2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

A Strong Duality Result for Cooperative Decentralized Constrained POMDPs

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***Abstract*— The work studies cooperative decentralized constrained POMDPs with asymmetric information. Using an extension of Sion's Minimax theorem for functions with positive infinity and results on weak-convergence of measures, strong duality and existence of a saddle point are established for the setting of infinite-horizon expected total discounted costs when the observations lie in a countable space, the actions are chosen from a finite space, the immediate constraint costs are bounded, and the immediate objective cost is bounded from below.**

I. Introduction

Single-Agent Markov Decision Processes (SA-MDPs) [1] and Single-Agent Partially Observable Markov Decision Processes (SA-POMDPs) [2] have long served as the basic building-blocks in the study of sequential decision-making. An SA-MDP is an abstraction in which an agent sequentially interacts with a fully-observable Markovian environment to solve a multi-period optimization problem; in contrast, in SA-POMDP, the agent only gets to observe a noisy or incomplete version of the environment. In 1957, Bellman proposed dynamic-programming as an approach to solve SA-MDPs [1], [3]. This combined with the characterization of SA-POMDP into an equivalent SA-MDP [4]–[6] (in which the agent maintains a belief about the environment's true state) made it possible to extend dynamic-programming results to SA-POMDPs. Reinforcement learning [7] based algorithmic frameworks use data-driven dynamic-programming approaches to solve such single-agent sequential decision-making problems when the environment is unknown.

In many engineering systems, there are multiple decision-makers that collectively solve a sequential decision-making problem but with safety constraints: e.g., a team of robots performing a joint task, a fleet of automated cars navigating a city, multiple traffic-light controllers in a city, etc. Bandwidth constrained communications and communication delays in such systems lead to a decentralized team problem with information asymmetry. In this work, we study a fairly general abstraction of such systems, namely that of a cooperative decentralized constrained POMDP, hereon referred to as Dec-C-POMDP. The special cases of Dec-C-POMDP when there are no constraints, when there is only one agent, or when the environment is fully observable to each agent are referred to as Dec-POMDP[1], SA-C-POMDP, and Dec-C-MDP, respectively. The relationships among these models are shown in Figure 1.

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[1]For a good introduction to Dec-POMDPs, see [8].

A. Related Work

1) Single-Agent Settings: Prior work on planning and learning under constraints has primarily focused on single-agent constrained MDP (SA-C-MDP) where unlike in SA-MDPs, the agent solves a constrained optimization problem. For this setup, a number of fundamental results from the planning perspective have been derived – for instance, [9]–[15]; see [16] for details of the convex-analytic approach for SA-C-MDPs. These aforementioned results have led to the development of many algorithms in the learning setting: see [17]–[23]. Unlike SA-C-MDPs, rigorous results for SA-C-POMDPs are limited; few works include [24]–[27].

2) Multi-Agent Settings: Challenges arising from the combination of partial observability of the environment and information-asymmetry[2] have led to difficulties in developing general solutions to Dec-POMDPs: e.g., solving a finite-horizon Dec-POMDP with more than two agents is known to be NEXP-complete [28]. Nevertheless, conceptual approaches exist to establish solution methodologies and structural properties in (finite-horizon) Dec-POMDPs namely: i) the person-by-person approach [29]; ii) the designer's approach [30]; and iii) the common-information (CI) approach [31], [32]. Using a fictitious coordinator that only uses the common information to take actions, the CI approach allows for the transformation of the problem to a SA-POMDP which can be used to solve for an optimal control. The CI approach has also led to the development of a multi-agent reinforcement learning (MARL) framework [33] where agents learn good compressions of common and private information that can suffice for approximate optimality. On the empirical front, worth-mentioning works include [34], [35]. Finally, as far as we know, work on Dec-C-POMDPs is non-existent.

B. Contribution

For Dec-C-POMDPs, the technical challenges increase even more from those of Dec-POMDPs because restriction of the search space to deterministic policy-profiles is no longer an option[3]. Therefore, the coordinator in the equivalent SA-C-POMDP has an uncountable prescription space, which leads to an uncountable state-space in its equivalent SA-C-MDP. This is an issue because most fundamental results in the theory of SA-C-MDPs (largely based on occupation-measures) rely heavily on the state-space being countably-

[2]Mismatch in the information of the agents.

[3]Restricting to deterministic policies can be sub-optimal in SA-C-MDPs and SA-C-POMDPs: see [16] and [24].

infinite; see [16]. Due to these reasons, the study of Dec-C-POMDPs calls for a new methodology—one which avoids this transformation and directly studies the decentralized problem. Our work takes the first steps in this direction and presents a rigorous approach for Dec-C-POMDPs which is based on structural characterization of the set of behavioral policies and their performance measures, and using measure theoretic results. The main result in this paper, namely Theorem 1, establishes strong duality and existence of a saddle-point for Dec-C-POMDPs, thus providing a firm theoretical basis for (future) development of primal-dual type planning and learning algorithms.

C. Organization

The rest of the paper is organized as follows. Mathematical formulation of (cooperative) Dec-C-POMDP is introduced in Section II. Results on strong duality and existence of a saddle point are then derived in Section III. Finally, concluding remarks are given in Section IV.

D. Notation

Before we present the model, we highlight the key notations in this paper.

- The sets of integers and positive integers are respectively denoted by $\mathbb{Z}$ and $\mathbb{N}$. For integers a and b, $[a,b]_{\mathbb{Z}}$ represents the set $\{a, a+1, \ldots, b\}$ if $a \leq b$ and $\emptyset$ otherwise. The notations [a] and $[a,\infty]_{\mathbb{Z}}$ are used as shorthands for $[1,a]_{\mathbb{Z}}$ and $\{a, a+1, \ldots\}$, respectively.
- For integers $a \leq b$ and $c \leq d$, and a quantity of interest q, $q^{(a:b)}$ is a shorthand for the vector $\left(q^{(a)}, q^{(a+1)}, \ldots, q^{(b)}\right)$ while $q_{c:d}$ is a shorthand for the vector $(q_c, q_{c+1}, \ldots, q_d)$. The combined notation $q_{a:b}^{(c:d)}$ is a shorthand for the vector $(q_i^{(j)} : i \in [a,b]_{\mathbb{Z}}, j \in [c,d]_{\mathbb{Z}})$. The infinite tuples $\left(q^{(a)}, q^{(a+1)}, \ldots, \right)$ and $(q_c, q_{c+1}, \ldots,)$ are respectively denoted by $q^{(a:\infty)}$ and $q_{c:\infty}$.
- For two real-valued vectors v_1 and v_2, the inequalitie $v_1 \leq v_2$ and $v_1 < v_2$ are meant to be element-wise inequalities.
- Probability and expectation operators are denoted by $\mathbb{P}$ and $\mathbb{E}$, respectively. Random variables are denoted by upper-case letters and their realizations by the corresponding lower-case letters. At times, we also use the shorthand $\mathbb{E}[\cdot|x] \triangleq \mathbb{E}[\cdot|X=x]$ and $\mathbb{P}(y|x) \triangleq \mathbb{P}(Y=y|X=x)$ for conditional quantities.
- Topological spaces are denoted by upper-case calligraphic letters. For a topological-space $\mathcal{W}$, $\mathcal{B}(\mathcal{W})$ denotes the Borel σ-algebra, measurability is determined with respect to $\mathcal{B}(\mathcal{W})$, and $\mathcal{M}_1(\mathcal{W})$ denotes the set of all probability measures on $\mathcal{B}(\mathcal{W})$ endowed with the topology of weak convergence. Also, unless stated otherwise, "measure" means a non-negative measure.
- Unless otherwise stated, if a set $\mathcal{W}$ is countable, as a topological space it will be assumed to have the discrete topology. Therefore, the corresponding Borel σ-algebra $\mathcal{B}(\mathcal{W})$ will be the power-set $2^{\mathcal{W}}$.
- Unless stated otherwise, the product of a collection of topological spaces will be assumed to have the product topology.

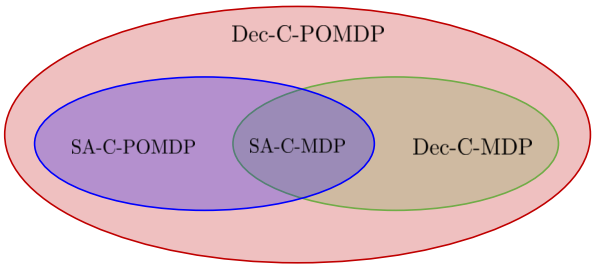

Fig. 1: Relationships between Models of Cooperative Sequential Decision-Making under Constraints.

II. Model

Let $(N, \mathcal{S}, \mathcal{O}, \mathcal{A}, \mathcal{P}_{tr}, (c,d), P_1, \mathcal{U}, \alpha)$ denote a (cooperative) Dec-C-POMDP with N agents, state space $\mathcal{S}$, joint-observation space $\mathcal{O}$, joint-action space $\mathcal{A}$, transition-law $\mathcal{P}_{tr}$, immediate-cost functions c and d, (fixed) initial distribution P_1, space of decentralized policy-profiles $\mathcal{U}$, and discount factor $\alpha \in (0,1)$. The decision problem (to be detailed later on) has the following attributes and notations.

- **State Process**: The state-space $\mathcal{S}$ is some topological space with a Borel σ-algebra $\mathcal{B}(\mathcal{S})$. The state-process is denoted by $\{S_t\}_{t=1}^{\infty}$.
- **Joint-Observation Process**: The joint-observation space $\mathcal{O}$ is a countable discrete space of the form $\mathcal{O} = \prod_{n=0}^{N} \mathcal{O}^{(n)}$, where $\mathcal{O}^{(0)}$ denotes the common observation space of all agents and $\mathcal{O}^{(n)}$ denotes the private observation space of agent $n \in [N]$. The joint-observation process is denoted by $\{O_t\}_{t=1}^{\infty}$ where $O_t = O_t^{(0:N)}$ and is such that at time t, agent $n \in [N]$ observes $O_t^{(0)}$ and $O_t^{(n)}$ only.
- **Joint-Action Process**: The joint-action space $\mathcal{A}$ is a finite discrete space of the form $\mathcal{A} = \prod_{n=1}^{N} \mathcal{A}^{(n)}$, where $\mathcal{A}^{(n)}$ denotes the action space of agent $n \in [N]$. The joint-action process is denoted by $\{A_t\}_{t=1}^{\infty}$ where $A_t = A_t^{(1:N)}$ and $A_t^{(n)}$ denotes the action of agent n at time t.[4] Since all $\mathcal{A}^{(n)}$'s and $\mathcal{A}$ are finite, they are all compact metric spaces.[5]
- **Transition-law**: At time $t \in \mathbb{N}$, given the current state S_t and current joint-action A_t, the next state S_{t+1} and the next joint-observation O_{t+1} are determined in a time-homogeneous manner, independent of all previous states, all previous and current joint-observations, and all previous joint-actions. The transition-law is given by

$$\mathcal{P}_{tr} \triangleq \{P_{saBo} : s \in \mathcal{S}, a \in \mathcal{A}, B \in \mathcal{B}(\mathcal{S}), o \in \mathcal{O}\}, \qquad (1)$$

where for all $t \in \mathbb{N}$,

$$\begin{aligned} &\mathbb{P}(S_{t+1} \in B, O_{t+1} = o | S_{1:t-1} = s_{1:t-1}, \\ &\quad O_{1:t} = o_{1:t}, A_{1:t-1} = a_{1:t-1}, S_t = s, A_t = a) \\ &= \mathbb{P}(S_{t+1} \in B, O_{t+1} = o | S_t = s, A_t = a) \qquad (2) \\ &\triangleq P_{saBo}. \end{aligned}$$

[4]The results in this work also hold if for every $(h_t^{(0)}, h_t^{(n)}) \in \mathcal{H}_t^{(0)} \times \mathcal{H}_t^{(n)}$, agent n is allowed to take action from a separate finite discrete space $\mathcal{A}^{(n)}(h_t^{(0)}, h_t^{(n)})$.

[5]Hence, also complete and separable.

• **Immediate-costs**: The immediate cost $c : \mathcal{S} \times \mathcal{A} \mapsto \mathbb{R}$ is a real-valued function whose expected discounted aggregate (to be defined later) we would like to minimize. On the other hand, the immediate cost $d : \mathcal{S} \times \mathcal{A} \mapsto \mathbb{R}^K$ is $\mathbb{R}^K$-valued function whose expected discounted aggregate we would like to keep within a specified threshold. For these reasons, we call c and d as the immediate objective and constraint costs respectively. We shall make use of the following assumption on immediate-costs in Theorem 1.

Assumption 1. *The immediate objective cost is bounded from below and the immediate constraint costs are bounded, i.e., there exist* $\underline{c} \in \mathbb{R}$ *and* $\underline{d}, \overline{d} \in \mathbb{R}^K$ *such that*

$$\underline{c} \leq c(\cdot,\cdot) \text{ and } \underline{d} \leq d(\cdot,\cdot) \leq \overline{d}. \tag{3}$$

Let $\bar{\underline{d}} = \|\underline{d}\|_\infty \vee \|\overline{d}\|_\infty$ *so that* $\|d(\cdot,\cdot)\|_\infty \leq \bar{\underline{d}} < \infty$.

• **Initial Distribution**: P_1 is a (fixed) probability measure for the initial state and initial joint-observation, i.e., $P_1 \in \mathcal{M}_1(\mathcal{S} \times \mathcal{O})$ and

$$P_1(B, o) \triangleq \mathbb{P}(S_1 \in B, O_1 = o). \tag{4}$$

• **Space of Policy-Profiles**: At time $t \in \mathbb{N}$, the common history of all agents is defined as all the common observations received thus far, i.e., $H_t^{(0)} \triangleq \left(O_{1:t}^{(0)}\right)$. Similarly, the private history of agent $n \in [N]$ at time t is defined as all observations received and all the actions taken by the agent thus far (except for those that are part of the common information), i.e.,

$$\begin{aligned} &H_1^{(n)} \triangleq O_1^{(n)} \setminus O_1^{(0)}, \text{ and} \\ &H_t^{(n)} \triangleq \left(H_{t-1}^{(n)}, (A_{t-1}^{(n)}, O_t^{(n)}) \setminus O_t^{(0)}\right) \ \forall t \in [2,\infty]_{\mathbb{Z}}. \end{aligned} \tag{5}$$

Finally, the joint history at time t is defined as the tuple of the common history and all the private histories at time t, i.e., $H_t \triangleq H_t^{(0:n)}$.

With the above setup, we define a (decentralized) behavioral policy-profile u as a tuple $u^{(1:N)} \in \mathcal{U} \triangleq \prod_{n=1}^N \mathcal{U}^{(n)}$ where $u^{(n)}$ denotes some behavioral policy used by agent n, i.e., $u^{(n)}$ itself is a tuple of the form $u_{1:\infty}^{(n)}$ where $u_t^{(n)}$ maps $\mathcal{H}_t^{(0)} \times \mathcal{H}_t^{(n)}$ to $\mathcal{M}_1\left(\mathcal{A}^{(n)}\right)$, and where agent n uses the distribution $u_t^{(n)}(H_t^{(0)}, H_t^{(n)})$ to choose its action $A_t^{(n)}$. We pause to emphasize that at any time t, each agent randomizes over its action-set independently of all other agents (*no common randomness*). Thus, given a joint-history $h_t \in \mathcal{H}_t$ at time t, the probability that joint-action $a_t \in \mathcal{A}$ is taken is given by

$$\begin{aligned} u_t(a_t|h_t) &\triangleq \prod_{n=1}^N u_t^{(n)}\left(h_t^{(0)}, h_t^{(n)}\right)\left(a_t^{(n)}\right) \\ &= \prod_{n=1}^N u_t^{(n)}\left(a_t^{(n)} \middle| h_t^{(0)}, h_t^{(n)}\right). \end{aligned} \tag{6}$$

Remark 1. *With Assumption 1, the conditional expectations* $\mathbb{E}_{P_1}[c(S_t, A_t) \mid H_t = h_t, A_t = a_t]$ *and* $\mathbb{E}_{P_1}[d(S_t, A_t) \mid H_t = h_t, A_t = a_t]$ *exist, are unique, and are bounded from below. Furthermore, the latter are element-wise finite.*

• **Optimization Problem**: Let $\mathbb{P}_{P_1}^{(u)}$ be the probability measure corresponding to policy-profile $u \in \mathcal{U}$ and initial-distribution P_1, and let $\mathbb{E}_{P_1}^{(u)}$ denote the corresponding expectation operator.[6] We define *infinite-horizon expected total discounted costs* $C : \mathcal{U} \to \mathbb{R} \cup \{\infty\}$ and $D : \mathcal{U} \to \mathbb{R}^K$ as

$$C(u) = C^{(P_1,\alpha)}(u) \triangleq \mathbb{E}_{P_1}^{(u)}\left[\sum_{t=1}^\infty \alpha^{t-1} c(S_t, A_t)\right], \tag{7}$$

$$\text{and } D(u) = D^{(P_1,\alpha)}(u) \triangleq \mathbb{E}_{P_1}^{(u)}\left[\sum_{t=1}^\infty \alpha^{t-1} d(S_t, A_t)\right]. \tag{8}$$

Remark 2. *Assumption 1 ensures that* $C(u) \in \mathbb{R} \cup \{\infty\}$*, and* $D(u) \in \mathbb{R}^K$ *with (absolute) element-wise bound* $\bar{\underline{d}}/(1-\alpha)$.

The decision process proceeds as follows: *i*) At time $t \in \mathbb{N}$, the current state S_t and observations O_t are generated; *ii*) Each agent $n \in [N]$ chooses an action $a^{(n)} \in \mathcal{A}^{(n)}$ based on $H_t^{(0)}, H_t^{(n)}$; *iii*) the immediate-costs $c(S_t, A_t), d(S_t, A_t)$ are incurred[7]; *iv*) The system moves to the next state and observations according to the transition-law $\mathcal{P}_{tr}$. Under these rules, the goal of the agents is to work cooperatively to solve the following constrained optimization problem.

$$\left.\begin{aligned} &\text{minimize } C(u) \\ &\text{subject to } u \in \mathcal{U} \text{ and } D(u) \leq \dot{D}. \end{aligned}\right\} \quad \textit{(Dec-C-POMDP)}$$

Here, $\dot{D}$ is a fixed K-dimensional real-valued vector. We refer to the solution of (*Dec-C-POMDP*) as its optimal value and denote it by $\underline{C} = \underline{C}^{(P_1,\alpha)}$. In particular, if the set of feasible policy-profiles is empty, we set $\underline{C}$ to ∞ and with slight abuse of terminology will consider any policy-profile in $\mathcal{U}$ to be optimal.

The following assumption about feasibility of (*Dec-C-POMDP*) will be used in one of the parts of Theorem 1.

Assumption 2 (Slater's Condition)**.** *There exists a policy-profile* $\overline{u} \in \mathcal{U}$ *and* $\zeta > 0$ *for which*

$$D(\overline{u}) \leq \dot{D} - \zeta 1. \tag{9}$$

III. Characterization of Strong Duality

To solve (*Dec-C-POMDP*), let us define the Lagrangian function $L : \mathcal{U} \times \mathcal{Y} \mapsto \mathbb{R} \cup \{\infty\}$ as follows.

$$\begin{aligned} L(u, \lambda) = L^{(P_1,\alpha)}(u, \lambda) &\triangleq C(u) + \langle \lambda, D(u) - \dot{D} \rangle \\ &= C(u) + \sum_{k=1}^K \lambda_k \left(D_k(u) - \dot{D}_k\right), \end{aligned} \tag{10}$$

Here, $\mathcal{Y} \triangleq \{\lambda \in \mathbb{R}^K : \lambda \geq 0\}$ is the set of tuples of K non-negative real-numbers, each commonly known as

[6] The existence and uniqueness of $\mathbb{P}_{P_1}^{(u)}$ can be ensured by an adaptation of the Ionesca-Tulcea theorem [36].

[7] In the planning context, the immediate-costs are known by all agents.

a Lagrange-multiplier. Our main result shows that the the solution $\underline{C}$ satisfies

$$\underline{C} = \inf_{u\in\mathcal{U}} \sup_{\lambda\in\mathcal{Y}} L\left(u,\lambda\right), \tag{11}$$

and that the inf and sup can be interchanged, i.e.,

$$\underline{C} = \sup_{\lambda\in\mathcal{Y}} \inf_{u\in\mathcal{U}} L\left(u,\lambda\right). \tag{12}$$

Theorem 1 (Strong Duality and Existence of Saddle Point)**.** *Under Assumption 1, the following statements hold.*

(a) The optimal value satisfies

$$\underline{C} = \inf_{u\in\mathcal{U}} \sup_{\lambda\in\mathcal{Y}} L\left(u,\lambda\right). \tag{13}$$

(b) A policy-profile $u^\star \in \mathcal{U}$ is optimal if and only if $\underline{C} = \sup_{\lambda\in\mathcal{Y}} L\left(u^\star,\lambda\right)$.

(c) Strong duality holds for (*Dec-C-POMDP*)*, i.e.,*

$$\underline{C} = \inf_{u\in\mathcal{U}} \sup_{\lambda\in\mathcal{Y}} L\left(u,\lambda\right) = \sup_{\lambda\in\mathcal{Y}} \inf_{u\in\mathcal{U}} L\left(u,\lambda\right). \tag{14}$$

Moreover, there exists a $u^\star \in \mathcal{U}$ such that $\underline{C} = \sup_{\lambda\in\mathcal{Y}} L\left(u^\star,\lambda\right)$ and $u^\star$ is optimal for (*Dec-C-POMDP*)*.*

(d) If Assumption 2 holds, then there also exists $\lambda^\star \in \mathcal{Y}$ such that the following saddle-point condition holds for all $(u,\lambda) \in \mathcal{U}\times\mathcal{Y}$,

$$L\left(u^\star,\lambda\right) \le L\left(u^\star,\lambda^\star\right) = \underline{C} \le L\left(u,\lambda^\star\right). \tag{15}$$

i.e., $u^\star$ minimizes $L\left(\cdot,\lambda^\star\right)$ and $\lambda^\star$ maximizes $L\left(u^\star,\cdot\right)$. In addition to this, the primal dual pair $(u^\star,\lambda^\star)$ satisfies the complementary-slackness condition:

$$\langle \lambda^\star, D\left(u^\star\right) - \dot{D}\rangle = 0. \tag{16}$$

Proof. (a) If $u\in\mathcal{U}$ is feasible (i.e., it satisfies $D\left(u\right) \le \dot{D}$), then the sup is obtained by choosing $\lambda = 0$, so

$$\sup_{\lambda\in\mathcal{Y}} L\left(u,\lambda\right) = C\left(u\right). \tag{17}$$

If $u\in\mathcal{U}$ is not feasible, then

$$\sup_{\lambda\in\mathcal{Y}} L\left(u,\lambda\right) = \infty. \tag{18}$$

Indeed, suppose WLOG that the k^{th} constraint is violated, i.e., $D_k\left(u\right) > \dot{D}_k$, then ∞ can be obtained by choosing λ_k arbitrarily large and setting other λ_k's to 0.

From (17), (18), and our convention that $\underline{C} = \infty$ whenever the feasible-set is empty, it follows that

$$\underline{C} = \inf_{u\in\mathcal{U}} \sup_{\lambda\in\mathcal{Y}} L\left(u,\lambda\right). \tag{19}$$

(b) By our convention on the value of $\underline{C}$ (when there is no feasible policy-profile), $u^\star$ is optimal if and only if $C\left(u^\star\right) = \underline{C}$, i.e., $\sup_{\lambda\in\mathcal{Y}} L\left(u^\star,\lambda\right) = \underline{C}$.

(c) To establish strong duality, we use [37][Proposition 4] which requires $\mathcal{U}$ and $\mathcal{Y}$ to be convex[8] topological spaces (with $\mathcal{U}$ being compact also). It is clear that $\mathcal{Y}$ is convex and we can endow it with the usual subspace topology of $\mathbb{R}^K$. For $\mathcal{U}$ however, we need to endow it with a suitable topology in which it is compact and then also show that it is convex. To achieve compactness, we can use the finiteness of the action-space $\mathcal{A}^{(n)}$ and the countability of observation-space $\mathcal{O}$ to associate $\mathcal{U}$ with a product of compact sets that are parameterized by (countable number of) all possible histories. Tychonoff's theorem (see [37][Proposition 4]) then helps achieve compactness under the product topology. (Convexity comes trivially). Now, we make this idea precise. For $t\in\mathbb{N}$ and $n \in [0,N]_{\mathbb{Z}}$, let $\mathcal{H}_t^{(n)}$ denote the set of all possible realizations of $H_t^{(n)}$. Then, by countability of observation and action spaces, the sets

$$\begin{aligned}
\mathcal{H}_t &\triangleq \prod_{n=0}^{N} \mathcal{H}_t^{(n)},\\
\mathcal{H}^{(n)} &\triangleq \bigcup_{t=1}^{\infty} \mathcal{H}_t^{(0)} \times \mathcal{H}_t^{(n)}, \text{ and}\\
\mathcal{H} &\triangleq \bigcup_{t=1}^{\infty} \mathcal{H}_t,
\end{aligned} \tag{20}$$

are countable. Here, $\mathcal{H}_t$ is the set of all possible joint-histories at time t, $\mathcal{H}^{(n)}$ is the set of all possible histories of agent n, and $\mathcal{H}$ is the set of all possible joint-histories. With this in mind, one observes that $\mathcal{U}$ is in one-to-one correspondence with the set $\mathcal{X}_{\mathcal{U}} \triangleq \prod_{n=1}^{N} \mathcal{X}_{\mathcal{U}^{(n)}}$, where

$$\mathcal{X}_{\mathcal{U}^{(n)}} \triangleq \prod_{h\in\mathcal{H}^{(n)}} \mathcal{M}_1\left(\mathcal{A}^{(n)}; h\right), \tag{21}$$

and $\mathcal{M}_1\left(\mathcal{A}^{(n)}; h\right)$ is a copy of $\mathcal{M}_1\left(\mathcal{A}^{(n)}\right)$ dedicated for agent-n's history h. For example, a given policy u would correspond to a point $x\in\mathcal{X}_{\mathcal{U}}$ such that $x_{n,\left(h_t^{(0)},h_t^{(n)}\right)} = u_t^{(n)}\left(\cdot|h_t^{(0)},h_t^{(n)}\right)$, and similarly, vice versa.

Since $\mathcal{A}^{(n)}$ is a complete separable (compact) metric space, by Prokhorov's Theorem (see [37][Proposition 6]), each $\mathcal{M}_1\left(\mathcal{A}^{(n)}; h\right)$ is a compact (and convex[9]) metric space (with the topology of weak-convergence). Therefore, endowing $\mathcal{X}_{\mathcal{U}^{(n)}}$ and $\mathcal{X}_{\mathcal{U}}$ with the product topology makes each a compact (and convex) metric space via Tychonoff's theorem (see [37][Proposition 4] which is also metrizable (via [37][Proposition 6]). Given the one-to-one correspondence, **from now onward, we assume that $\mathcal{U}^{(n)}$ and $\mathcal{U}$ have the same topology as that of $\mathcal{X}_{\mathcal{U}^{(n)}}$ and $\mathcal{X}_{\mathcal{U}}$ respectively**. Henceforth, we will consider C, D_k, and L as functions on topological spaces. Furthermore, since $\mathcal{U}^{(n)}$'s and $\mathcal{U}$ have been shown to be compact metric spaces (hence, also complete and separable), we can also define $\mathcal{B}\left(\mathcal{U}^{(n)}\right)$, $\mathcal{B}\left(\mathcal{U}\right) = \otimes_{n=1}^{N}\mathcal{B}\left(\mathcal{U}^{(n)}\right)$[10], and $\mathcal{M}_1\left(\mathcal{U}\right)$, where $\mathcal{M}_1\left(\mathcal{U}\right)$ is compact (and convex) metrizable space by Prokhorov's theorem (see [37][Proposition 6]).

To establish part (c), it will be helpful to work with (decentralized) mixtures of behavioral policy-profiles – wherein

[8]Convexity is a set property rather than a topological property. In the rest of the paper, by a "convex topological space", we mean convexity of the set on which the topology is defined.

[9]Convexity of $\mathcal{M}_1\left(\mathcal{A}^{(n)}\right)$ is trivial.

[10]For separable metric spaces $\mathcal{W}_1,\mathcal{W}_2,\ldots$, $\mathcal{B}\left(\mathcal{W}_1\times\mathcal{W}_2\times\ldots\right) = \mathcal{B}\left(\mathcal{W}_1\right)\otimes\mathcal{B}\left(\mathcal{W}_2\right)\otimes\ldots$. See [38][Lemma 1.2].

each agent $n \in [N]$ first uses a measure $\mu^{(n)} \in M_1(\mathcal{U}^{(n)})$[11] to choose its policy-profile $u^{(n)}$ and then proceeds with it from time 1 onward. We denote this set of mixtures by $\mathcal{U}_{\text{mixed}} \triangleq \prod_{n=1}^{N} M_1(\mathcal{U}^{(n)})$, whose typical element, denoted by $\mu \triangleq \times_{n=1}^{N} \mu^{(n)}$, is a factorized measure on $\mathcal{U}$, i.e., $\mu^{(n)} \in M_1(\mathcal{U}^{(n)})$. Since $\mathcal{U}_{\text{mixed}} \subseteq \mathcal{M}_1(\mathcal{U})$, we endow it with the same metric as that of $\mathcal{M}_1(\mathcal{U})$. Now, we can extend the definitions of C, D, and L to $\widehat{C} : \mathcal{U}_{\text{mixed}} \to \mathbb{R} \cup \{\infty\}$, $\widehat{D} : \mathcal{U}_{\text{mixed}} \to \mathbb{R}^K$, and $\widehat{L} : \mathcal{U}_{\text{mixed}} \times \mathcal{Y} \to \mathbb{R} \cup \{\infty\}$ as follows:

$$\begin{aligned}
\widehat{C}(\mu) = \widehat{C}_{P_1}(\mu) &\triangleq \mathbb{E}_{P_1}^{(\mu)}\left[\sum_{t=1}^{\infty} \alpha^{t-1} c(S_t, A_t)\right], \\
\widehat{D}(\mu) = \widehat{D}_{P_1}(\mu) &\triangleq \mathbb{E}_{P_1}^{(\mu)}\left[\sum_{t=1}^{\infty} \alpha^{t-1} d(S_t, A_t)\right], \text{ and} \\
\widehat{L}(\mu, \lambda) = \widehat{L}^{(P_1,\alpha)}(\mu, \lambda) &= \widehat{C}(\mu) + \langle \lambda, \widehat{D}(u) \rangle.
\end{aligned} \tag{22}$$

In [37][Lemma 4] it is shown that any $\mu \in \mathcal{U}_{\text{mixed}}$ can be replicated by a behavioral policy-policy $u \in \mathcal{U}$. [37][Corollary 4.1] then shows that

$$\begin{aligned}
\inf_{u \in \mathcal{U}} \sup_{\lambda \in \mathcal{Y}} L(u, \lambda) &= \inf_{\mu \in \mathcal{U}_{\text{mixed}}} \sup_{\lambda \in \mathcal{Y}} \widehat{L}(\mu, \lambda), \text{ and} \\
\sup_{\lambda \in \mathcal{Y}} \inf_{u \in \mathcal{U}} L(u, \lambda) &= \sup_{\lambda \in \mathcal{Y}} \inf_{\mu \in \mathcal{U}_{\text{mixed}}} \widehat{L}(\mu, \lambda).
\end{aligned} \tag{23}$$

In light of (23), it suffices to prove part (c) for $\widehat{L}$. By definition, $\widehat{L}$ is affine and thus trivially concave in λ. [37][Proposition 8] implies that $\widehat{L}$ is convex in μ and Lemma 2 shows that $\widehat{L}$ is lower semi-continuous[12] in μ. From [37][Proposition 11], it then follows that

$$\inf_{u \in \mathcal{U}_{\text{mixed}}} \sup_{\lambda \in \mathcal{Y}} \widehat{L}(u, \lambda) = \sup_{\lambda \in \mathcal{Y}} \inf_{\mathcal{U}_{\text{mixed}}} \widehat{L}(u, \lambda),$$

and that there exists $\mu^\star \in \mathcal{U}_{\text{mixed}}$ such that

$$\sup_{\lambda \in \mathcal{Y}} \widehat{L}(\mu^\star, \lambda) = \inf_{u \in \mathcal{U}_{\text{mixed}}} \sup_{\lambda \in \mathcal{Y}} \widehat{L}(u, \lambda).$$

The optimality of $\mu^\star$ is implied by parts (b) and (a).

(d) This follows from Lagrange-multiplier theory.

This concludes the proof. □

Lemma 2 (Lower Semi-Continuity of $\widehat{L}$ on $\mathcal{U}_{\text{mixed}}$)**.** *Under Assumption 1, $\widehat{L}$ is lower semi-continuous on $\mathcal{U}_{\text{mixed}}$.*

Proof. Fix $\lambda \in \mathcal{Y}$ and $\mu \in \mathcal{U}_{\text{mixed}}$. Let $\{\mu_i\}_{i=1}^{\infty}$ be a sequence of (factorized) measures in $\mathcal{U}_{\text{mixed}}$ that converges to $\mu \in \mathcal{U}_{\text{mixed}}$. Since $\mathcal{U}_{\text{mixed}} \subseteq \mathcal{M}_1(\mathcal{U})$ and has the same metric as $\mathcal{M}_1(\mathcal{U})$, it means that $\{\mu_i\}_{i=1}^{\infty}$ also converges to μ in $\mathcal{M}_1(\mathcal{U})$. We want to show

$$\liminf_{i \to \infty} \mathbb{E}_{P_1}^{(U \sim \mu_i)}[L(U, \lambda)] \geq \mathbb{E}_{P_1}^{(U \sim \mu)}[L(U, \lambda)].$$

By Lemma 3, L is point-wise lower semi-continuous on $\mathcal{U}$. Therefore, [37][Proposition 9] applies on $\mathcal{M}_1(\mathcal{U})$ and the above inequality follows. □

[11] $M_1(\cdot)$ denotes the set of all probability measures on $\cdot$.

[12] For definition of lower semi-continuity, see [37][Definition 1].

Lemma 3 (Lower Semi-Continuity of L on $\mathcal{U}$)**.** *Under Assumption 1, the functions C and D_k's are lower semi-continuous on $\mathcal{U}$. Hence, L is lower semi-continuous on $\mathcal{U}$.*

Proof. We will prove the statement for C. The proof of lower semi-continuity of D_k's is similar. For brevity, let

$$\begin{aligned}
p(u, t, h_t, a_t) &= p_{P_1}(u, t, h_t, a_t) \triangleq \mathbb{P}_{P_1}^{(u)}(H_t = h_t, A_t = a_t), \\
W(u, t, h_t, a_t) &= W_{P_1}(u, t, h_t, a_t) \\
&\triangleq p(u, t, h_t, a_t)\, \mathbb{E}_{P_1}[c(S_t, A_t) | H_t = h_t, A_t = a_t].
\end{aligned}$$

Then,

$$\begin{aligned}
C(u) &= \mathbb{E}_{P_1}^{(u)}\left[\sum_{t=1}^{\infty} \alpha^{t-1} c(S_t, A_t)\right] \\
&= \mathbb{E}_{P_1}^{(u)}\left[\sum_{t=1}^{\infty} \alpha^{t-1} (c(S_t, A_t) - \underline{c})\right] + \sum_{t=1}^{\infty} \alpha^{t-1} \underline{c} \\
&\overset{\text{(a)}}{=} \sum_{t=1}^{\infty} \alpha^{t-1} \mathbb{E}_{P_1}^{(u)}[c(S_t, A_t) - \underline{c}] + \sum_{t=1}^{\infty} \alpha^{t-1} \underline{c} \\
&\overset{\text{(b)}}{=} \sum_{t=1}^{\infty} \alpha^{t-1} \mathbb{E}_{P_1}^{(u)}[\mathbb{E}_{P_1}[c(S_t, A_t) | H_t, A_t]] \\
&= \sum_{t=1}^{\infty} \sum_{h_t \in \mathcal{H}_t} \sum_{a_t \in \mathcal{A}} \alpha^{t-1} W(u, t, h_t, a_t).
\end{aligned}$$

Here, (a) follows from applying the Monotone-Convergence Theorem to the (increasing non-negative) sequence $\{\sum_{t=1}^{i} \alpha^{t-1} (c(S_t, A_t) - \underline{c})\}_{i=1}^{\infty}$ (see [37][Proposition 1] and (b) uses the tower property of conditional expectation.[13]

Let $\{{}^i u\}_{i=1}^{\infty}$ be a sequence in $\mathcal{U}$ that converges to u. By Fatou's Lemma (see [37][Proposition 3]),

$$\liminf_{i \to \infty} C({}^i u) \geq \sum_{t=1}^{\infty} \sum_{h_t \in \mathcal{H}_t} \sum_{a_t \in \mathcal{A}} \alpha^{t-1} \liminf_{i \to \infty} W({}^i u, t, h_t, a_t). \tag{24}$$

Following [37][Lemma 5], $p({}^i u, t, h_t, a_t)$ converges to $p(u, t, h_t, a_t)$. Therefore,

$$\lim_{i \to \infty} W({}^i u, t, h_t, a_t) = W(u, t, h_t, a_t). \tag{25}$$

From (24) and (25), it follows that $\liminf_{i \to \infty} C({}^i u) \geq C(u)$, which establishes the lower semi-continuity of $C(u)$. □

IV. Conclusion

In this work, we studied a (cooperative) decentralized constrained POMDP in the setting of infinite-horizon expected total discounted costs. We established strong duality and existence of a saddle point using an extension of Sion's Minimax Theorem which required giving a suitable topology to the space of all possible policy-profiles and then establishing lower semi-continuity of the Lagrangian function. The strong duality result provides a firm theoretical footing for future development of primal-dual type planning and

[13] The conditional expectations $\mathbb{E}_{P_1}[c(S_t, A_t) | H_t, A_t]$ exist and are unique because $c(\cdot, \cdot)$ is bounded from below.

learning algorithms for Dec-C-POMDPs—see [39] for one such algorithm.

V. ACKNOWLEDGMENTS

This work was funded by NSF via grants ECCS2038416, EPCN1608361, EARS1516075, CNS1955777, CCF2008130, and CMMI2240981 for V. Subramanian, and grants EARS1516075, CNS1955777, CCF2008130, and CMMI2240981 for N. Khan. The authors would also like to thank Dr. Hsu Kao for helpful discussions.

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