condition in Theorem 1 to analyze the stability margin for the case when ϕ is the scalar ReLU. The Lurье system in Figure 2 is equivalent to the LFT representation $F_U(G, \phi)$ in Figure 1 using:

$$G(z) = \begin{bmatrix} -G_{11}(z) \alpha & 1 \\ 1 & 0 \end{bmatrix} \quad (20)$$

The Lurье system with ReLU is stable for a given value of α if the $LMI(P, M, \gamma^2) < 0$, defined in (17), is feasible for some $\gamma < \infty$. We solved for the largest feasible value of

²The results depend on the number of terms included in the Zames-Falb finite impulse response filter. This is the best (largest) reported value.

α using bisection using the ReLU QCs.³ The feasibility at each bisection step is a semidefinite program and was solved using CVX [34] as a front-end and SDPT3 [35], [36] as the solver. The bisection is initialized with $\underline{\alpha} = 0$ and $\bar{\alpha} = 200$. The bisection is terminated when $\bar{\alpha} - \underline{\alpha} \leq 10^{-3}(1 + \bar{\alpha})$. This stopping condition implies that the absolute and relative error are both less than 10^{-3} .

The top row of Table II shows the results as a function of the lifting horizon. For comparison, we also computed the stability margin using QCs defined by $M = \begin{bmatrix} 0 & Q_0^\top \\ Q_0 & -(Q_0 + Q_0^\top) \end{bmatrix}$ where Q_0 is a doubly hyperdominant matrix. These results are shown in the second row of Table II. Computational times are also given for each bisection to the stopping tolerance.

lift size N	1	2	5	8	12
ReLU: max α	0.6516	0.6516	4.2999	33.472	181.543
DH: max α	0.6516	0.6516	0.8636	0.8820	0.8911
ReLU Comp. (s)	4.73	4.87	5.15	5.10	5.32
DH Comp. (s)	3.80	3.83	4.00	4.05	4.33

TABLE II: *Stability margins (top)*: Results using the doubly hyperdominant (DH) and ReLU QCs as a function of lifting horizon N . The ReLU QC exploits specific ReLU properties to significantly improve the stability margin. *Computational times (bottom)*: given for each stability margin calculation.

The doubly hyperdominant QC is the strongest possible constraint for $[0,1]$ slope restricted nonlinearities as discussed in Section IV-A. This test corresponds to the circle criterion when $N = 1$. The stability margin improves with the lifting horizon, getting to a maximum value of 0.8911, but stays below the Nyquist value as expected. The Zames-Falb condition, given in Table I, provides the larger stability margin of 1.0870 than our lifted Lyapunov condition with the doubly hyperdominance QC for $N = 12$.

More interestingly, the ReLU QC provides a stability margin that far exceeds the Nyquist value of 1.0870 as the lifting horizon N increases, giving a maximum value of 181.543. This is possible because the ReLU QC exploits specific properties of the repeated ReLU that appears in the lifted system.

To empirically verify this result, we simulated the Lurье system with ϕ as the scalar ReLU, $\alpha = 100$, and 20 initial conditions for the state drawn from a 2×1 Gaussian distribution with standard deviation of 10. The simulations of the Lurье system with the ReLU are shown in Figure 3. The simulations have a large initial transient but all trajectories converge back to the origin. This provides some independent validation of our stability margin results for the ReLU Lurье system.

B. Gain Example

In this section, we used the condition in Theorem 1 to analyze the induced ℓ_2 -norm for a ReLU RNN $F_U(G, \Phi)$.

³Note that γ appears in the (3,3) block of the LMI in (17). A Schur complement argument can be used to conclude that feasibility for some $\gamma < \infty$ is equivalent to feasibility of the upper left two blocks. Hence our numerical implementation used bisection with only these upper left two blocks of the LMI.

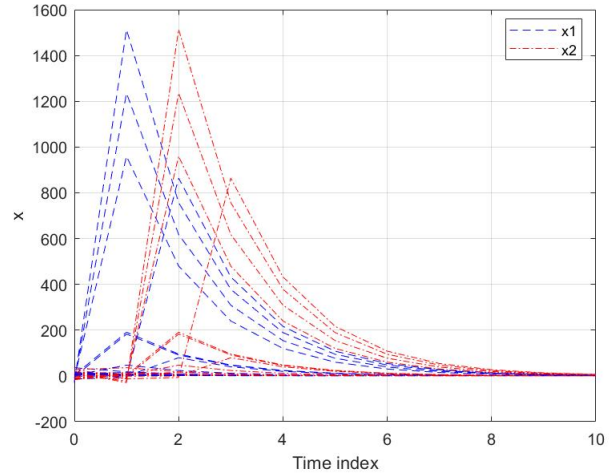


Fig. 3: State trajectories of the Lurье system $F_U(G, \phi)$ with G defined in (20) and $\alpha = 100$. Trajectories are shown for 20 random initial conditions.

The nominal part G is a discrete-time, linear time-invariant (LTI) system given in (1) with the following state matrices:

$$A = \begin{bmatrix} 0.84 & -0.17 & 0.10 & -0.04 \\ 0.17 & 0.80 & -0.11 & -0.04 \\ 0.05 & -0.11 & 0.90 & -0.08 \\ -0.04 & 0.18 & 0.01 & 0.74 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -0.08 & -0.18 \\ -0.11 & 0.11 \\ -0.01 & -0.05 \\ -0.04 & 0.03 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.04 & 0.11 & 0.14 \\ -0.06 & 0.03 & -0.02 \\ 0.02 & -0.03 & 0.01 \\ 0.04 & -0.08 & 0.20 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -0.35 & -0.40 & -1.04 & 1.36 \\ 0.90 & 0.55 & 1.13 & -0.46 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0.79 & -0.62 & -1.87 & -0.09 \\ -0.67 & -1.05 & 1.80 & -0.06 \\ 2.25 & -1.08 & -0.36 & 1.08 \end{bmatrix}$$

The nonlinearity Φ has $n_v = 2$ inputs and $n_w = 2$ outputs. The dimensions of the disturbance and error channels are $n_d = 3$ and $n_e = 3$, respectively. The “best” gain bound γ for a specific lifting horizon N is obtained by minimizing γ^2 subject to the LMI constraint $LMI(P, M, \gamma^2) < 0$, as defined in (17). There are additional (linear) constraints on the QC matrices in M , e.g. \tilde{Q} is a Metzler matrix. This minimization is a semidefinite program and was solved using CVX [34] as a front-end and SDPT3 [35], [36] as the solver. The top row of Table III shows the results as a function of the lifting horizon. For comparison, we also computed a bound on the ℓ_2 gain using QCs defined by $M = \begin{bmatrix} 0 & Q_0^\top \\ Q_0 & -(Q_0 + Q_0^\top) \end{bmatrix}$ where Q_0 is a doubly hyperdominant matrix. These results are shown in the second row of Table III. The doubly hyperdominant QC is the strongest possible constraint for $[0,1]$ slope restricted nonlinearities as discussed in Section IV-A. The results gain bound obtained using the ReLU QC is better (smaller) as it exploits specific properties of the ReLU. Moreover, both results improve with the lifting horizon as the QCs can exploit couplings between the input/output data for the repeated nonlinearity. Computational times are also given to run each minimization. There is a mild growth in computation time with increasing horizon N for this problem.

lift size N	1	2	3	4	5	6
ReLU: gain bound	7.556	5.530	3.932	3.466	3.247	3.128
DH: gain bound	34.379	13.450	7.992	5.844	4.811	4.263
ReLU Comp. (s)	0.82	0.44	0.48	0.52	0.91	1.25
DH Comp. (s)	0.28	0.28	0.29	0.36	0.47	0.61

TABLE III: *Gain (top):* ℓ_2 -gain upper-bounds given using ReLU and the doubly hyperdominant (DH) QCs as a function of lifting horizon N . The ReLU QC exploits specific ReLU properties to provide a significantly less conservative upper-bound on the gain. *Computational times (bottom):* provided for each upper-bound solution.

VII. CONCLUSIONS

This paper presents sufficient conditions for the stability and ℓ_2 -gain performance of RNNs with ReLU activation functions. These conditions are derived by combining Lyapunov/dissipativity theory with Quadratic Constraints (QCs) satisfied by repeated ReLUs. We use a “lifted” representation for the ReLU RNN to derive our stability and performance condition. Future work will consider the computational cost of this condition and scalability to larger RNNs. We will also study the theoretical properties as the lifting horizon N tends to infinity.

VIII. ACKNOWLEDGMENTS

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