2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Homogeneous Finite/Fixed-time Stabilization with Quantization

Yu Zhou, Andrey Polyakov, Gang Zheng, Xubin Ping

Abstract— **The paper investigates the homogeneous stabilization of linear plants with uniform and logarithmic quantization of state measurements. To achieve quadratic-like stability, the homogeneous control system with quantized state measurements is transformed into a standard homogeneous one to obtain a sufficient stability condition. It is shown that a homogeneous feedback stabilization with a uniform quantizer may achieve only practical stability, while, in the case of a logarithmic quantizer, a global (finite-time, nearly fixed-time, exponential) stability is preserved, provided that the quantization density is sufficiently high. Theoretical results are demonstrated in simulations.**

I. INTRODUCTION

Quantization is the process of digitizing a continuous signal. In network control, signals can be quantized before transmission to overcome time delays and save communication resources. In recent years, the problem of controlling systems through quantized signals has garnered significant interest. Most feedback control designs that utilize digitized data are developed under the emulation method, wherein the controller is first designed in a continuous time and then implemented as a digitalized-data controller [1], [2]. This paper develops the mentioned approach for homogeneous control systems.

The homogeneity is a dilation symmetry useful for control design and stability analysis of the continuous-time model. Homogeneous methods have been extensively studied in the last two decades (see e.g., [3], [4], [5], [6], [7], [8]). In the context of continuous-time homogeneous systems, many important properties can help for a control design (e.g., the existence of homogeneous Lyapunov functions, equivalence between local and global stability, and the finite/fixed-time convergence dependently on a homogeneity degree, etc.). However, when these methods are applied to real-world systems, they often need to be implemented using digital control techniques, such as quantization. Quantization can be modeled as a mapping from a continuous domain to a finite (or countable) set of discrete values, which can result in the loss of some important properties that were discovered in the continuous-time case.

There are two classical quantizers: uniform quantizer and logarithmic quantizer (see, e.g., [9], [10] or Section II of this paper). The linear stabilization using state measurements quantized uniformly is studied in [11], [12]. The interaction between control design and quantizer design was demonstrated in [13], where an optimal logarithmic quantizer for a single-input-single-output linear system is introduced and studied. For the logarithmic quantizers, the quantization step increases exponentially as the input increases. This approach follows the intuitive idea that the farther from the origin the state is, the less precise control action and knowledge about the state are needed. If the number of quantization levels is not limited, a logarithmic quantization does not destroy the quadratic stability[1] in some cases [10].

Various results about feedback control using quantized state measurements can be found in the literature (*e.g.* [12], [13], [14], [15]). The simplest approach is proposed in [9], where the logarithmic quantization error is represented as sector-bounded non-linearity. Most of the results are obtained for linear controllers, and the corresponding analysis is essentially based on quadratic stability. Since the asymptotically stable homogeneous system is topologically equivalent to a quadratically stable [7], then the aforementioned observation motivated us to investigate the stability of homogeneous stabilization with quantized state measurement. Specifically, we focused on homogeneous control design based on the canonical homogeneous norm ([16], [7], [8]). This paper makes a twofold contribution. Firstly, we derive a sufficient stability condition for stabilization with a general quantization function. Secondly, we investigate emulated homogeneous stabilization using uniform and logarithmic quantizers. It is shown that, for a uniform quantization, just practical stability can be guaranteed, while, for a logarithmic quantization, the global finite-time, exponential, and nearly fixed-time stability of the homogeneous system can be preserved.

The paper is organized as follows. In Section II, we provide a brief introduction to homogeneous control systems analysis and design. The detailed problem statement is given in Section III. Section IV presents the main results of the paper, which include a stability condition with a general quantization function; stability of a homogeneous control system with uniform and logarithmic quantization. Section V presents simulation results.

Notations

$\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$; $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$; $\mathbf{0}$ denotes the zero vector from $\mathbb{R}^n$; $\operatorname{diag}\{\lambda_i\}_{i=1}^n$ is the diagonal matrix with elements

This work is partially supported by Chinese Science Council (CSC) and and by National Natural Science Foundation of China under Grant 62050410352.

Y. Zhou, A. Polyakov and G. Zheng are with Inria, Univ. Lille, CNRS, Centale Lille, France, e-mail: yu.zhou(andrey.polyakov, gang.zheng)@inria.fr.

Xubin Ping is with Xidian University, Xi'an, Shanxi China, e-mail: pingxubin@126.com.

[1] A system is quadratically stable if it admits the quadratic Lyapunov function

$\lambda_i; P \succ 0 (\prec 0, \succeq 0, \preceq 0)$ for $P \in \mathbb{R}^{n\times n}$ means that the matrix P is symmetric and positive (negative) definite (semidefinite); $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ represent the minimal and maximal eigenvalue of a matrix $P = P^\top$; for $P \succeq 0$ the square root of P is a matrix $M = P^{\frac{1}{2}}$ such that $M^2 = P$; Let denote by $\mathcal{K}$ the set of continuous strictly increasing functions map $\mathbb{R}_+$ to $\mathbb{R}_+$.

II. PRELIMINARIES: HOMOGENEOUS STABILIZATION

Homogeneity refers to a class of dilation symmetries, which have been shown to possess several useful properties for control design and analysis [17], [18], [19], [20], [6]. In finite-dimensional systems, the linear dilation is widely used.

Definition 1: [7] A mapping $\mathbf{d}(s) : \mathbb{R}^n \mapsto \mathbb{R}^n$, $s \in \mathbb{R}$ is said to be a group of linear dilation in $\mathbb{R}^n$ if

- $\mathbf{d}(0) = I_n$, $\mathbf{d}(s+t) = \mathbf{d}(s)\mathbf{d}(t) = \mathbf{d}(t)\mathbf{d}(s), \forall s, t \in \mathbb{R}$;
- $\lim\limits_{s\to-\infty} \|\mathbf{d}(s)x\| = 0$ and $\lim\limits_{s\to\infty} \|\mathbf{d}(s)x\| = \infty$.

The dilation $\mathbf{d}$ is continuous if the mapping $s \mapsto \mathbf{d}(s)$ is continuous. Any linear continuous dilation can be represented as

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}} := \sum_{i=0}^{\infty} \frac{s^i G_{\mathbf{d}}^i}{i!}.$$

The matrix $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ is the generator of the dilation $\mathbf{d}$, and is anti-Hurwitz, meaning that all its eigenvalues have non-negative real parts. For the standard dilation, $G_{\mathbf{d}} = I_n$, and for weighted dilation, $G_{\mathbf{d}}$ is diagonal. Monotonicity is essential in homogeneous dynamic analysis. Let us recall the definition of a monotone dilation.

Definition 2: A dilation $\mathbf{d}$ is strictly monotone in $\mathbb{R}^n$ if $\exists \beta > 0$ such that

$$\|\mathbf{d}(s)\| \le e^{\beta s}, \quad \forall s \le 0.$$

The monotonicity of the continuous dilation guarantees the uniqueness of the homogeneous projection of $x \neq \mathbf{0}$ on the unit sphere. This property is essential in defining a homogeneous norm.

Definition 3: [7] The functional $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \to (0, +\infty)$ defined as $\|\mathbf{0}\|_{\mathbf{d}} = 0$ and

$$\|x\|_{\mathbf{d}} = e^s, \text{ where } s \in \mathbb{R} : \|\mathbf{d}(-s)\, z\| = 1, \quad x \neq \mathbf{0}$$

is called the canonical homogeneous norm in $\mathbb{R}^n$, where $\mathbf{d}$ is a continuous monotone linear dilation on $\mathbb{R}^n$.

The functional $\|\cdot\|_{\mathbf{d}}$ is single-valued and continuous at the origin. Although the canonical homogeneous norm does not satisfy the triangle inequality in $\mathbb{R}^n$, it is a norm in a special Euclidean space $\mathbb{R}^n_{\mathbf{d}}$ being homeomorphic to $\mathbb{R}^n$.

Proposition 1: A dilation $\mathbf{d}$ is strictly monotone in $\mathbb{R}^n$ equipped with the norm $\|z\| = \sqrt{z^\top P z}$ if and only if the following linear matrix inequality holds

$$PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0,$$

where $G_{\mathbf{d}} \in \mathbb{R}^n$ is the generator of the dilation $\mathbf{d}$. Moreover, one has

$$\begin{cases} e^{\alpha s} \le \|\mathbf{d}(s)\| \le e^{\beta s}, & s \le 0, \\ e^{\beta s} \le \|\mathbf{d}(s)\| \le e^{\alpha s}, & s \ge 0, \end{cases} \tag{1}$$

where $\alpha = \frac{1}{2}\lambda_{\max}\left(P^{\frac{1}{2}} G_{\mathrm{d}} P^{-\frac{1}{2}} + P^{-\frac{1}{2}} G_{\mathrm{d}}^\top P^{\frac{1}{2}}\right) > 0$, $\beta = \frac{1}{2}\lambda_{\min}\left(P^{\frac{1}{2}} G_{\mathrm{d}} P^{-\frac{1}{2}} + P^{-\frac{1}{2}} G_{\mathrm{d}}^\top P^{\frac{1}{2}}\right) > 0$. and there exists $M \ge 1$ such that

$$\begin{cases} \frac{1}{M}\|x\|_{\mathbf{d}}^{\alpha} \le \|x\| \le \|x\|_{\mathbf{d}}^{\beta}, & \|x\| \le 1, \\ \|x\|_{\mathbf{d}}^{\beta} \le \|x\| \le M\|x\|_{\mathbf{d}}^{\alpha}, & \|x\| > 1. \end{cases} \tag{2}$$

The canonical homogeneous norm can be utilized as an implicit Lyapunov function in homogeneous control design, thereby simplifying both the control design and analysis procedures. The partial derivative of the canonical homogeneous norm induced by $\|x\| = \sqrt{x^\top P x}$ can be calculated as follows [7]:

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \|x\|_{\mathbf{d}} \frac{x^\top \mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}}) P \mathbf{d}(-\ln\|x\|_{\mathbf{d}})}{x^\top \mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d}(-\ln\|x\|_{\mathbf{d}}) x}. \tag{3}$$

Definition 4: [19] A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ (resp., a function $h : \mathbb{R}^n \to \mathbb{R}$) is said to be $\mathbf{d}$-homogeneous of degree $\mu \in \mathbb{R}$ if

$$f(\mathbf{d}(s)) = e^{\mu s}\mathbf{d}(s) f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R},$$
$$(\text{resp., } h(\mathbf{d}(s)) = e^{\mu s} h(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}),$$

where $\mathbf{d}$ is a linear continuous dilation in $\mathbb{R}^n$.

In this paper, we investigate the stability of a homogeneous control system under quantized state measurement. The control algorithm for continuous-time systems is taken from the paper [8], which considers the linear time-invariant system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \tag{4}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u : \mathbb{R}^n \to \mathbb{R}^m$ is the feedback control to be designed, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are system matrices.

Theorem 1: [8] If the linear equation

$$AG_0 + BY_0 = G_0 A + A, \quad G_0 B = \mathbf{0}. \tag{5}$$

has a solution $G_0 \in \mathbb{R}^{n\times n}$ and $Y_0 \in \mathbb{R}^{m\times n}$ such that $G_0 - I_n$ is invertible, then for any $\mu \ge -1$ such that $G_{\mathbf{d}} = I_n + \mu G_0$ is anti-Hurwitz, the linear system (4) with the control

$$u(x) = K_0 x + \|x\|_{\mathbf{d}}^{1+\mu} K \mathbf{d}\left(-\ln\|x\|_{\mathbf{d}}\right) x, \quad K = YX^{-1} \tag{6}$$

is $\mathbf{d}$-homogeneous of degree μ and globally asymptotically stable provided that $K_0 = Y_0(G_0 - I_n)^{-1}$, $A_0 = A + BK_0$, $X \in \mathbb{R}^{n\times n}, Y \in \mathbb{R}^{m\times n}$ satisfy the following algebraic system

$$\begin{cases} XA_0^\top + A_0 X + Y^\top B^\top + BY + \rho\left(XG_{\mathbf{d}}^\top + G_{\mathbf{d}} X\right) = \mathbf{0} \\ XG_{\mathbf{d}}^\top + G_{\mathbf{d}} X \succ 0, \quad X \succ 0 \end{cases} \tag{7}$$

and the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is induced by the norm $\|x\| = \sqrt{x^\top X^{-1} x}$. Moreover, the closed-loop system is

- globally uniformly finite-time stable[2] for $\mu < 0$;
- globally uniformly exponentially stable for $\mu = 0$;
- globally uniformly nearly fixed-time stable[3] for $\mu > 0$.

If the matrix A is nilpotent and $m = 1$ then $K_0 = \mathbf{0}$.

[2]The system (4), (6) is finite-time stable if it is Lyapunov stable and $\exists T(x_0) : \|x(t)\| = 0, \forall t \ge T(x_0), \forall x_0 \in \mathbb{R}^n$.

[3]The system (4), (6) is uniformly nearly fixed-time stable if it is Lyapunov stable and $\forall r > 0, \exists T_r > 0 : \|x(t)\| < r, \forall t \ge T_r$ independently of $x_0 \in \mathbb{R}^n$.

III. Problem Statement

Let us consider the single-input system

$$\dot{x} = Ax + Bu(x), \quad u(x) = \|x\|_{\mathbf{d}}^{1+\mu} K\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x, \quad (8)$$

where $A = \begin{pmatrix} \mathbf{0} & I_{n-1} \\ 0 & \mathbf{0} \end{pmatrix} \mathbb{R}^{n\times n}$, $B = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \mathbb{R}^{n\times 1}$, and the feedback control u is designed using Theorem 1. In this case, $\mathbf{d}(s) = \mathrm{diag}(e^{r_1 s}, e^{r_2 s}, c\cdots, e^{r_n s})$, $r_i > 0$, $i = 1, 2, \cdots, n$ is a weighted dilation.

We assume that the state measurement are quantized. Mathematically, the latter means the state x in the control is replaced with $\mathfrak{q}(x)$, where

$$\mathfrak{q} : \mathbb{R}^n \mapsto Q = \{i \in \mathbb{N} : \; q_i \in \mathbb{R}^n\}, \quad (9)$$

is a piecewise constant function that maps $\mathbb{R}^n$ to a discrete subset of $\mathbb{R}^n$.

The aim of this paper is to investigate the stability of closed-loop control system in the case of quantized state measurements.

We consider two commonly used quantizers:

- *Uniform quantizer*: uniform quantizer with a quantization step size equal to some bounded value Δ is given by

$$q(y) = \Delta \cdot \left\lfloor \frac{y}{\Delta} + \frac{1}{2} \right\rfloor, \quad y \in \mathbb{R}$$

  where the notation $\lfloor\rfloor$ denotes the floor function. For the vector case, we define $\mathfrak{q}(x) = (q(x_1), ..., q(x_n))^\top$. The quantization error is bounded by the quantization step:

$$\|\mathfrak{q}(x) - x\|_\infty \leq \Delta. \quad (10)$$

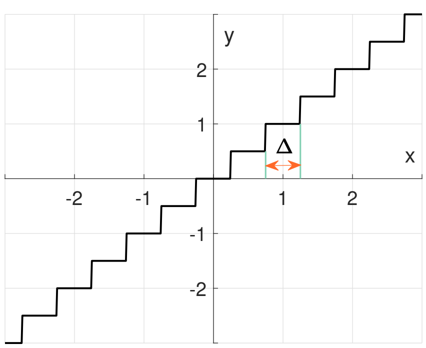

Fig. 1. Uniform quantization

- *Logarithmic quantizer*: a static logarithmic quantizer is described by

$$q(y) = \begin{cases} \rho^i, & \text{if } \frac{1}{1+\delta}\rho^i < y \leq \frac{1}{1-\delta}\rho^i, \\ & i = 0, \pm 1, \pm 2, \ldots \\ 0, & \text{if } y = 0 \\ -q(-y), & \text{if } y < 0 \end{cases}$$

  where $\rho \in (0,1)$ represents the quantization density and $\delta = (1-\rho)/(1+\rho)$. A small ρ (or large δ) implies coarse quantization, and a large ρ (or small δ) means dense quantization.

For the logarithmis quantizer, the quantization error is sector bounded:

$$\|\mathfrak{q}(x) - x\|_\infty \leq \delta\|x\|_\infty, \; \delta \in (0,1) \quad (11)$$

the quantization is vanishing as state goes to the origin. In order to do not change the equilibrium of system, we invoke following assumption.

Assumption 1: Assume $q(0) = 0$ for both quantizers .

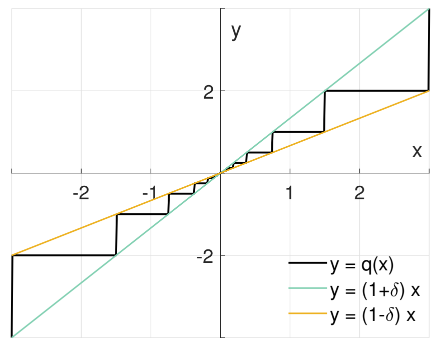

Fig. 2. Logarithmic quantization

IV. Main results

The closed-loop system with quantized state measurements can be represented as follows:

$$\dot{x} = Ax + B\|\mathfrak{q}(x)\|_{\mathbf{d}}^{1+\mu} K\mathbf{d}(-\ln\|\mathfrak{q}(x)\|_{\mathbf{d}})\mathfrak{q}(x). \quad (12)$$

The latter system is discontinuous. Its solution is understood in the sense of Filippov [21].

A. Stability of homogeneous control system with quantized state measurements

Since $\|x\|\mathbf{d}$ is a Lyapunov function for the quantization-free system (8), Theorem 4 in [7] shows that the quantization-free system (8) can be transformed to a standard homogeneous system (*i.e.* $G_{\mathbf{d}} = I$) of degree μ. Moreover, the corresponding transformation of the system (8) gives a quadratically stable quasi-linear system, so the analysis of the robustness with respect to the quatization of the state can follow the conventional ideas known for linear quantized systems.

The mentioned transformation is given by $z = \|x\|_{\mathbf{d}}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x$. It results in

$$\begin{aligned} \dot{z} &= (I - G_{\mathbf{d}})\mathbf{d}(-s)x\tfrac{d\|x\|_{\mathbf{d}}}{dt} + \|x\|_{\mathbf{d}}\mathbf{d}(-s)\dot{x} \\ &= \|x\|_{\mathbf{d}}\tfrac{(I-G_{\mathbf{d}})\mathbf{d}(-s)xx^\top\mathbf{d}^\top(-s)P\mathbf{d}(-s)\dot{x}}{x^\top\mathbf{d}^\top(-s)PG_{\mathbf{d}}\mathbf{d}(-s)x} + \|x\|_{\mathbf{d}}\mathbf{d}(-s)\dot{x}, \end{aligned} \quad (13)$$

where $s = \ln\|x\|_{\mathbf{d}}$. Taking the derivative of $\|x\|_{\mathbf{d}}$, we have

$$\dot{z} = \|x\|_{\mathbf{d}} \left(\tfrac{(I-G_{\mathbf{d}})\mathbf{d}(-s)xx^\top\mathbf{d}^\top(-s)P}{x^\top\mathbf{d}^\top(-s)PG_{\mathbf{d}}\mathbf{d}(-s)x} + I \right) \mathbf{d}(-s)\dot{x} \quad (14)$$

Taking $z = \|x\|_{\mathbf{d}}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x$ and $\|z\| = \|x\|_{\mathbf{d}}$, then $\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x = \frac{z}{\|z\|}$. Moreover, for the quantization-free system (8), we have

$$\dot{z} = \|z\| H_z \mathbf{d}(-s)\left(Ax + \|x\|_{\mathbf{d}}^{1+\mu} BK\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x\right) \tag{15}$$

where $H_z = \frac{(I-G_{\mathbf{d}})zz^\top P}{z^\top PG_{\mathbf{d}}z} + I$. Since $A\mathbf{d}(-s) = e^{\mu s}A\mathbf{d}(-s)$ and $\mathbf{d}(-s)B = e^{-s}B$, we have

$$\begin{aligned}\dot{z} &= \|z\|H_z\left(\|z\|^{\mu}A\tfrac{z}{\|z\|} + \|z\|^{\mu}BK\mathbf{d}(-\ln\|z\|_{\mathbf{d}})z\right)\\ &= \|z\|^{\mu}H_z\left(A_0 + BK\right)\tfrac{z}{\|z\|}\end{aligned} \tag{16}$$

The latter transformed system is standard homogeneous of degree μ. If (8) is asymptotically stable, then the transformed system is also asymptotically stable.

According to the definition of canonical homogenous norm, we have $\|x\|_{\mathbf{d}} = \|z\|$, then the quadratic form $z^\top Pz$ is Lyapunov function of system (16). Besides

$$\begin{aligned}z^\top P\dot{z} &= \|z\|^{1+\mu}z^\top P\dot{z}\\ &= \|z\|^{\mu}z^\top P\left(\tfrac{(I-G_{\mathbf{d}})zz^\top P}{z^\top PG_{\mathbf{d}}z} + I\right)(A+BK)z\end{aligned} \tag{17}$$

Then

$$z^\top P\dot{z} = \tfrac{1}{2}\|z\|^{\mu}\left(\tfrac{z^\top Pz}{z^\top PG_{\mathbf{d}}z}\right)z^\top P(A+BK)z \tag{18}$$

Since P is positive definite, and from the monotonicity of dilation we have $PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0$, then there exists a positive constant a_1, such that

$$z^\top P\dot{z} \le a_1\|z\|^{\mu}z^\top P(A+BK)z \tag{19}$$

where $a_1 = \frac{\lambda_{\min}(P)}{\lambda_{\max}(G_{\mathbf{d}}^\top P + PG_{\mathbf{d}})}$. Since system (16) is asymptaltically stable, we conclude that $a_1\|z\|^{\mu}z^\top P(A+BK)z$ is negative definite.

Based on the quantization-free standard homogeneous system and the quadratic Lyapunov function, the stability of the system with control using a quantized state is investigated.

Theorem 2: Let $\mathbf{d}, \|\cdot\|_{\mathbf{d}}, \|\cdot\|$ be as in Theorem 1. Let $\mathfrak{q} : \mathbb{R}^n \mapsto Q = \{i \in \mathbb{N} : \ q_i \in \mathbb{R}^n\}$ be a quantization function, where Q is a discrete set. If $\sup_{x\neq\mathbf{0}}\|\mathbf{d}(-\ln\|x\|_{\mathbf{d}})(q(x) - x)\|$ is sufficiently small, then the system (12) is globally asymptotically stable.

Proof: Using the coordinate transformation considered above we derive

$$\dot{z} = \|z\|H_z\left(\|z\|^{\mu}A_0z + \mathbf{d}(-s)\|\mathfrak{q}\|_{\mathbf{d}}^{1+\mu}BK\mathbf{d}(-\ln\|\mathfrak{q}\|)\mathfrak{q}\right), \tag{20}$$

where $s = \ln\|x\|_{\mathbf{d}}$. Let quantization error denote by $\sigma = q(x) - x$. Using the homogeneity, we obtain

$$\begin{aligned}&\mathbf{d}(-\ln\|x\|_{\mathbf{d}})\|\mathfrak{q}\|_{\mathbf{d}}^{1+\mu}BK\mathbf{d}(-\ln\|\mathfrak{q}\|)\mathfrak{q}\\ &= \mathbf{d}(-\ln\|x\|_{\mathbf{d}})\|x+\sigma\|_{\mathbf{d}}^{1+\mu}BK\mathbf{d}(-\ln\|x+\sigma\|_{\mathbf{d}})(x+\sigma)\\ &= \|z\|^{\mu}\left\|\tfrac{z}{\|z\|}+\sigma_z\right\|_{\mathbf{d}}^{1+\mu}BK\mathbf{d}\left(-\ln\left\|\tfrac{z}{\|z\|}+\sigma_z\right\|_{\mathbf{d}}\right)\left(\tfrac{z}{\|z\|}+\sigma_z\right)\end{aligned} \tag{21}$$

where $\sigma_z = \mathbf{d}(-s_z)\sigma$. Thus,

$$\begin{aligned}\dot{z} =& \|z\|^{1+\mu}H_z\left(A\tfrac{z}{\|z\|} + BK\tfrac{z}{\|z\|}\right)\\ &+ \|z\|H_z\mathbf{d}(-s)\|\mathfrak{q}\|_{\mathbf{d}}^{1+\mu}BK\mathbf{d}(-\ln\|\mathfrak{q}\|)\mathfrak{q}\\ &- \|z\|^{1+\mu}H_zBK\tfrac{z}{\|z\|}\\ =& \|z\|^{\mu}H_z(Az + BKz)\\ &+ \|z\|^{1+\mu}H_z\|z_\sigma\|_{\mathbf{d}}^{1+\mu}BK\mathbf{d}\left(-\ln\|z_\sigma\|_{\mathbf{d}}\right)z_\sigma\\ &- \|z\|^{1+\mu}H_z\left\|\tfrac{z}{\|z\|}\right\|_{\mathbf{d}}^{1+\mu}BK\mathbf{d}(-\ln\left\|\tfrac{z}{\|z\|}\right\|_{\mathbf{d}})\tfrac{z}{\|z\|}\end{aligned} \tag{22}$$

where $z_\sigma = \frac{z}{\|z\|} + \sigma_z$, and $z_\sigma - \frac{z}{\|z\|} = \sigma_z$. Considering the control using quantized states (22) and the Lyapunov function $z^\top Px$, we have

$$\begin{aligned}z^\top P\dot{z} =& \|z\|^{\mu}z^\top PH_z(A+BK)z\\ &+ \|z\|^{1+\mu}z^\top PH_z\|z_\sigma\|_{\mathbf{d}}^{1+\mu}BK\mathbf{d}\left(-\ln\|z_\sigma\|_{\mathbf{d}}\right)z_\sigma\\ &- \|z\|^{1+\mu}z^\top PH_z\left\|\tfrac{z}{\|z\|}\right\|_{\mathbf{d}}^{1+\mu}BK\mathbf{d}(-\ln\left\|\tfrac{z}{\|z\|}\right\|_{\mathbf{d}})\tfrac{z}{\|z\|}\end{aligned} \tag{23}$$

Notice that $\tilde{u} = \|x\|_{\mathbf{d}}^{1+\mu}K\mathbf{d}(-\ln\|x\|)x$, it is continuous outside the origin. By assumption, $\sigma_z = \mathbf{d}(-\ln\|x\|_{\mathbf{d}})(q(x) - x) \le \bar{\sigma}$ for all $x \neq \mathbf{0}$ with a sufficiently small $\bar{\sigma} > 0$. Since $\frac{z}{\|z\|}$ belongs to the unit sphere for any $z \neq \mathbf{0}$ then for sufficiently small $\bar{\sigma}$ there exists a compact set (with does not contain the origin), such that $\frac{z}{\|z\|}$ and z_σ always belongs to this compact. Since $\tilde{u}$ is continuous on the compact, then it is uniformly continuous on this compact, i.e., there exists a class $\mathcal{K}$ function γ such that

$$\|\tilde{u}(z_\sigma) - \tilde{u}(\tfrac{z}{\|z\|})\| \le \gamma(\|\sigma_z\|). \tag{24}$$

Taking (19) and using Cauchy–Schwarz inequality we have

$$z^\top P\dot{z} \le -\|z\|^{\mu}z^\top Pz\left(a_1a_2 - \gamma(\|\sigma_z\|)\right), \tag{25}$$

where $a_2 = -\lambda_{\min}(PA + A^\top A + PBK + K^\top B^\top P)$. It is clear that $z^\top P\dot{z}$ is negative definite for a sufficiently small $\bar{\sigma}$. The proof is complete. ■

Recall that $x \mapsto \mathbf{d}(-\ln\|x\|_{\mathbf{d}})x$ is the homogeneous projector on the unit sphere, so the latter Theorem reveals that if the projected quantization error is sufficiently small, then the system can maintain the same stability as a quantization-free system.

Corollary 1: Let q be uniform quantizer, then the system (12) is globally asymptotically practically stable.

Proof: According to the Theorem 2, and the quantization error of uniform quantizer (10), we have

$$\|\mathbf{d}(-\ln\|x\|_{\mathbf{d}})\sigma\| \le \|\mathbf{d}(-\ln\|x\|_{\mathbf{d}})\|\sqrt{n}\Delta. \tag{26}$$

Using the estimate (1), it yields that

$$\|\mathbf{d}(-\ln\|x\|_{\mathbf{d}})\sigma\| \le \begin{cases}\|x\|_{\mathbf{d}}^{-\beta}\sqrt{n}\Delta, & \|x\|_{\mathbf{d}} \ge 1\\ \|x\|_{\mathbf{d}}^{-\alpha}\sqrt{n}\Delta, & \|x\|_{\mathbf{d}} \le 1\end{cases} \tag{27}$$

the latter estimate indicates that if $\|x\|_{\mathbf{d}} \to \infty$, then $\|\mathbf{d}(-\ln\|x\|_{\mathbf{d}})\sigma\| \to 0$. Then, the system has no finite-time escape.

Meanwhile, taking account (25), it implies that the attractive set of system is $\{x \in \mathbb{R}^n : \|x\|_{\mathbf{d}} \leq \max\{\left(\frac{\sqrt{n}\Delta}{\gamma^{-1}(a_1a_2)}\right)^{1/\beta}, 1, \left(\frac{\sqrt{n}\Delta}{\gamma^{-1}(a_1a_2)}\right)^{1/\alpha}\}\}$. ■

In the case of the uniform quantization, only practical stability can be guaranteed since the quantization error is non-vanishing. The distinguishing property of a logarithmic quantizer is that the quantization error decreases as the states approach the origin. This allows the convergence/stability property of the original system to be preserved.

Corollary 2: Let q be a logarithmic quantization function. Then, the system (12) is globally asymptotically stable provided δ is sufficiently small (see (11)). Moreover,

- globally uniformly finite-time stable for $\mu < 0$;
- globally uniformly exponentially stable for $\mu = 0$;
- globally uniformly nearly fixed-time stable for $\mu > 0$.

Proof: For weighted homogeneous system (i.e. $G_{\mathbf{d}}$ is diagnal), let $G_{\mathbf{d}} = \text{diag}(r_1, r_2, \cdots, r_n)$, $x = [x_1, x_2, \cdots, x_n]^\top$, and $\sigma = [q_1(x_1) - x_1, q_2(x_2) - x_2, \cdots, q_n(x_n) - x_n]$. According to the stability condition (25) outlined in Theorem 2, a sufficient criterion for the global finite-time/exponential, nearly fixed-time stability is:

$$\sum_{i=1}^{n} \|x\|_{\mathbf{d}}^{-2r_i} |\sigma_i|^2 \leq \frac{\bar{\sigma}^2}{\lambda_{\max}(P)} \tag{28}$$

where $\bar{\sigma} > 0$ is small enough. On the other hand, from the definition of canonical homogeneous norm we have

$$\frac{1}{\lambda_{\max}(P)} \leq x^\top \mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x \leq \frac{1}{\lambda_{\min}(P)}. \tag{29}$$

it is equavalent to

$$\frac{1}{\lambda_{\max}(P)} \leq \sum_{i=1}^{n} \|x\|_{\mathbf{d}}^{-2r_i} |x_i|^2 \leq \frac{1}{\lambda_{\min}(P)}. \tag{30}$$

Then, it yields that

$$\|x\|_{\mathbf{d}}^{-2r_i} |x_i|^2 \leq \frac{1}{\lambda_{\min}(P)}, \ \forall i \in \mathbb{N}. \tag{31}$$

Since, due to logarithmic quantization, $|\sigma_i|^2 \leq \delta^2 |x_i|^2$ then

$$|\sigma_i| \leq \delta^2 \frac{\|x\|_{\mathbf{d}}^{2r_i}}{\lambda_{\min}(P)} \tag{32}$$

then for sufficiently small δ we have

$$\sum_{i=1}^{n} \|x\|_{\mathbf{d}}^{-2r_i} |\sigma_i|^2 \leq \frac{1}{\lambda_{\min}(P)} \sum_{i=1}^{n} \delta^2 \leq \frac{\epsilon^2}{\lambda_{\max}(P)}, \tag{33}$$

and the stability condition is fullfilled. Consider the quantization error for a vector, then the latter inequation becomes:

$$\sum_{i=1}^{n} \|x\|_{\mathbf{d}}^{-2r_i} |\sigma_i|^2 \leq \frac{n\delta^2}{\lambda_{\min}(P)} \leq \frac{\epsilon^2}{\lambda_{\max}(P)} \tag{34}$$

The proof is complete ■

Corollary 2 extends result known for linear control system to a class of generalized homogeneous control systems.

V. VALIDATION AND SIMULATION

For validation, a simulation is created in simulink. Let consider a second-order system

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Following the design in Theorem 1, we have

$$G_{\mathbf{d}} = \begin{bmatrix} (1-\mu)I & 0 \\ 0 & I \end{bmatrix}.$$

A negative homogeneity degree is considered, and the control parameters are obtained by solving the LMIs and listed as follows.

$$\mu = -0.5, \ K = [-5.9429, -2.5], \ P = \begin{bmatrix} 8.9359 & 2.2554 \\ 2.2554 & 1.5036 \end{bmatrix}.$$

In the simulationm, we set $\Delta = 1$, $\delta = 0.5$. The simulation

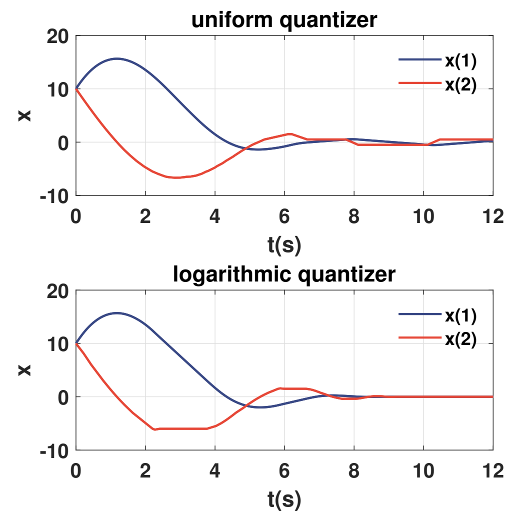

Fig. 3. The states of system (continuous time signal).

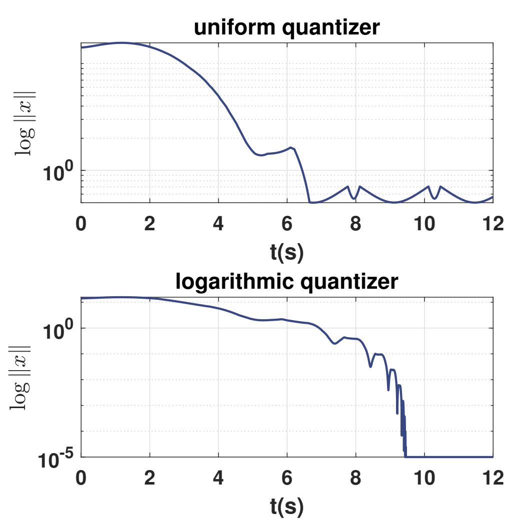

Fig. 4. The logarithm of norm state of system(continuous time signal).

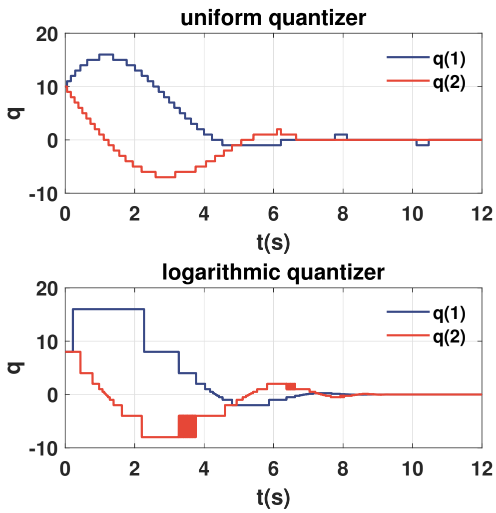

Fig. 5. Quantized state.

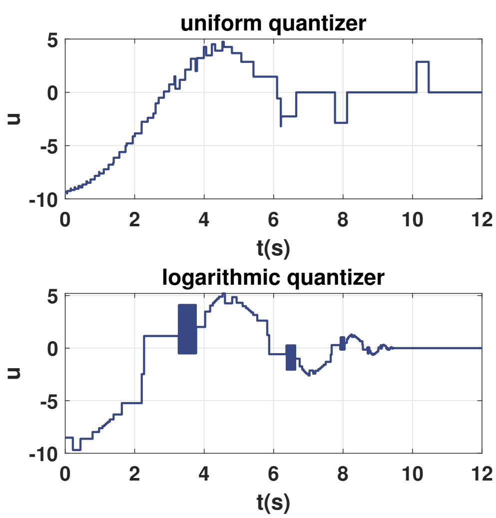

Fig. 6. Control signal using quantized state.

results are presented in Figures 3 to 6. The continuous-time state signal shown in Figures 3 and 4 demonstrates that the control using uniformly quantized states drives the system towards a neighborhood of the origin. On the other hand, the control law with logarithmic quantization achieves finite-time convergence to the origin. The quantized states and control signals obtained using the logarithmic quantizer and uniform quantizer are shown in Figures 5 and 6.

VI. Conclusions

In this study, we investigate the stability of homogeneous control systems under quantization of the measured states. Sufficient stability for the preservation of the stability properties of the original (quantization-free) system is derived. It is shown that in the case of uniform quantization, only global practical stability can be guaranteed, while in the case of a logarithmic quantizer, the system may preserve finite-time/exponential/nearly fixed-time stability (dependently on the homogeneity degree). Since the logarithmic quantization is less dense far from zero, the obtained results open a possibility to use a homogeneity-based design of event-trigger systems, which are aimed a minimization for information transmission through communication networks.

References

[1] D. Nesic and D. Liberzon. A unified framework for design and analysis of networked and quantized control systems. *IEEE Transactions on Automatic control*, 54(4):732–747, 2009.

[2] E. Bernuau, E. Moulay, and P. Coirault. Stability of homogeneous nonlinear systems with sampled-data inputs. *Automatica*, 85:349–355, 2017.

[3] M. Kawski. Homogeneous stabilizing feedback laws. *Control Theory and advanced technology*, 6(4):497–516, 1990.

[4] Y. Hong, J. Huang, and Y. Xu. On an output feedback finite-time stabilization problem. *IEEE Transactions on Automatic Control*, 46(2):305–309, 2001.

[5] N. Nakamura, H. Nakamura, Y. Yamashita, and H. Nishitani. Homogeneous stabilization for input affine homogeneous systems. *IEEE Transactions on Automatic Control*, 54(9):2271–2275, 2009.

[6] S. P Bhat and D. S Bernstein. Geometric homogeneity with applications to finite-time stability. *Mathematics of Control, Signals and Systems*, 17(2):101–127, 2005.

[7] A. Polyakov. Sliding mode control design using canonical homogeneous norm. *International Journal of Robust and Nonlinear Control*, 29(3):682–701, 2019.

[8] K. Zimenko, A. Polyakov, D. Efimov, and W. Perruquetti. Robust feedback stabilization of linear mimo systems using generalized homogenization. *IEEE Transactions on Automatic Control*, 65(12):5429–5436, 2020.

[9] M. Fu and L. Xie. The sector bound approach to quantized feedback control. *IEEE Transactions on Automatic control*, 50(11):1698–1711, 2005.

[10] Z-P Jiang and L Teng-Fei. Quantized nonlinear control—a survey. *Acta Automatica Sinica*, 39(11):1820–1830, 2013.

[11] D. F Delchamps. Stabilizing a linear system with quantized state feedback. *IEEE transactions on automatic control*, 35(8):916–924, 1990.

[12] R. W Brockett and D. Liberzon. Quantized feedback stabilization of linear systems. *IEEE transactions on Automatic Control*, 45(7):1279–1289, 2000.

[13] N. Elia and S. K Mitter. Stabilization of linear systems with limited information. *IEEE transactions on Automatic Control*, 46(9):1384–1400, 2001.

[14] H. Ishii and B. A Francis. Quadratic stabilization of sampled-data systems with quantization. *Automatica*, 39(10):1793–1800, 2003.

[15] D. Liberzon. Hybrid feedback stabilization of systems with quantized signals. *Automatica*, 39(9):1543–1554, 2003.

[16] A. Polyakov, D. Efimov, and W. Perruquetti. Finite-time and fixed-time stabilization: Implicit lyapunov function approach. *Automatica*, 51:332–340, 2015.

[17] V. I. Zubov. Systems of ordinary differential equations with generalized-homogeneous right-hand sides. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, (1):80–88, 1958.

[18] VV Khomenyuk. Systems of ordinary differential equations with generalized homogeneous right-hand sides. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, (3):157–164, 1961.

[19] M. Kawski. Families of dilations and asymptotic stability. In *Analysis of controlled dynamical systems*, pages 285–294. Springer, 1991.

[20] L. Rosier. Homogeneous lyapunov function for homogeneous continuous vector field. *Systems & Control Letters*, 19(6):467–473, 1992.

[21] A. F. Filippov. *Differential equations with discontinuous righthand sides: control systems*, volume 18. Springer Science & Business Media, 2013.