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Local Input-to-State Stability for Consensus in the Presence of Intermittent Communication and Input Saturation

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Abstract— **This paper addresses the problem of reaching consensus under input saturation and intermittent communication, which can hinder the convergence of the system. We propose a method that translates the consensus into an equivalent stability problem. Then, we compute bounded sets that enclose the initial conditions and the evolution of trajectories leading to local input-to-state stability for systems interconnected over directed intermittent topologies. Our contributions include sufficient conditions for stability and stabilization of multi-agent systems under intermittent interactions and saturating inputs, with the ability to evaluate disturbance tolerance and rejection based on the regions that enclose the system's trajectories. We define disturbance rejection in terms of the $\mathscr{L}_2$ gain, and formulate stability and controller design conditions as convex optimization problems. Our method enable the maximization of regions that ensure local input-to-state stability, we provide numerical examples highlighting the trade-offs between mean frequency of intermittent interactions, disturbance energy, and convergence region size.**

I. Introduction

Multi-agent systems consist of multiple agents that can interact, coordinate, cooperate, and/or compete with one another to perform complex tasks (see [1], [2] for recent advances). These systems have wide applicability across different disciplines, such as robotic swarms [3], environmental monitoring [4], and surveillance [5]. In many applications, the agents need to reach agreement on the value of a variable of interest, resulting in the multi-agent system achieving *consensus*. There is a vast and increasing literature on the consensus problem [6], [7], [8], [9].

In distributed consensus, agents within a team must interact with each other, and thus the majority of the literature assumes all-time network connectivity between agents. Nevertheless, all-time connectivity, either by control or by assumption, can be overly conservative, and even impractical, especially for monitoring large-scale environments like the ocean [10], underground tunnels [11], or urban cities where wireless signal can be severely attenuated due to obstacle occlusions and other environmental factors.

In real-world applications, multi-agent systems must contend with intermittent network connectivity, input saturation, modeling errors, and exogeneous disturbances, all of which can lead to significant performance degradation of the team. In terms of consensus, these conditions can give rise to deteriorating convergence rate, creation of limit cycles, and even instability [12], [13]. Existing work in multi-agent systems mostly address these challenges individually. Works focused on the effects intermittent network connectivity on multi-agent systems include [14], [15], [16]. Unfortunately, these strategies are only developed for individual agents with either first- or second-order dynamics, with some exceptions [17], [18], and thus do not generalize to systems composed of agents with more complex dynamics. Literature that focus on input saturation effects include [19], [20], [21], [22]. However, these works either assumes directed [19] or strongly connected [20], [21] network topologies. And while [22] proposed an adaptive fault-tolerant control in the presence of $\mathcal{L}_2$-limited disturbances and input saturation, all these works assume *asymptotically null controllable agents*, *i.e.*, open-loop linear systems with no poles on the right-half plane in the continuous-time domain.

An objective of this work is to investigate the impact of intermittent communication topologies on a multi-agent system's ability to achieve consensus. In particular, we are interested in systems composed of individuals with more general dynamics models subject to constraints and exogenous disturbances. The current limitations in existing literature is partly due to the fact that [23], [24] has shown that *global* convergence of linear systems with saturating inputs can only be achieved by asymptotically null controllable systems. As such, the system's dynamics needs to be open-loop stable, in the Lyapunov sense, due to saturation. Hence, global asymptotic convergence in these systems can only happen under overly conservative conditions. To overcome this limitation, it is necessary to characterize regions for initial conditions from which the convergence can be guaranteed, especially when the objective is to design stabilizing feedback strategies for open-loop *unstable* linear systems. From a consensus perspective, this means that it is critical to estimate relative regions that encompass the agents' states from which the group's convergence can be achieved. This is a fundamentally more general and challenging problem than assuming asymptotic null controllability.

Towards this end, we propose a methodology capable of characterizing local stability for the general case of potentially open-loop unstable agents (*e.g.*, agents with poles of the open-loop system on the right-half plane), subject to stochastic nonsynchronous link formations, unlike most existing work [14], [15], [16], [17], [18]. To the best of the authors' knowledge, the combined impact of saturating inputs and intermittent communications have only been addressed in [25], [26], [27] but only for asymptotically null controllable agents and assume communication link activation and deactivation happen in a synchronized fashion [25], [26],

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[27]. We consider the synthesis of distributed stabilizing controllers for multi-agent systems subject to saturating inputs over time-varying stochastic communication topologies, with agents subject to disturbances limited in energy and non-zero initial conditions. This is a particularly challenging task since saturating inputs require the characterization of regions that guarantee the system convergence for stability [12], while disturbances and intermittent communication among agents directly affects the structure of these regions. We propose a convex approach to address this problem and ensure that trajectories starting within a specific set remain inside a larger outer set. Disturbance rejection is then defined as the measurement of relative sizes of these two sets for whenever the initial conditions of the network are different from zero, building upon previous work on single systems [13].

Notation: For a matrix M, $M > 0$ ($M \geq 0$) denotes that M is positive definite (semi-definite). M' stands for the transpose of M. The ith row (element) of the matrix (vector) N is denoted by $N_{(i)}$, while $N'_{(i)}$ represents the transpose of the ith row of N. $\mathrm{diag}(M_1, \dots, M_k)$ represents a block diagonal matrix with matrices $M_1, \dots, M_k$ in the main diagonal. Transposed entries in a symmetric matrix are represented by $*$. The Kronecker product of matrices M_1 and M_2 is denoted by $M_1 \otimes M_2$.

II. Preliminaries and Problem Formulation

This section introduces the representation for agent interaction and the control model for closed-loop agents, including the transformation of consensus into a stability problem.

A. Algebraic Graph Theory

A graph is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, in which $\mathcal{V} = \{1, ..., N\}$ is a set of N vertices and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of directed edges. Each element of the edge set, e_{ij}, represents a directed edge from i to j if $(i,j) \in \mathcal{E}$. A graph $\mathcal{G}$ is called *undirected graph* if $(i,j) \in \mathcal{E} \iff (j,i) \in \mathcal{E}$ and *directed graph* if the equivalence does not hold. In this work, we are interested in *directed graphs*. The set of agents that have the ith agent as the child vertex is $\mathcal{N}_i = \{j \in \mathcal{V} : (j,i) \in \mathcal{E}\}$ and we call it neighborhood of the ith vertex, and we call neighbor of i an agent j that belongs to the neighborhood $\mathcal{N}_i$. The adjacency matrix $\mathcal{A} = [a_{ij}]$ associated with the graph $\mathcal{G}$ is defined by

$$a_{ij} = \begin{cases} 0, & \text{if } i = j \text{ or } \nexists\ (j,i) \in \mathcal{E} \\ 1, & \text{if } (j,i) \in \mathcal{E}. \end{cases}$$

We can represent the graph $\mathcal{G}$ through the Laplacian matrix, defined by $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where the diagonal degree matrix, $\mathcal{D}$, has elements $d_{ii} = \sum_{j=1}^{N} a_{ij}$.

B. Dynamical Network

Consider N agents with the following open-loop dynamics

$$\dot{x}_i(t) = Ax_i(t) + B\mathrm{sat}(u_i(t)) + Dw_i(t), \text{ for } i = 1, ..., N, \quad (1)$$

where $x_i(t) \in \mathbb{R}^m$ is the state variable of the ith agent, $u_i(t) \in \mathbb{R}^p$ is its input, $w_i(t) \in \mathbb{R}^q$ is an exogenous signal, and A, B, and D are the system matrices with appropriate dimensions. The function $\mathrm{sat}(\cdot)$ is given by,

$$\mathrm{sat}(u_i(t)) = [\mathrm{sat}(u_{i(1)}(t)) \ \cdots \ \mathrm{sat}(u_{i(p)}(t))]',$$
$$\mathrm{sat}(u_{i(r)}(t)) = \mathrm{sign}(u_{i(r)}(t)) \min(u_{\max}, |u_{i(r)}(t)|),$$

for $r = 1, ..., p$, where the scalar $u_{\max}$ is the limit of the actuator. This study investigates the impact of saturation on networked systems subject to stochastic communication topologies and input disturbances. The time-varying interconnections are modeled by a dependency of the elements of the communication graph in a continuous-time Markov process $\{\theta_t : t \geq 0\}$ over a finite set of states $\mathcal{S} = \{1, \dots, s\}$, where the random variable θ_t is the state of the Markov chain at time t. This dependency is represented by $a_{ij}(\theta_t)$. We consider a distributed control setting and adopt the following variant of the conventional consensus protocol [28]:

$$u_i(t) = -\sum_{j=1}^{N} a_{ij}(\theta_t) K \big(x_i(t) - x_j(t)\big), \quad (2)$$

in which $K \in \mathbb{R}^{p \times m}$ is a constant gain matrix. The value $a_{ij}(k) = 1$ if, and only if, $(j,i) \in \mathcal{E}$ when $\theta_t = k$, for $k \in \mathcal{S}$. An implicit assumption on θ_t in equation (2) is that it models accurately enough intermittent interactions in the networked system, these parameters can be computed using identification techniques [29]. In addition, we work under the following assumption:

Assumption 2.1: The union of graphs, $\bigcup_{\theta_t \in \mathcal{S}} \mathcal{G}(\theta_t)$, associated with the time-varying communication topology of the network has a directed spanning tree, *i.e.,* there is a node with a directed path to every other node in the network.

The Markov process is defined by a time-varying transition rate matrix $\Pi(t) = [\pi_{ij}(t)]$, $i, j \in \mathcal{S}$, which evolution is described by the following infinitesimal generator,

$$\Pr\{\theta_{t+\Delta} = j | \theta_t = i\} = \begin{cases} \pi_{ij}(t)\Delta + o(\Delta), & i \neq j, \\ 1 + \pi_{ii}(t)\Delta + o(\Delta), & i = j, \end{cases} \quad (3)$$

where $\Pr\{\theta_{t+\Delta} = j | \theta_t = i\}$ stands for the probability of the next state be $\theta_{t+\Delta} = j$ given that the current state is $\theta_t = i$, $\Delta > 0$ and $\lim_{\Delta \to 0} \frac{o(\Delta)}{\Delta} = 0$, $\pi_{ij}(t) \geq 0$ for $i \neq j$ is the transition rate from state i to j, and $\pi_{ii}(t) = -\sum_{j=1, j\neq i}^{s} \pi_{ij}(t)$. In addition, the probability distribution of the Markov chain is denoted by $\mu = \{\mu_1, \dots, \mu_s\}$. The time-varying transition matrix $\Pi(t)$ is not precisely known, but belonging to a polytopic uncertainty domain denoted by

$$\Pi(\alpha(t)) = \sum_{i=1}^{r} \alpha_i \Pi^i, \ \alpha_i \in \Xi_r, \quad (4)$$

where the unit simplex Ξ_r is defined as $\Xi_r = \{\boldsymbol{\alpha} \in \mathbb{R}^r : 0 \leq \alpha_i \leq 1, \ \sum_{i=1}^{r} \alpha_i = 1\}$, in which r is the number of vertices of the polytope which are given by known matrices Π^i with elements π^i_{jq}, for $i \in \{1, ..., r\}$. This approach allows us to robustly tackle the problem of time-varying and uncertain switching topologies.

The $\mathscr{L}_2$ gain is traditionally used to evaluate the disturbance rejection of a system [30]. However, for the multi-agent system (1), the $\mathscr{L}_2$ gain might not be well-defined, since sufficiently large disturbances might lead to unbounded states [22], [31]. To address this, we focus on disturbances whose energy is bounded, belonging to the following set:

$$\mathcal{W} = \left\{ w_i \in \mathbb{R}^q : \int_0^\infty w_i(\tau)' w_i(\tau) d\tau < \rho \right\},$$

where ρ is a positive constant representing the energy bound of the disturbance. Our goal is to find the largest ρ for which the trajectories of the networked agents remain bounded in the presence of switching topologies and input saturation, by seeking the $\mathscr{L}_2$ gain as the ratio between a bounded output of interest and input limited in energy. We provide a formal definition in Sec. III-A.2.

Now, formalize the problem investigated in this work:

Definition 1: Under stochastic switching topology, the networked system (1) in closed-loop with (2), for all $i \in \mathcal{V}$, reaches the mean-square consensus if, for all $i \neq j$, $\lim_{t\to\infty} \mathbb{E}\big(\|x_i(t) - x_j(t)\|^2\big) \to 0$ holds in the mean square sense, for any initial distribution μ.

Here, $\mathbb{E}(\cdot)$ stands for the mathematical expectation.

C. Mean-Square consensus as a stability problem

We translate the consensus problem into a stability analysis and derive sufficient conditions that guarantee the stabilization of the equivalent system. Consider the following disagreement transformation [32] and its equivalent stacked form,

$$
\begin{aligned}
z_i(t) &= x_1(t) - x_{i+1}, \text{ for } i = 1, ..., N-1,\\
\boldsymbol{z}(t) &= (U \otimes I_m)\boldsymbol{x}(t),
\end{aligned} \tag{5}
$$

where $\otimes$ denotes the Kronecker product, I_m is an identity matrix, $\boldsymbol{z}(t) = [z_1(t)' \ \cdots \ z_{N-1}(t)']'$, $\boldsymbol{x}(t) = [x_1(t)' \ \cdots \ x_N(t)']'$, and $U = [\mathbf{1}_{N-1} \ \ -I_{N-1}]$ with $\mathbf{1}_{N-1}$ being an all-ones vector of size $N-1$. The transformation from the disagreement variables $\boldsymbol{z}(t)$ back to the stacked agent states can be computed by

$$
\boldsymbol{x}(t) = \mathbf{1}_N \otimes x_1(t) + (W \otimes I_m)\boldsymbol{z}(t), \tag{6}
$$

in which $W = \begin{bmatrix}\mathbf{0}_{N-1} & -I_{N-1}\end{bmatrix}'$, and $\mathbf{0}_{N-1}$ is an all-zeros vector of size $N-1$. The disagreement variable $\boldsymbol{z}(t)$ provides the errors between a pivot agent and all other agents. Such a transformation allows us to reformulate the consensus problem in Definition 1 according to the following lemma:

Lemma 2.1 ([32]): The multi-agent system (1), in closed-loop with the control law (2), asymptotically reaches the mean-square consensus if, and only if, $\lim_{t\to\infty} \mathbb{E}\big(\|z_i(t)\|^2\big) \to 0$, for all $i = 1, \ldots, N-1$.

We compute the equivalent multi-agent system by expressing (1) in stacked form with the closed-loop control law (2) as follows,

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) = &\big(I_N \otimes A\big)\boldsymbol{x}(t) - (I_N \otimes B)\mathrm{sat}\big((\mathcal{L}_{\theta_t} \otimes K)\boldsymbol{x}(t)\big)\\
&+ \big(I_N \otimes D\big)\boldsymbol{w}(t),
\end{aligned} \tag{7}
$$

where $\boldsymbol{w}(t) = [w_1(t)' \ \cdots \ w_N(t)']'$ and $\mathcal{L}_{\theta_t}$ is a shorthand for $\mathcal{L}(\theta_t)$ which represents the Laplacian matrix associated to the current state of the Markov process $\{\theta_t : t \geq 0\}$. Taking the time-derivative of (5) and substituting $\dot{\boldsymbol{x}}(t)$ by (7) gives

$$
\begin{aligned}
\dot{\boldsymbol{z}}(t) = &(U \otimes I_m)\Big[\big(I_N \otimes A\big)\boldsymbol{x}(t) - (I_N \otimes B)\\
&\times \mathrm{sat}\big((\mathcal{L}_{\theta_t} \otimes K)\boldsymbol{x}(t)\big) + \big(I_N \otimes D\big)\boldsymbol{w}(t)\Big],
\end{aligned}
$$

then, considering $\boldsymbol{x}(t)$ in (6) and applying the Kronecker identity $\big(M \otimes N\big)\big(P \otimes Q\big) = \big(MP \otimes NQ\big)$, for matrices M, N, P, and Q with appropriate dimensions, we have

$$
\begin{aligned}
\dot{\boldsymbol{z}}(t) = &\Big(U\mathbf{1}_N \otimes Ax_1(t) + (UW \otimes A)\boldsymbol{z}(t)\Big)\\
&- (U \otimes B)\mathrm{sat}\Big(\big(\mathcal{L}_{\theta_t}\mathbf{1}_N \otimes Kx_1(t)\big) + (\mathcal{L}_{\theta_t}W \otimes K)\boldsymbol{z}(t)\Big)\\
&+ \big(U \otimes D\big)\boldsymbol{w}(t).
\end{aligned}
$$

Finally, noticing that $U\mathbf{1}_N = 0$, $\mathcal{L}_{\theta_t}\mathbf{1}_N = 0$, and $UW = I_{N-1}$, we have

$$
\begin{aligned}
\dot{\boldsymbol{z}}(t) = &\big(I_{N-1} \otimes A\big)\boldsymbol{z}(t) - \big(U \otimes B\big)\mathrm{sat}\Big(\big(\mathcal{L}_{\theta_t}W \otimes K\big)\boldsymbol{z}(t)\Big)\\
&+ \big(U \otimes D\big)\boldsymbol{w}(t).
\end{aligned} \tag{8}
$$

Hence, we study the consensus problem of the multi-agent system (1) by analysing the stability of (8), where agents' coordinates are transformed to a relative coordinate system.

The non-linearity produced by the saturation satisfies a sector condition, which allows us to analyze the dynamics of the multi-agent system as a Lur'e problem [30]. To do so, we can use the following dead-zone function [12],

$$
\Phi(\boldsymbol{u}(t)) = \boldsymbol{u}(t) - \mathrm{sat}(\boldsymbol{u}(t)), \tag{9}
$$

with $\boldsymbol{u}(t) = (\mathcal{L}_{\theta_t}W \otimes K)\boldsymbol{z}(t)$. Then, by summing and subtracting $(U \otimes B)\boldsymbol{u}(t)$ in equation (8) we get

$$
\begin{aligned}
\dot{\boldsymbol{z}}(t) = &\big(I_{N-1} \otimes A\big)\boldsymbol{z}(t) - \big(U\mathcal{L}_{\theta_t}W \otimes BK\big)\boldsymbol{z}(t)\\
&+ \big(U \otimes B\big)\Phi\Big(\big(\mathcal{L}_{\theta_t}W \otimes K\big)\boldsymbol{z}(t)\Big) + \big(U \otimes D\big)\boldsymbol{w}(t).
\end{aligned} \tag{10}
$$

With the multi-agent system represented as in (10), we can define the following polyhedral set together with a sector condition.

Lemma 2.2 (Generalised Sector Condition [33]): For a given auxiliary signal $\boldsymbol{\vartheta}(t) \in \mathbb{R}^{Np}$ and saturation limits $u_{\max}$, define the set

$$
\begin{aligned}
\mathbb{S}(\boldsymbol{\vartheta}(t), u_{\max}) = \big\{&\boldsymbol{u}(t) \in \mathbb{R}^{Np} : |(\boldsymbol{u}(t) - \boldsymbol{\vartheta}(t))_{(q)}| \leq u_{\max},\\
&\text{for } q = 1, ..., Np\big\}.
\end{aligned} \tag{11}
$$

If $\boldsymbol{u}(t)$ belongs to $\mathbb{S}(\boldsymbol{\vartheta}(t), u_{\max})$, then $\Phi\big(\boldsymbol{u}(t)\big)'T\big[\Phi\big(\boldsymbol{u}(t)\big) + \boldsymbol{\vartheta}(t)\big] \leq 0$ is satisfied for any positive diagonal matrix $T \in \mathbb{R}^{Np \times Np}$, where $\Phi\big(\boldsymbol{u}(t)\big)$ is given by (9).

Since it might not be possible to achieve convergence from every initial conditions for a system subject to input saturation [24], [23], the state space under stabilizing linear feedback control can be partitioned into a region that leads to convergence and other that may lead to divergence. Therefore, we define the region of convergence as follows.

Definition 2: The region of convergence is a subset of the state space, $\mathscr{C} \subseteq \mathbb{R}^{Nm}$, such that for any initial conditions starting from $\mathscr{C}$, the multi-agent system without disturbances attain the mean-square consensus. Namely,

$$
\begin{aligned}
\mathscr{C} = \big\{&\boldsymbol{x}(0) \in \mathbb{R}^{Nm} : \lim_{t\to\infty} \mathbb{E}\big(\|x_i(t) - x_j(t)\|^2\big) \to 0,\\
&\forall i, j \in \mathcal{V}\big\}.
\end{aligned} \tag{12}
$$

Exact characterization the region of convergence is a challenging, even for single systems [12]. We address this issue by defining a convex optimization problem to compute a subset of $\mathscr{C}$ less conservative as possible.

III. Main Results

In this section, we derive stochastic stability criteria for the disagreement system (8). In addition, we tackle the following problems: i) the disturbance tolerance from the consensus point, ii) the disturbance tolerance from arbitrary initial conditions, iii) the $\mathscr{L}_2$ gain, and iv) the maximization of the region of convergence and the region with guaranteed bounded states for the agents.

Theorem 3.1: Let the multi-agent system (1) in closed-loop with the consensus protocol (2) subject to input saturation and stochastic time-varying communication topology be represented by equation (10). For given positive scalars ρ, η, and time-varying transition matrix $\Pi(t)$, if there exist symmetric positive definite matrices $S \in \mathbb{R}^{Np \times Np}$, $Y_\ell \in \mathbb{R}^{m(N-1) \times m(N-1)}$, and matrices $X_\ell \in \mathbb{R}^{Np \times m(N-1)}$ for all $\ell \in \{1, ..., s\}$, such that the following matrix inequalities hold for all $i \in \{1, \ldots, r\}$:

$$
\begin{bmatrix}
\Lambda & * & * & *\\
S\big(U \otimes B\big)' - X_\ell & -2S & * & *\\
\sqrt{\rho}\big(U \otimes D\big)' & 0 & \frac{1-\gamma}{\gamma}I & *\\
R_\ell' & 0 & 0 & -Q_\ell
\end{bmatrix} \leq 0, \tag{13}
$$

$$
\begin{bmatrix}
Y_\ell & *\\
(\mathcal{L}_\ell W \otimes K)_{(q)}Y_\ell - X_{\ell(q)} & u_{\max}^2\gamma
\end{bmatrix} \geq 0, \tag{14}
$$

for $q = 1, ..., Np$, where $Y_\ell = P_\ell^{-1}$,

$\Lambda = \mathrm{He}\Big(\big(I_{N-1} \otimes A - U\mathcal{L}_\ell W \otimes BK\big)Y_\ell\Big) + \pi^i_{\ell\ell} Y_\ell,$

$R_\ell = \big[\sqrt{\pi^i_{\ell 1}}Y_\ell \ \cdots \ \sqrt{\pi^i_{\ell(\ell-1)}}Y_\ell \ \sqrt{\pi^i_{\ell(\ell+1)}}Y_\ell \ \cdots \ \sqrt{\pi^i_{\ell s}}Y_\ell\big]$, and

$Q_\ell = \mathrm{diag}\big(Y_1, \ \cdots, \ Y_{\ell-1}, \ Y_{\ell+1}, \ \cdots \ Y_s\big),$

then:

(i) every trajectory of the multi-agent system starting from the region $\mathscr{R}(\boldsymbol{z}(t), 1)$ remains within $\mathscr{R}(\boldsymbol{z}(t), \gamma^{-1})$ for all $t \geq 0$, with $\gamma = 1/(1 + N\rho\eta)$;

(ii) every trajectory of the multi-agent system starting from the origin will remain within the region $\mathscr{R}(\boldsymbol{z}(t), \gamma^{-1})$ for all $t \geq 0$, with $\gamma = 1/(N\rho\eta)$; and

(iii) in absence of disturbances, $\boldsymbol{w}(t) = 0$, the region $\mathscr{R}(\boldsymbol{z}(t), 1)$ is an estimate included in the region of convergence $\mathscr{C}$, and the multi-agent system attains the mean-square consensus asymptotically.

Moreover, the set $\mathscr{R}(\boldsymbol{z}(t), \sigma)$, for constant σ, is defined as

$$\mathscr{R}(\boldsymbol{z}(t), \sigma) = \bigcap_{\ell \in \mathcal{S}} \mathscr{E}(P_\ell, \sigma), \text{ with}$$

$$\mathscr{E}(P_\ell, \sigma) = \big\{\boldsymbol{z}(t) \in \mathbb{R}^{m(N-1)} : \boldsymbol{z}(t)' P_\ell \boldsymbol{z}(t) \leq \sigma\big\}. \quad (15)$$

Proof: Consider the following stochastic Lyapunov candidate functional:

$$V(\boldsymbol{z}(t), \theta_t = \ell) = \boldsymbol{z}(t)' P_\ell \boldsymbol{z}(t), \text{ with } \ell \in \mathcal{S}, \quad (16)$$

where P_ℓ is a constant positive definite matrix for each $\theta_t \in \mathcal{S}$. Let $\mathscr{D}$ be the weak infinitesimal generator of the random process $\{\theta_t : t \geq 0\}$. Following [34], [35], the difference of (16) along the trajectories of (10) yields

$$\begin{aligned}
\mathscr{D}V(\boldsymbol{z}(t)) &= \lim_{\Delta \to 0} \frac{1}{\Delta}\left[V(\boldsymbol{z}(t+\Delta), \theta_{t+\Delta}) - V(\boldsymbol{z}(t), \theta_t = \ell)\right] \\
&= 2\dot{\boldsymbol{z}}(t)' P_\ell \boldsymbol{z}(t) + \boldsymbol{z}(t)' \sum_{j=1}^{s} \pi_{\ell j}(\alpha(t)) P_j \boldsymbol{z}(t) \\
&= 2\boldsymbol{z}(t)' P_\ell \Big[\big(I_{N-1} \otimes A - U\mathcal{L}_\ell W \otimes BK\big)\boldsymbol{z}(t) \\
&\quad + \big(U \otimes B\big)\Phi\Big(\big(\mathcal{L}_\ell W \otimes K\big)\boldsymbol{z}(t)\Big) + \big(U \otimes D\big)\boldsymbol{w}(t)\Big] \\
&\quad + \boldsymbol{z}(t)' \sum_{j=1}^{s} \pi_{\ell j}(\alpha(t)) P_j \boldsymbol{z}(t),
\end{aligned}$$

noting that $2\boldsymbol{z}(t)' P_\ell \big(U \otimes D\big)\boldsymbol{w}(t) \leq \frac{1}{\eta}\boldsymbol{z}(t)' P_\ell \big(U \otimes D\big)\big(U \otimes D\big)' P_\ell \boldsymbol{z}(t) + \eta \boldsymbol{w}(t)' \boldsymbol{w}(t)$, with $\eta > 0$, we have

$$\begin{aligned}
\mathscr{D}V(\boldsymbol{z}(t)) &\leq 2\boldsymbol{z}(t)' P_\ell \Big[\big(I_{N-1} \otimes A - U\mathcal{L}_\ell W \otimes BK\big)\boldsymbol{z}(t) \quad (17) \\
&\quad + \big(U \otimes B\big)\Phi\Big(\big(\mathcal{L}_\ell W \otimes K\big)\boldsymbol{z}(t)\Big)\Big] \\
&\quad + \frac{1}{\eta}\boldsymbol{z}(t)' P_\ell \big(U \otimes D\big)\big(U \otimes D\big)' P_\ell \boldsymbol{z}(t) \\
&\quad + \eta \boldsymbol{w}(t)' \boldsymbol{w}(t) + \boldsymbol{z}(t)' \sum_{j=1}^{s} \pi_{\ell j}(\alpha(t)) P_j(\alpha(t)) \boldsymbol{z}(t).
\end{aligned}$$

For the sake of continuity, we assume that $\boldsymbol{u}(t) \in \mathbb{S}(\boldsymbol{\vartheta}(t), u_{\max})$, a sufficient condition for this assumption is given in the sequence. As a result, the inequality $-2\Phi\big(\boldsymbol{u}(t)\big)' T\big[\Phi\big(\boldsymbol{u}(t)\big) + \boldsymbol{\vartheta}(t)\big] \geq 0$ holds, in accordance with Lemma 2.2. Considering the auxiliary signal $\boldsymbol{\vartheta}(t) = G\boldsymbol{z}(t)$ and adding the previous inequality to equation (17) gives,

$$\mathscr{D}V(\boldsymbol{z}(t)) \leq \chi(t)' \Psi \chi(t) + \eta \boldsymbol{w}(t)' \boldsymbol{w}(t), \quad (18)$$

with $\chi(t)' = [\boldsymbol{z}(t)' \ \Phi(\boldsymbol{u}(t))']$ and

$$\Psi = \begin{bmatrix} \Omega^1 & * \\ \big(U \otimes B\big)' P_\ell - TG & -2T \end{bmatrix}.$$

With $\Omega^1 = \mathrm{He}\Big(P_\ell\big(I_{N-1} \otimes A - U\mathcal{L}_\ell W \otimes BK\big)\Big) + \frac{1}{\eta}P_\ell\big(U \otimes D\big)\big(U \otimes D\big)' P_\ell + \sum_{j=1}^{s} \pi_{\ell j}(\alpha(t)) P_j(\alpha(t))$. By left- and right-multiplying Ψ by $\mathrm{diag}(P_\ell^{-1}, T^{-1})$, making the change of variables $Y_\ell = P_\ell^{-1}$, $S = T^{-1}$, $X_\ell = GP_\ell^{-1}$, also noticing from (4) that $\pi_{\ell j}(\alpha(t)) = \sum_{i=1}^{r} \alpha_i \pi^i_{\ell j}$, it is sufficient to test the conditions on the vertices of $\Pi(t)$ [36], then applying Schur complement two consecutive times on the non-linear terms gives inequality (13). Therefore, if inequality (13) is satisfied, we have that $\mathscr{D}V(\boldsymbol{z}(t)) \leq \eta \boldsymbol{w}(t)' \boldsymbol{w}(t)$. From Dynkin's formula, we have that for all $\ell \in \mathcal{S}$,

$$\begin{aligned}
\mathbb{E}\big(V(\boldsymbol{z}(t))\big) &\leq V(\boldsymbol{z}(0)) + \eta \mathbb{E}\left(\int_0^t \boldsymbol{w}(\tau)' \boldsymbol{w}(\tau) d\tau\right) \\
&< V(\boldsymbol{z}(0)) + N\eta\rho.
\end{aligned}$$

Finally, we establish the three cases in the Theorem 3.1, namely: (i) if for each $\ell \in \mathcal{S}$ we have $V(\boldsymbol{z}(0)) \leq 1$, that is $\boldsymbol{z}(0) \in \mathscr{E}(P_\ell, 1)$, all trajectories will remain within $\mathscr{E}(P_\ell, 1 + N\rho\eta) = \big\{\boldsymbol{z}(t) \in \mathbb{R}^{m(N-1)} : \boldsymbol{z}(t)' P_\ell \boldsymbol{z}(t) \leq 1 + N\rho\eta\big\}$ for all $\ell \in \mathcal{S}$ and $t \geq 0$; (ii) if, for each $\ell \in \mathcal{S}$, $V(\boldsymbol{z}(0)) = 0$, which implies that $\boldsymbol{z}(0) = 0$, then $\boldsymbol{z}(t) \in \mathscr{E}(P_\ell, N\rho\eta) = \big\{\boldsymbol{z}(t) \in \mathbb{R}^{m(N-1)} : \boldsymbol{z}(t)' P_\ell \boldsymbol{z}(t) \leq N\rho\eta\big\}$, $\forall \ell \in \mathcal{S}$, and finally; (iii) in absence of disturbances $\mathscr{D}V(\boldsymbol{z}(t)) \leq 0$, and $V(\boldsymbol{z}(t)) = 0$ if, and only if $\boldsymbol{z}(t) = 0$, whenever for each $\ell \in \mathcal{S}$, $\boldsymbol{z}(0) \in \mathscr{E}(P_\ell, 1)$. Hence, the set $\mathscr{E}(P_\ell, 1)$ is an estimate included in the region of convergence.

Henceforth, we demonstrate that if inequality (14) is satisfied, then our previous assumption that $\boldsymbol{u}(t) \in \mathbb{S}(\boldsymbol{\vartheta}(t), u_{\max})$ holds. Replacing Y_ℓ and X_ℓ by P_ℓ^{-1} and $P_\ell^{-1}G'$ in (14), respectively, left- and right-multiplying the result by $\mathrm{diag}(P_\ell, 1)$, and recalling that γ has a different description according the cases (i)-(iii), we get

$$\begin{bmatrix} P_\ell & * \\ (\mathcal{L}_\ell W \otimes K)_{(q)} - G_{(q)} & \gamma u^2_{\max} \end{bmatrix} \geq 0.$$

By applying Schur complement and left- and right-multiplying the result by $\boldsymbol{z}(t)'$ and $\boldsymbol{z}(t)$, respectively, we get that

$$\boldsymbol{z}(t)'\Big((\mathcal{L}_\ell W \otimes K)_{(q)} - G_{(q)}\Big)' \gamma^{-1} u^{-2}_{\max} \Big((\mathcal{L}_\ell W \otimes K)_{(q)} - G_{(q)}\Big)$$
$$\times \boldsymbol{z}(t) \leq \boldsymbol{z}(t)' P_\ell \boldsymbol{z}(t) \leq V(\boldsymbol{z}(t)).$$

Hence, condition (14) ensures that $\boldsymbol{u}(t) \in \mathbb{S}(\boldsymbol{\vartheta}, u_{\max})$, and therefore the inequality in Lemma 2.2 is satisfied. ■

Theorem 3.1 establishes conditions for bounded trajectories and mean-square consensus in networked systems, according to agents disturbances. However, important questions remain unanswered, including identifying the largest disturbance bounds for bounded trajectories, minimizing the difference between estimated and actual convergence regions, and estimating the $\mathscr{L}_2$ gain on $\mathcal{W}$. Addressing these problems is crucial for designing and deploying multi-agent systems. The following sections will discuss each of these issues.

A. Optimization methods for design

1) Maximization of the disturbance tolerance: Maximizing the disturbance tolerance involves maximizing value of ρ, such that computing estimates for the convergence set and the set that contains the trajectories of the network is feasible. We formulate this as the following optimization problem:

$$\begin{cases} \sup\limits_{\gamma \in (0,1)} & \bar{\rho} \\ \text{s.t.} & (a)\ \mathscr{E}(Z, 1) \subseteq \mathscr{E}(P_\ell, 1), \\ & (b)\ \text{LMIs (13), (14),} \end{cases} \quad (19)$$

for all $\ell \in \{1, \ldots, s\}$, with $\bar{\rho} = \sqrt{\rho}$. The constraint (a) corresponds to placing an ellipsoid inside the intersection of ellipsoids defined by P_ℓ for all $\ell \in \{1,\ldots,s\}$, and it is equivalent to

$$\begin{bmatrix} Z & * \\ I & Y_\ell \end{bmatrix} \geq 0, \text{ for } \ell = 1,\ldots,s,$$

with $Y_\ell = P_\ell^{-1}$. Clearly, all constraints in (19) are linear matrix inequalities for fixed values of $\gamma \in (0,1)$.

Another relevant issue is finding the maximal disturbance tolerance when the initial condition is the equilibrium point, *i.e.*, $\boldsymbol{z}(0) = 0$. This value is critical to estimating the $\mathscr{L}_2$ gain of the network. This can be cast similarly as the optimization problem (19), by letting $\eta = 1$ and $\gamma = 1/N\rho$. By applying Schur complement on inequality (13) we get,

$$\begin{bmatrix} \Lambda + (U\otimes D)(U\otimes D)' & * & * \\ S(U\otimes B)' - X_\ell & -2S & * \\ R_\ell' & 0 & -Q_\ell \end{bmatrix} \leq 0. \tag{20}$$

Therefore, the problem of estimating the maximal disturbance tolerance from the consensus point can be cast as,

$$\begin{cases} \min & \gamma \\ \text{s.t.} & (a)\ \mathscr{E}(Z,1) \subseteq \mathscr{E}(P_\ell,1), \\ & (b)\ \text{LMIs (20), (14),} \end{cases} \tag{21}$$

for all $\ell \in \{1,\ldots,s\}$, and ρ is given by $\rho = 1/N\gamma$.

2) Estimation of the $\mathscr{L}_2$ gain: The $\mathscr{L}_2$ gain represents the ratio between a bounded output and a bounded input for a system starting from the origin. In our scenario, we define an output for the multi-agent system (1) on the disagreement variable as, $\boldsymbol{y}(t) = C\boldsymbol{z}(t)$, in which C is an matrix with appropriate dimension that properly weight disagreement variables of interest. Therefore, we compute the $\mathscr{L}_2$ gain of the multi-agent system, subject to energy bounded disturbances according the following corollary:

Corollary 3.2 ($\mathscr{L}_2$ gain): For given positive scalars ρ and ϱ, and time-varying transition matrix $\Pi(t)$, if there exist symmetric positive definite matrices $S \in \mathbb{R}^{Np\times Np}$, $Y_\ell \in \mathbb{R}^{m(N-1)\times m(N-1)}$, and matrices $X_\ell \in \mathbb{R}^{Np\times m(N-1)}$ for all $\ell \in \{1,...,s\}$, such that the following matrix inequalities hold for all $i \in \{1,\ldots,r\}$:

$$\begin{bmatrix} \Lambda + (U\otimes D)(U\otimes D)' & * & * & * \\ S(U\otimes B)' - X_\ell & -2S & * & * \\ R_\ell' & 0 & -Q_\ell & * \\ CY_\ell & 0 & 0 & -\varrho^2 I \end{bmatrix} \leq 0, \tag{22}$$

$$\begin{bmatrix} Y_\ell & * \\ (\mathcal{L}_\ell W\otimes K)_{(q)} - X_{\ell(q)} & u_{\max}^2\gamma \end{bmatrix} \geq 0, \tag{23}$$

for $q = 1,...,Np$, where $\gamma = 1/N\rho$, $Y_\ell = P_\ell^{-1}$, and Λ, R_ℓ, and Q_ℓ are defined in 13. Then, the $\mathscr{L}_2$ gain from $\boldsymbol{w}(t)$ to $\boldsymbol{y}(t)$ for all $w_i(t) \in \mathcal{W}$ is no larger than ϱ.

Proof: Similarly to Theorem 3.1, inequality (23) ensures that $\boldsymbol{u}(t) \in \mathbb{S}(\boldsymbol{\vartheta}(t), u_{\max})$ whenever $\boldsymbol{z}(t) \in \mathscr{E}(P_\ell, N\rho)$. We have that (22) is equivalent to

$$\begin{aligned} &2\boldsymbol{z}(t)'P_\ell\Big[\big(I_{N-1}\otimes A - U\mathcal{L}_\ell W\otimes BK\big)\boldsymbol{z}(t) \\ &+(U\otimes B)\Phi\big(\boldsymbol{u}(t)\big)\Big] + \boldsymbol{z}(t)'P_\ell(U\otimes D)(U\otimes D)'P_\ell\boldsymbol{z}(t) \\ &+\frac{1}{\varrho^2}\boldsymbol{z}(t)'C'C\boldsymbol{z}(t) + \boldsymbol{z}(t)'\sum_{j=1}^{s}\pi_{\ell j}(\alpha(t))P_j(\alpha(t))\boldsymbol{z}(t) \\ &-2\Phi\big(\boldsymbol{u}(t)\big)'T\big[\Phi\big(\boldsymbol{u}(t)\big) + G\boldsymbol{z}(t)\big] \leq 0, \end{aligned} \tag{24}$$

which can be shown by applying Schur complement on (22) two times consecutively, making the change of variables $Y_\ell = P_\ell^{-1}$, $S = T^{-1}$, $X_\ell = GP_\ell^{-1}$, left- and right-multiplying it by $\operatorname{diag}(P_\ell, T)$ and by $[\boldsymbol{z}(t)'\ \Phi(\boldsymbol{u}(t))']$ and its transpose, respectively. Notice that this condition contains inequality (13) with $\eta = 1$. Therefore, admitting that (22) is satisfied and considering inequalities (18) and (24) we have that

$$\mathscr{D}V(\boldsymbol{z}(t)) \leq -\frac{1}{\varrho^2}\boldsymbol{z}(t)'C'C\boldsymbol{z}(t) + \boldsymbol{w}(t)'\boldsymbol{w}(t),$$

which from Dynkin's formula yields

$$\mathbb{E}\big(V(\boldsymbol{z}(t))\big) \leq$$

$$-\frac{1}{\varrho^2}\mathbb{E}\left(\int_0^t \boldsymbol{z}(\tau)'C'C\boldsymbol{z}(\tau)d\tau\right) + \mathbb{E}\left(\int_0^t \boldsymbol{w}(\tau)'\boldsymbol{w}(\tau)d\tau\right).$$

Noticing that $V(\boldsymbol{z}(t)) \geq 0$ and $w_i(t) \in \mathcal{W}$, for all $i \in \mathcal{V}$, implies

$$\mathbb{E}\left(\int_0^t \boldsymbol{y}(\tau)'\boldsymbol{y}(\tau)d\tau\right) < \varrho^2 N\rho,$$

which concludes the demonstration. ∎

3) Synthesis of the controllers' gains: The problem of synthesizing the gains of the feedback matrices can be tackled by letting the matrix K be an additional variable on the previous formulations. We provide this procedure as a corollary, since a particular choice for the structure of some variables is necessary to produce linear convex problems. The adaptation of Theorem 3.1 to compute the feedback gains is given as follows.

Corollary 3.3: Let the networked system (1) in closed-loop with the consensus protocol (2) subject to input saturation and stochastic time-varying communication topology be represented by equation (10). For given positive scalars ρ and η, time-varying transition matrix $\Pi(t)$, if there exist symmetric positive definite matrices $S \in \mathbb{R}^{Np\times Np}$, $\bar{Y}_\ell = I_{N-1}\otimes F \in \mathbb{R}^{m(N-1)\times m(N-1)}$, and matrices $X_\ell \in \mathbb{R}^{Np\times m(N-1)}$ for all $\ell \in \{1,...,s\}$, such that the following matrix inequalities hold for all $i \in \{1,\ldots,r\}$:

$$\begin{bmatrix} \bar{\Lambda} & * & * & * \\ S(U\otimes B)' - X_\ell & -2S & * & * \\ \sqrt{\rho}(U\otimes D)' & 0 & \frac{1-\gamma}{\gamma}I & * \\ R_\ell' & 0 & 0 & -Q_\ell \end{bmatrix} \leq 0, \tag{25}$$

$$\begin{bmatrix} \bar{Y}_\ell & * \\ (\mathcal{L}_\ell W\otimes \bar{K})_{(q)} - X_{\ell(q)} & u_{\max}^2\gamma \end{bmatrix} \geq 0, \text{for } q = 1,...,Np, \tag{26}$$

where,

$$\bar{\Lambda} = \mathrm{He}\Big(\big(I_{N-1}\otimes A\big)\bar{Y}_\ell - U\mathcal{L}_\ell W\otimes B\bar{K}\Big) + \pi_{\ell\ell}^i\bar{Y}_\ell$$

$$R_\ell = \big[\sqrt{\pi_{\ell 1}^i}\bar{Y}_\ell\ \cdots\ \sqrt{\pi_{\ell(\ell-1)}^i}\bar{Y}_\ell\ \sqrt{\pi_{\ell(\ell+1)}^i}\bar{Y}_\ell\cdots\sqrt{\pi_{\ell s}^i}\bar{Y}_\ell\big], \text{ and}$$

$$Q_\ell = \operatorname{diag}\big(\bar{Y}_1,\ \cdots,\ \bar{Y}_{\ell-1},\ \bar{Y}_{\ell+1},\ \cdots,\ \bar{Y}_s\big),$$

then items (i)-(iii) in Theorem 3.1 hold with agents in closed-loop with feedback matrix $K = \bar{K}F^{-1}$.

Proof: The demonstration follows the same lines of Theorem 3.1. Replacing the matrices variables Y_ℓ by the structured variables $\bar{Y}_\ell = I_{N-1}\otimes F$ for all $\ell \in \mathcal{S}$. Then, the conditions are obtained by applying the Kronecker identity in all products with the gain matrix K and, subsequently, making the change of variables $\bar{K} = KF$. ∎

Remark 1: The conditions proposed in Theorem 3.1 and Corollary 3.3 are Linear Matrix Inequalities, hence their computational complexity grow with the number of decision variables [37]. In our scenario, the number of agents, the dimension of their state-space, as well as the number of vertices of the convex hull of the the time-varying transition matrix determine the computational complexity of the conditions. Precisely, we have $\frac{\ell}{2}\big((Np)^2+(m(N-1))^2+2N^2pm+$

$Np + m(N-1) - 2Npm\big)$ variables in the conditions of Theorem 3.1, and $\frac{\ell}{2}\big((Np)^2 + 2N^2pm + \frac{m^2+m}{\ell} - 2Npm\big)$ variables in the conditions of Corollary 3.3. Although these numbers might be high for some systems, the proposed results achieve less conservative results in a more general context than similar works from the literature. In addition, all conditions are computed off-line.

IV. Simulation Results

Example 4.1: We consider the problem of maximizing the disturbance tolerance of a multi-agent system starting from the equilibrium point. The network is described by (1) with

$$A = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -3.0 \end{bmatrix}, \; B = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, \; K = \begin{bmatrix} 0.1 & 0.5 \\ 0 & 0 \end{bmatrix},$$

and $D = I_2$, where K is computed using Corollary 3.3 and then fixed. In addition, we assume that the switching of the time-varying communication topology is captured by the Markov process defined by the following transition matrix:

$$\Pi = \begin{bmatrix} -2 & 1 & 1 \\ 2 & -4 & 2 \\ 1 & 1 & -2 \end{bmatrix},$$

and the topologies associated with each state of the Markov process are given by

$$\mathcal{L}_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}; \mathcal{L}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \mathcal{L}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Notice that no single topology is connected, however, the union of the graphs has a directed spanning tree.

By setting a limit for the actuators and solving optimization problem (21) for the multi-agent system starting from the equilibrium point, we obtain the maximum energy bound, $N\rho$, for disturbances affecting the network shown in Figure 1. Unsurprisingly, this indicates that the ability of disturbance rejection of the network grows with the actuation limit.

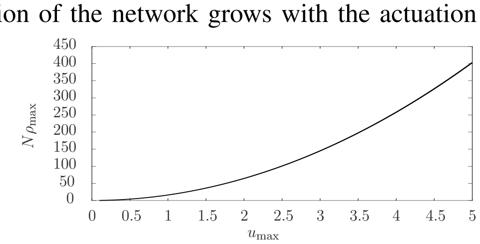

Fig. 1. Variation of disturbance level rejection according saturation limit.

For $u_{\max} = 3$ we have $N\rho = 145.1$, hence if the energy of the disturbances that impacts the network is less than 145.1, the agents' states are guaranteed to remain within the designed region.

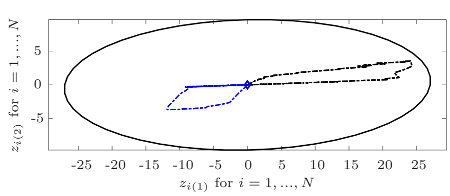

Fig. 2. Two realizations of the multi-agent system trajectories subject to ramp (in black line) and constant (in blue line) disturbances with energy limited by 145.

Figure 2 depicts a realization of the trajectories of the multi-agent system from the equilibrium point subject to a ramp disturbance impacting only agent 1 in black line, *i.e.*, $w_1(t) = [t\ t]'$ and $w_i(t) = [0\ 0]'$ for $i = 2, 3$, and a constant disturbance impacting only agent 2 in blue line, *i.e.*, $w_2(t) = [10\ 10]'$ and $w_i(t) = [0\ 0]'$ for $i = 1, 3$, both with specific duration such that $N\rho_{\max} \leq 145$. The control inputs and switching topologies are shown in Figure 3 for both cases.

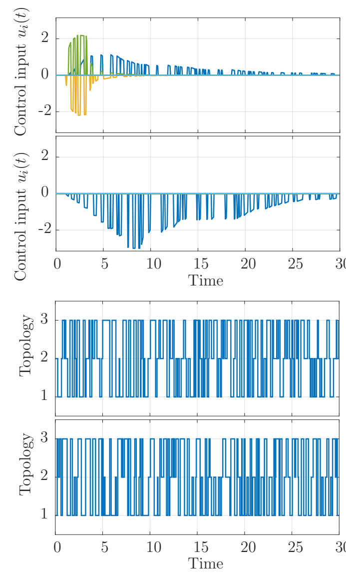

Fig. 3. Control signals and switching topologies for the multi-agent system subject to ramp (in the bottom) and constant (in the top) disturbances with energy limited by 145.

Example 4.2: We show the impact of the switching frequency of the stochastic communication on the convergence region size. Using the same system as in Example 4.1, we optimize the region $\mathscr{R}(\boldsymbol{z}(t), 1)$ for different transition matrices, according to case (i) of Theorem 3.1. By replacing the objective function in (19) by $\min\ \mathrm{Trace}(Z)$ and setting $\gamma = 0.8$, we solve optimization problem with ρ as variable. Figure 4 shows the trace of Z as function of ε, a positive constant that scales the transition matrix as follow

$$\Pi = \varepsilon \begin{bmatrix} -2 & 1 & 1 \\ 2 & -4 & 2 \\ 1 & 1 & -2 \end{bmatrix}.$$

Figure 4 shows a non-linear relation between the size of of the estimate of the region of convergence and the probability of link formation. Decreasing the frequency of interactions reduces the regions of convergence, however the biggest region for this network occurs when $\varepsilon = 0.51$. For larger values the size starts to contract. The estimate of the region

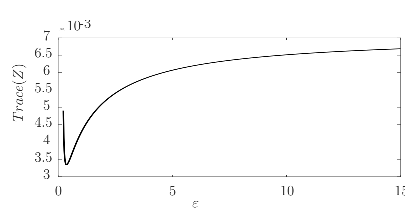

Fig. 4. Trace of Z as function of different scales parameters of the transition matrix Π.

of convergence are shown in Figure 5 for $\varepsilon = 0.25$, $\varepsilon = 0.51$, and $\varepsilon = 15.00$.

V. Conclusions

In this paper, we investigated the effects of disturbances and stochastic intermittent communications on agents performing consensus under saturating inputs. We formulated

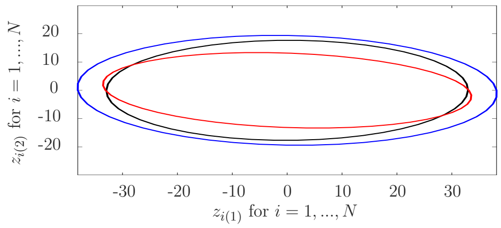

Fig. 5. Estimates of the region of convergence for three different values of ε, in black for $\varepsilon = 0.25$, in blue for $\varepsilon = 0.51$, and in $\varepsilon = 15.00$ in red line.

the problem as a stability problem of Markov jump linear systems, proposed conditions to estimate the region of convergence and the region that bounds the trajectories of the agents' states, and demonstrated the formulation of optimization convex methods to design networks with optimized parameters. Numerical simulations illustrate our proposed approach. Future work includes finding specific control gains for different network topologies, formulating the consensus problem for heterogeneous networks, and investigating event-triggered control approaches.

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