2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Robust adaptive step-tracking with exponential stability and convolution bounds using Supervisory Control

Craig J. Lalumiere, Daniel E. Miller *
Dept. of Elect. and Comp. Eng. University of Waterloo
Waterloo, ON, Canada N2L 3G1

Abstract

Supervisory Control has been shown to be a very effective approach to adaptive control which ensures step-tracking, exponential stability, and a degree of robustness to unmodeled dynamics. Here we apply the technique in the discrete-time setting and prove a new linear-like convolution bound on the effect of the noise/disturbance. This property is then leveraged to prove robustness to slow time-variations.

Keywords: Adaptive control, Supervisory Control, Exponential stability, Bounded gain, Convolution bounds.

1. Introduction

Adaptive control is an approach used to deal with systems with uncertain and possibly time-varying parameters. The first general proofs that parameter adaptive controllers work came around 1980, e.g. see [1]–[5]. However, these original adaptive controllers are typically not robust to unmodeled dynamics, do not tolerate time-variations well, have poor transient behavior, and do not handle noise or disturbances well, e.g. see [6]. During the 1980s subsequent research was able to alleviate these shortcomings to some degree: controller/estimator modifications include the use of signal normalization, deadzones, $\sigma-$modification, and projection onto a convex set of admissible parameters. While these modified controllers provide such highly desirable LTI-like properties as tolerance to a degree of time-variations and unmodeled dynamics, in general they do not provide the even more desirable properties of exponential stability, a bounded gain on the noise, nor a convolution-like bound on the input-output behavior.

In the 1990s a new continuous-time adaptive control technique labeled Supervisory Control was shown in [7], [8] to provide exponential stability and a bounded gain on the noise in a particular non-standard norm. While robustness to a degree of unmodeled dynamics is proven, tolerance to time variations is not, although in [9] a modified version using hysteresis tolerates a degree of time-variations which depends on the size of the initial condition. Last of all, no convolution-like bound on the input-output behavior is proven.

*This research was supported by the Natural Sciences and Engineering Research Council of Canada via Discovery Grant No. RGPIN-04219.

More recently, in [10]–[15] it is shown that highly desirable LTI-like input-output behavior can be achieved via a carefully designed recursive parameter estimation algorithm. This is expressed as a convolution bound, which confers exponential stability, a bounded gain on the noise in every p-norm, and using a modular technique analyzed in [16], [17], robustness to a degree of unmodeled dynamics and slowly time-varying parameters. In this paper our goal is to prove that the Supervisory Control approach enjoys the same properties. To proceed, we translate Supervisory Control into the discrete-time domain, combined with a pole placement control law with an integrator, and show that all of these properties do in fact hold. In so doing, we have proven desirable LTI-like properties never before seen in the Supervisory Control literature, in particular, a global tolerance to a degree of time-variations and nonlinearities, a convolution bound on the input-output behavior, and a bounded gain in every p-norm. This work is based on the MASc thesis of the first author [18], which also considers the related d-step-ahead adaptive control problem. The analysis of this latter problem has been submitted and is under review [19]; while the analysis and proofs provided here are similar to those of [19] at a high level, the details differ.

We use the Euclidean 2-norm for vectors and the corresponding induced norm for matrices, and denote the norm of a vector or matrix by $\|\cdot\|$. We let l_∞ denote the set of real-valued bounded sequences. For an arbitrary signal $a:\mathbf{Z}\to\mathbf{R}$, we let $\Delta a(t)$ denote $a(t)-a(t-1)$.

2. The Setup

Let $t_0\in\mathbf{Z}$ denote the initialization time. We start with a linear time-invariant discrete-time plant described by

$$y(t+1)=\underbrace{\begin{bmatrix} y(t)\\ \vdots \\ y(t-n+1)\\ u(t)\\ \vdots\\ u(t-m+1)\end{bmatrix}^\top}_{=:\phi(t)}\underbrace{\begin{bmatrix} a_1^*\\ \vdots\\ a_n^*\\ b_1^*\\ \vdots\\ b_m^*\end{bmatrix}}_{=:\theta^*}+w(t),\quad t\ge t_0, \qquad (1)$$

with known initial condition $\phi(t_0)$, $y(t) \in \mathbf{R}$ the measured output, $u(t) \in \mathbf{R}$ the control input, and $w \in l_\infty$ the disturbance (or noise) input. We assume that θ^* is unknown but belongs to a known set $\mathscr{S} \subset \mathbf{R}^{n+m}$. Associated with this plant are the polynomials

$$A_{\theta^*}(z^{-1}) = 1 - a_1^* z^{-1} - \ldots - a_n^* z^{-n},$$
$$B_{\theta^*}(z^{-1}) = b_1^* z^{-1} + \ldots + b_m^* z^{-m},$$

and the transfer function ${}^{B_{\theta^*}(z^{-1})}/_{A_{\theta^*}(z^{-1})}$.

The goal is closed-loop stability and exponential tracking of an exogenous, constant reference input r in the presence of a constant disturbance; we define the **tracking error** by $\varepsilon(t) := y(t) - r$. This is achieved using an indirect adaptive control scheme composed of a certainty equivalence control law and a parameter estimator, as illustrated in Figure 1. The estimator attempts to generate an estimate $\hat{\theta}$ of the unknown parameter vector θ^* using the observed input-output data from the plant. Since it is expected that the disturbance w includes a constant component, the observed data is first passed through what is referred to as a 'data filter' in [20, Chapter 11]. The zero at $z = 1$ eliminates any bias caused by a disturbance with non-zero mean. The parameter estimate is then used to synthesize a pole-placement control law which generates the 'control difference' Δu, which is then integrated to find the control effort u.

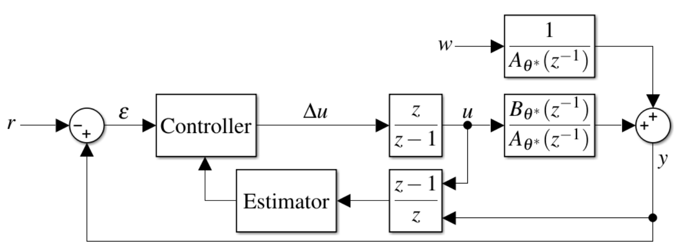

Figure 1: Block diagram of the Supervisory Control scheme.

We impose several assumptions on the set of admissible plant parameters.

Assumption 1 *n and m are known;*

Assumption 2 *The polynomials $A_{\theta^*}(z^{-1})$ and $B_{\theta^*}(z^{-1})$ corresponding to each θ^* in $\mathscr{S}$ are coprime.*

Assumption 3 *The polynomial $B_{\theta^*}(z^{-1})$ corresponding to each θ^* in $\mathscr{S}$ is nonzero at $z = 1$.*

Assumption 4 *For each $\bar{n} \in \{0, \ldots, n\}$ and $\bar{m} \in \{1, \ldots, m\}$, the set of all $\theta^* \in \mathscr{S}$ with $A_{\theta^*}(z^{-1})$ of degree $\bar{n}$ and $B_{\theta^*}(z^{-1})$ of degree $\bar{m}$ is compact.*

Remark 1 *Since we do not require $a_n \neq 0$ nor $b_n \neq 0$, Assumption 1 can be interpreted as assuming that an upper bound on the degrees of $A_{\theta^*}(z^{-1})$ and $B_{\theta^*}(z^{-1})$ are known.*

Remark 2 *Assumption 2 ensures that a controller can be constructed to place all the closed-loop poles at will.*

Remark 3 *Assumption 3 ensures that the plant does not have a zero at $z = 1$, which is necessary to achieve step tracking using a bounded input.*

Remark 4 *Assumption 4 is used to ensure that the pole placement control law parameters are uniformly bounded for all admissible plant models. Two natural examples are a finite set of plant models, and the case for which n and m are fixed, such as for a mechanical system in which masses, spring constants, and damper coefficients are uncertain but lie in a range.*

2.1. Parameter Estimation

Following [7], the procedure of choosing the parameter estimate $\hat{\theta}(t)$ is performed by what is referred to as the 'supervisor' via an optimization procedure. It is well known that optimizing a function over a convex region is easier than doing the same over a non-convex region, so before proceeding we use the following result. This provides a cover of $\mathscr{S}$ in $\mathbf{R}^{n+m}$; since some elements of this cover may lie outside of $\mathscr{S}$, we label its elements θ rather than θ^* (which we have assumed to lie in $\mathscr{S}$), and adopt the natural definition of $A_\theta(z^{-1})$ and $B_\theta(z^{-1})$.

Lemma 1 *There exists a finite number p of compact, convex sets $\hat{\mathscr{S}}_i \subset \mathbf{R}^{n+m}$ so that $\hat{\mathscr{S}} := \bigcup_{i=1}^{p} \hat{\mathscr{S}}_i \supset \mathscr{S}$, where for every element $\theta \in \hat{\mathscr{S}}_i$, the corresponding polynomials $A_\theta(z^{-1})$ and $B_\theta(z^{-1})$ are coprime, the degrees of those polynomials are constant on the set $\hat{\mathscr{S}}_i$, and $B_\theta(z^{-1})$ is nonzero at $z = 1$.*

Proof: This follows from a straight forward extension of [21, Proposition 1]. □

To create the parameter estimator, we introduce the matrix-valued dynamical system called the 'performance weight generator', defined as

$$W(t+1) = \lambda W(t) + \begin{bmatrix} \Delta\phi(t) \\ \Delta y(t+1) \end{bmatrix} \begin{bmatrix} \Delta\phi(t) \\ \Delta y(t+1) \end{bmatrix}^\top, \quad t \geq t_0, \tag{2}$$

where $\lambda \in (0,1)$ is the 'forgetting factor', and initial condition $W(t_0) \in \mathbf{R}^{(n+m+1)\times(n+n+1)}$ is a positive semidefinite and symmetric matrix, both chosen by the designer. The supervisor uses the following cost function in selecting the parameter estimate:

$$J(\theta, t) = \begin{bmatrix} \theta \\ -1 \end{bmatrix}^\top W(t) \begin{bmatrix} \theta \\ -1 \end{bmatrix}, \quad \theta \in \hat{\mathscr{S}}, \quad t \geq t_0 + 1. \tag{3}$$

Since $\hat{\mathscr{S}}$ is made up of a finite union of p compact convex sets, minimizing J over $\hat{\mathscr{S}}$ reduces to p straightforward convex optimization problems. The motivation for this cost function is that if we define the **prediction error** corresponding to a parameter estimate θ as

$$e_\theta(t) := \Delta y(t) - \Delta\phi(t-1)^\top \theta, \quad \theta \in \hat{\mathscr{S}}, \quad t \geq t_0 + 1, \tag{4}$$

then the cost function (3) may be equivalently expressed as

$$J(\theta,t) := \sum_{i=t_0}^{t-1} \lambda^{t-i-1} |e_\theta(i+1)|^2 + \lambda^{t-t_0} \begin{bmatrix} \theta \\ -1 \end{bmatrix}^\top W(t_0) \begin{bmatrix} \theta \\ -1 \end{bmatrix}, \quad \theta \in \hat{\mathscr{S}}, \quad t \geq t_0 + 1.$$

We now introduce the concept of 'dwelling', which is a constraint placed upon the supervisor which prohibits it from changing $\hat{\theta}$ too rapidly. This is atypical in the adaptive control literature, but is commonplace in the Supervisory Control literature, and will be needed in the later stability proof. This is accomplished via a new dynamic variable $\tau(t)$, dubbed the 'dwell-timer', constrained to the set $\{0,\dots,n+m\}$. Whenever τ takes a positive value, the supervisor is said to be 'dwelling'. With initial condition $\tau(t_0) \in \{0,\dots,n+m\}$ and an initial parameter guess $\hat{\theta}(t_0) \in \hat{\mathscr{S}}$, for $t \geq t_0$, the parameter estimation routine is defined by

$$\begin{bmatrix} \hat{\theta}(t+1) \\ \tau(t+1) \end{bmatrix} = \begin{cases} \begin{bmatrix} \arg\min_{\theta \in \hat{\mathscr{S}}} J(\theta,t+1) \\ n+m \end{bmatrix} & \text{if } \tau(t) = 0 \text{ and } J(\hat{\theta}(t),t+1) > \min_{\theta \in \hat{\mathscr{S}}} J(\theta,t+1) \\ \begin{bmatrix} \hat{\theta}(t) \\ \max\{\tau(t)-1,0\} \end{bmatrix} & \text{otherwise.} \end{cases} \tag{5}$$

The consequence of this is that $\hat{\theta}$ is piecewise-constant, changing no more than once every $n+m+1$ steps. The reason for this specific choice of dwell period will be made clear later. Next, define the constant $\bar{k} := 1 + \|\mathscr{S}\|$. An important consequence of the optimization routine described is that whenever the supervisor is not dwelling (i.e. when $\tau(t-1) = 0$), the following holds for all $\theta \in \hat{\mathscr{S}}$:

$$\begin{aligned} &\sum_{i=t_0}^{t-1} \lambda^{t-i-1} |e_{\hat{\theta}(t)}(i+1)|^2 + \lambda^{t-t_0} \begin{bmatrix} \hat{\theta}(t) \\ -1 \end{bmatrix}^\top W(t_0) \begin{bmatrix} \hat{\theta}(t) \\ -1 \end{bmatrix} \\ &\quad \leq \sum_{i=t_0}^{t-1} \lambda^{t-i-1} |e_\theta(i+1)|^2 + \lambda^{t-t_0} \begin{bmatrix} \theta \\ -1 \end{bmatrix}^\top W(t_0) \begin{bmatrix} \theta \\ -1 \end{bmatrix}. \end{aligned}$$

Most importantly, since $\theta^* \in \hat{\mathscr{S}}$, using (1) and (4) yields

$$\begin{aligned} &\sum_{i=t_0}^{t-1} \lambda^{t-i-1} |e_{\hat{\theta}(t)}(i+1)|^2 \\ &\quad \leq \sum_{i=t_0}^{t-1} \lambda^{t-i-1} |\Delta w(i)|^2 + \lambda^{t-t_0} \bar{k}^2 \|W(t_0)\|. \end{aligned} \tag{6}$$

3. Pole Placement Control Law

With Figure 1 in mind, we now describe the construction of the pole placement controller with input ε and output Δu, placed in feedback with the 'augmented plant' - the composition of the actual plant and the integrator. Define the transformation $\mathrm{h} : \mathbf{R}^{n+m} \to \mathbf{R}^{n+m+1}$ by

$$\begin{bmatrix} a_1 & \dots & a_n & b_1 & \dots & b_m \end{bmatrix}^\top \longmapsto \begin{bmatrix} 1+a_1 & a_2 - a_1 & \dots & a_n - a_{n-1} & -a_n & b_1 & \dots & b_m \end{bmatrix}^\top.$$

By adopting the alternative state vector

$$\varphi(t) := \begin{bmatrix} \varepsilon(t) & \dots & \varepsilon(t-n) & \Delta u(t) & \dots & \Delta u(t-m+1) \end{bmatrix}^\top \in \mathbf{R}^{n+m+1}, \tag{7}$$

the *augmented plant's* dynamics from input Δu to output ε may be expressed as

$$\varepsilon(t+1) = \varphi(t)^\top \mathrm{h}(\theta^*) + \Delta w(t), \quad t \geq t_0. \tag{8}$$

We now design a controller for each $\hat{\mathscr{S}}_i$. Fix $i \in 1,\dots,p$, and let $\bar{n} \in \{0,\dots,n\}$ and $\bar{m} \in \{1,\dots,m\}$ denote the degrees of $A_\theta(z^{-1})$ and $B_\theta(z^{-1})$, respectively, corresponding to (every) $\theta \in \hat{\mathscr{S}}_i$. We now choose

$$L_\theta(z^{-1}) := 1 - l_{\theta 1} z^{-1} - \ldots - l_{\theta \bar{m}} z^{-\bar{m}}$$

and

$$P_\theta(z^{-1}) := -p_{\theta 1} z^{-1} - \ldots - p_{\theta \bar{n}+1} z^{-\bar{n}-1}$$

to place all closed-loop poles at the origin:

$$(1 - z^{-1}) A_\theta(z^{-1}) L_\theta(z^{-1}) + B_\theta(z^{-1}) P_\theta(z^{-1}) = 1; \tag{9}$$

Lemma 1 ensures that $(1-z^{-1})A_\theta(z^{-1})$ and $B_\theta(z^{-1})$ are coprime, so a unique solution exists - see [18, Section 5.1] for more details, including how to compute the coefficients of $L_\theta(z^{-1})$ and $P_\theta(z^{-1})$. Since the coefficients of $L_\theta(z^{-1})$ and $P_\theta(z^{-1})$ are analytic functions of $\theta \in \hat{\mathscr{S}}_i$, it follows from the compactness of $\hat{\mathscr{S}}_i$ that they are uniformly bounded on the set. Finally, by substituting θ by the estimate $\hat{\theta}(t+1)$, the certainty equivalence control law is

$$\Delta u(t+1) = \begin{bmatrix} p_{\hat{\theta}(t+1)1} \\ \vdots \\ p_{\hat{\theta}(t+1)\bar{n}+1} \\ l_{\hat{\theta}(t+1)1} \\ \vdots \\ l_{\hat{\theta}(t+1)\bar{m}} \end{bmatrix}^\top \begin{bmatrix} \varepsilon(t) \\ \vdots \\ \varepsilon(t-\bar{n}) \\ \Delta u(t) \\ \vdots \\ \Delta u(t-\bar{m}+1) \end{bmatrix}, \quad t \geq t_0 + 1.$$

By padding the left vector with some extra zeros:

$$f_\theta := [p_{\theta 1} \quad \dots \quad p_{\theta \bar{n}+1} \quad \overbrace{0 \quad \dots \quad 0}^{n-\bar{n}} \quad l_{\theta 1} \quad \dots \quad l_{\theta \bar{m}} \quad \overbrace{0 \quad \dots \quad 0}^{m-\bar{m}}],$$

we can express the control law as

$$\Delta u(t+1) = f_{\hat{\theta}(t+1)} \varphi(t), \quad t \geq t_0. \tag{10}$$

4. State Space Representation

Now we form an update equation for the transformed state vector φ from (7). Define $\bar{A}_k \in \mathbf{R}^{k \times k}$, $k \in \mathbf{N}$, as the transpose of the matrix in Jordan form with all eigenvalues at zero, and subsequently define

$$A_{\theta_1 \theta_2} := \begin{bmatrix} \bar{A}_{n+1} & \mathbf{0} \\ \mathbf{0} & \bar{A}_m \end{bmatrix} + \begin{bmatrix} 1 \\ \mathbf{0}_{n+m} \end{bmatrix} \mathrm{h}(\theta_1)^\top + \begin{bmatrix} \mathbf{0}_{n+1} \\ 1 \\ \mathbf{0}_{m-1} \end{bmatrix} f_{\theta_2};$$

using the plant dynamics (8) and control law (10) we obtain

$$\varphi(t+1) = A_{\theta^*\hat{\theta}(t+1)}\varphi(t) + \begin{bmatrix} 1 \\ \mathbf{0}_{n+m} \end{bmatrix} \Delta w(t), \quad t \geq t_0. \quad (11)$$

Alternatively, (4) may be rearranged to show that $\varepsilon(t+1) = \varphi(t)^\top \mathrm{h}\left(\hat{\theta}(t+1)\right) + e_{\hat{\theta}(t+1)}(t+1)$. Combine this with the control law to yield another representation:

$$\varphi(t+1) = A_{\hat{\theta}(t+1)\hat{\theta}(t+1)}\varphi(t) + \begin{bmatrix} 1 \\ \mathbf{0}_{n+m} \end{bmatrix} e_{\hat{\theta}(t+1)}(t+1), \ t \geq t_0. \quad (12)$$

This form is useful because $A_{\theta\theta}$ represents the closed-loop system dynamics if $\hat{\theta}(\cdot) = \theta^* = \theta \in \hat{\mathscr{S}}$. This matrix has $n+m+1$ eigenvalues. The pole placement control law sets all of these eigenvalues to zero, which means that

$$\|A_{\theta\theta}^{n+m+1}\| = 0, \quad \theta \in \hat{\mathscr{S}}. \quad (13)$$

5. The Main Result

The following stability proof relies on the 'dwell time' to prove stability. The choice of dwell time ensures that $\hat{\theta}(t)$ may change no more than once every $n+m+1$ time steps. Combined with (13), it follows that the state transition matrix $\Phi(t_2,t_1)$ corresponding to $A_{\hat{\theta}(t)\hat{\theta}(t)}$ has finite support. Thus, for any choice of $\tilde{\lambda} \in (0,\lambda)$, there exists a $\hat{\gamma} \geq 1$, which is independent of $\hat{\theta}(\cdot)$, such that

$$\|\Phi(t_2,t_1)\| \leq \hat{\gamma}\tilde{\lambda}^{t_2-t_1}, \quad t_2 \geq t_1 \geq t_0. \quad (14)$$

Therefore, the dwell-time switching logic defined in (2) ensures that (12) is a stable system driven by the *internal* signal $e_{\hat{\theta}(\cdot)}(\cdot)$, which reveals the convolution bound, albeit in terms of an internal signal rather than exogenous ones:

$$\|\varphi(t)\| \leq \hat{\gamma}\tilde{\lambda}^{t-t_0}\|\varphi(t_0)\| + \hat{\gamma}\sum_{i=t_0}^{t-1}\tilde{\lambda}^{t-i-1}|e_{\hat{\theta}(i+1)}(i+1)|. \quad (15)$$

Thus, a stability proof relies on finding a meaningful bound on the summation above.

Theorem 1 *For every $\lambda \in (0,1)$, $\tilde{\lambda} \in (0,\lambda)$, and $\bar{\lambda} \in (\sqrt{\lambda},1)$, there exists a $\gamma \geq 1$ so that for every $\theta^* \in \mathscr{S}$, $t_0 \in \mathbf{Z}$, $\varphi(t_0) \in \mathbf{R}^{n+m+1}$, $W(t_0) \in \mathbf{R}^{(n+m+1)\times(n+m+1)}$ positive semidefinite and symmetric, $\hat{\theta}(t_0) \in \hat{\mathscr{S}}$, $\tau(t_0) \in \{0,\ldots,n+m\}$, $r \in \mathbf{R}$ and $w \in l_\infty$, when the supervisory controller given by (2), (3), (5) and (10) is applied to the plant (1), the following holds:*

$$\left\|\begin{bmatrix} \varphi(t) \\ \mathrm{vec}\left(W(t)^{\frac{1}{2}}\right) \end{bmatrix}\right\| \leq \gamma\bar{\lambda}^{t-t_0}\left\|\begin{bmatrix} \varphi(\bar{t}_0) \\ \mathrm{vec}\left(W(\bar{t}_0)^{\frac{1}{2}}\right) \end{bmatrix}\right\| + \gamma\sum_{i=\bar{t}_0}^{t-1}\bar{\lambda}^{t-i-1}|\Delta w(i)|, \quad t \geq \bar{t}_0 \geq t_0.$$

Remark 5 *A consequence of Theorem 1 is that when the disturbance is constant, ε goes exponentially to zero. Also, by applying Parseval's Theorem, we can obtain a bound on the energy of the tracking error which is proportional to the sum of the energy of the disturbance-difference Δw and the squared magnitude of the initial condition.*

The proof of Theorem 1 is very similar to that of Theorem 2 of our earlier work [19], although the details are different. To proceed, we need a modified version of Kreisselmeier's lemma from [22].

Lemma 2 *Consider the time-varying square matrix: $\bar{A}(t) = A(t) + \Delta(t)$. Let $\Phi(t,\tau)$ and $\bar{\Phi}(t,\tau)$ be the state transition matrices of $A(t)$ and $\bar{A}(t)$, respectively. Suppose there exists constants $c \geq 1$ and $\mu \in (0,1)$ such that*

$$\|\Phi(t,\tau)\| \leq c\mu^{t-\tau}, \quad t \geq \tau.$$

Then, for every $\bar{\mu} \in (\mu,1)$, $\alpha \geq 0$, and $\beta \in [0, \frac{\bar{\mu}-\mu}{c})$, there exists a $\bar{c} \geq 1$ such that if $\sum_{t=t_1}^{t_2-1}\|\Delta(t)\| \leq \alpha + \beta(t_2-t_1)$, $t_2 > t_1$, then the following bound holds:

$$\|\bar{\Phi}(t,\tau)\| \leq \bar{c}\bar{\mu}^{t-\tau}, \quad t \geq \tau.$$

Proof: A simple modification of the proof in [22]. □

Proof of Theorem 1

Fix $\lambda \in (0,1)$, $\tilde{\lambda} \in (0,\lambda)$, and $\bar{\lambda} \in (\sqrt{\lambda},1)$, and let $\theta^* \in \mathscr{S}$, $t_0 \in \mathbf{Z}$, $\phi(t_0) \in \mathbf{R}^{n+m}$, $\hat{\theta}(t_0) \in \hat{\mathscr{S}}$, $\tau(t_0) \in \{0,\ldots,n+m\}$, $r \in \mathbf{R}$, $w \in l_\infty$ and $\bar{t}_0 \geq t_0$ be arbitrary.

To prove this, we split up time into those for which the norm of the **parameter estimation error** $\tilde{\theta}(t) := \hat{\theta}(t) - \theta^*$ is small and those for which it is not. Before proceeding, recall from (13) that the matrix $A_{\theta\theta}$ is deadbeat. From standard linear systems theory there exists a $\sigma > 0$ and $\bar{\gamma} \geq 1$ so that the state transition matrix $\Phi_{\theta^*\hat{\theta}(t)}(t_2,t_1)$ corresponding to $A_{\theta^*\hat{\theta}(t)}$ satisfies

$$\|\Phi_{\theta^*\hat{\theta}(t)}(t_2,t_1)\| \leq \bar{\gamma}\sqrt{\lambda}^{t_2-t_1} \quad (16)$$

for $t_2 \geq t_1 \geq \bar{t}_0$ for which $\|\tilde{\theta}(t+1)\| \leq \sigma$. With $\delta \leq \sigma$ chosen sufficiently small, we now partition the timeline of $t \geq \bar{t}_0$ into two parts:

- intervals of the form $\{\underline{t},\ldots,\bar{t}\}$ satisfying $\|\tilde{\theta}(t)\| < \delta \leq \sigma, t \in \{\underline{t}+1,\ldots,\bar{t}\}$, in which case we can obtain a bound on $\|\varphi(t)\|$ in terms of exogenous inputs and $\|\varphi(\underline{t})\|$, and
- times $t \geq \bar{t}_0 + 1$ for which $\|\tilde{\theta}(t)\| \geq \delta$, in which case we obtain a bound on $\|\varphi(t)\|$ in terms of the exogenous inputs and $\|\varphi(\bar{t}_0)\|$.

Part 1: A bound on $\|\varphi(t)\|$ on intervals $\{\underline{t},\ldots,\bar{t}\}$, $\bar{t}_0 \leq \underline{t} < \bar{t} < \infty$ for which $\|\tilde{\theta}(t)\| < \delta, t \in \{\underline{t}+1,\ldots,\bar{t}\}$. For intervals of this sort, (16) holds, so from (11):

$$\|\varphi(t)\| \leq \bar{\gamma}\sqrt{\lambda}^{t-\underline{t}}\|\varphi(\underline{t})\| + \bar{\gamma}\sum_{i=\underline{t}}^{t-1}\sqrt{\lambda}^{t-i-1}|\Delta w(i)|, \quad t \in \{\underline{t},\ldots,\bar{t}\}. \quad (17)$$

Part 2: A bound on $\|\varphi(t)\|$ for $\|\tilde{\theta}(t)\| \geq \delta$ and $t \geq \bar{t}_0 + 1$. With $\hat{\theta}(t)$ any piecewise-constant signal with dwell time at least $n+m+1$, the system (12) is exponentially stable, which yields the convolution bound (15). Thus, the remainder of the stability proof relies on finding a meaningful bound on the summation in (15). To proceed, we borrow heavily from [8]. We shall use the following preliminary result:

Claim 1 *For every fixed $\bar{t}_0 \geq t_0$ and $t \geq \bar{t}_0$, there exists a projection operator $\psi : \{t \in \mathbf{Z} : t \geq \bar{t}_0\} \to \{0,1\}$ that satisfies*

$$\sqrt{\sum_{i=\bar{t}_0}^{t-1} \lambda^{t-i-1}(1-\psi(i))|e_{\hat{\theta}(t)}(i+1)|^2} \leq \bar{k}\sqrt{\lambda}^{t-\bar{t}_0}\|W(\bar{t}_0)^{\frac{1}{2}}\| + \sqrt{\sum_{i=\bar{t}_0}^{t-1} \lambda^{t-i-1}|w(i)|^2}$$

and

$$\sum_{i=\bar{t}_0}^{\infty} |\psi(i)| \leq n+m. \tag{18}$$

Proof: Let $\bar{t}$ be the most recent time when the supervisor was not dwelling ($\tau(\bar{t}-1)=0$), or $\bar{t}_0$ if such a time does not exist. Since the dwell time is $n+m+1$, we know that $t-\bar{t} \leq n+m$. Let

$$\psi(i) = \begin{cases} 1 & i \in \{\bar{t},\ldots,t-1\} \\ 0 & else; \end{cases}$$

then (18) holds, and using (6) we obtain

$$\begin{aligned} &\sqrt{\sum_{i=\bar{t}_0}^{t-1} \lambda^{t-i-1}|(1-\psi(i))e_{\hat{\theta}(t)}(i+1)|^2} \\ &= \sqrt{\lambda^{t-\bar{t}}\sum_{i=\bar{t}_0}^{\bar{t}-1} \lambda^{\bar{t}-i-1}|e_{\hat{\theta}(\bar{t})}(i+1)|^2} \\ &\leq \sqrt{\sum_{i=\bar{t}_0}^{t-1} \lambda^{t-i-1}|w(i)|^2} + \bar{k}\sqrt{\lambda}^{t-\bar{t}_0}\|W(\bar{t}_0)^{\frac{1}{2}}\|. \qquad \Box \end{aligned}$$

To proceed, we are going to approximate $\tilde{\theta}(i)$ as a linear combination of a finite set of specially chosen basis vectors. Since $\tilde{\theta}(i) \in \mathbf{R}^{n+m}$, we need at most $n+m$ basis vectors. They must be chosen such that we can make use of Claim 1 and such that the weights on the basis vectors are uniformly bounded. Claim 2 describes this construction; it is a reformulation of Section VIII B of [8].

Claim 2 *Let δ be a positive constant and let $\mathscr{X}$ be a list of vectors $x_1, x_2, \ldots, x_q \in \mathbf{R}^n$ whose last element satisfies $\|x_q\| \geq \delta$. Then there exists an ordered subset $\{x_{i_1}, x_{i_2}, \ldots, x_{i_{\bar{n}}}\} \subseteq \mathscr{X}$, with $1 \leq \bar{n} \leq \min\{n,q\}$ and*

$$1 \leq i_1 < i_2 < \ldots < i_{\bar{n}} = q,$$

as well as coefficients $g_j(i)$ which satisfy

$$g_j(i) = 0, \quad i = i_j+1,\ldots,q, \quad j = 1,\ldots,\bar{n}-1,$$

$$|g_j(i)| \leq \left(1+\frac{\|\mathscr{X}\|}{\delta}\right)^{\bar{n}}, \quad i = 1,\ldots,i_j, \quad j = 1,\ldots,\bar{n},$$

$$\left\| x_i - \sum_{j=1}^{\bar{n}} g_j(i)x_{i_j} \right\| \leq \delta, \quad i = 1,\ldots,q.$$

Proof: The proof and instructions for how to find the ordered subset are found in Section VIII B of [8]; an alternative analysis is presented in [18, Section 3.3]. The details are omitted due to space considerations. $\Box$

Now let $t \geq \bar{t}_0+1$ satisfy $\|\tilde{\theta}(t)\| \geq \delta$ and apply Claim 2 with $\mathscr{X} := \{\tilde{\theta}(\bar{t}_0+1),...,\tilde{\theta}(t)\}$. This shows that we can choose a set of basis vectors $\{\tilde{\theta}(i_1), \tilde{\theta}(i_2), \ldots, \tilde{\theta}(i_{\bar{n}})\}$ with $\bar{n} \leq n+m$, $\bar{t}_0+1 \leq i_1 < \ldots < i_{\bar{n}} = t$, and a set of g_j such that the 'approximation error'

$$\bar{c}(i) := \tilde{\theta}(i) - \sum_{j=1}^{\bar{n}} g_j(i)\tilde{\theta}(i_j), \quad i = \bar{t}_0+1,\ldots,t \tag{19}$$

satisfies $\|\bar{c}(i)\| \leq \delta$. Thus, each $\tilde{\theta}(i)$ for $i = \bar{t}_0+1,\ldots,t$ is approximated by a linear combination of these basis vectors. Moreover, each of the coefficients is bounded:

$$|g_j(i)| \leq \left(1+\frac{2\|\hat{\mathscr{S}}\|}{\delta}\right)^{n+m}. \tag{20}$$

Recall that our objective is to find a meaningful bound on the summation in (15). It is easy to construct a matrix R so that the prediction error (4) can be rewritten as

$$e_{\hat{\theta}(i+1)}(i+1) = \Delta w(i) - \varphi(i)^\top R\tilde{\theta}(i+1), \quad i = t_0,...,t-1.$$

Now we may use this and (19) to obtain

$$\begin{aligned} e_{\hat{\theta}(i+1)}(i+1) &= \Delta w(i) - \varphi(i)^\top R\tilde{\theta}(i+1) \\ &= \Delta w(i) - \sum_{j=1}^{\bar{n}} g_j(i+1)\varphi(i)^\top R\tilde{\theta}(i_j) - \varphi(i)^\top R\bar{c}(i+1) \\ &= \sum_{j=1}^{\bar{n}} g_j(i+1)e_{\hat{\theta}(i_j)}(i+1) + \left(1-\sum_{j=1}^{\bar{n}} g_j(i+1)\right)\Delta w(i) \\ &\quad - \varphi(i)^\top R\bar{c}(i+1), \quad i = \bar{t}_0,...,t-1. \end{aligned} \tag{21}$$

Now we want to find a bound on

$$\sum_{i=\bar{t}_0}^{t-1} \tilde{\lambda}^{t-i-1}|g_j(i+1)e_{\hat{\theta}(i_j)}(i+1)|, \quad j = 1,...,\bar{n}.$$

While we cannot do so directly, by making use of Claim 1 we can find a bound for something similar. We see that for each $j \in \{1,\ldots,\bar{n}\}$, there exists a projection operator $\psi_j : \{t \in \mathbf{Z} : t \geq \bar{t}_0\} \to \{0,1\}$ satisfying (18) such that

$$\begin{aligned} &\sqrt{\sum_{i=\bar{t}_0}^{t-1} \lambda^{t-i-1}|(1-\psi_j(i))g_j(i+1)e_{\hat{\theta}(i_j)}(i+1)|^2} \\ &\leq \left(1+\frac{2\|\hat{\mathscr{S}}\|}{\delta}\right)^{n+m} \sqrt{\sum_{i=\bar{t}_0}^{t-1} \lambda^{t-i-1}|\Delta w(i)|^2} \\ &\quad + \bar{k}\left(1+\frac{2\|\hat{\mathscr{S}}\|}{\delta}\right)^{n+m} \sqrt{\lambda}^{t-\bar{t}_0}\|W(\bar{t}_0)^{\frac{1}{2}}\|. \end{aligned}$$

Now if we create a new projection operator $\Psi : \{\bar{t}_0, \ldots, t-1\} \to \{0,1\}$ whose support is precisely the union of the supports of $\psi_j, j \in \{1, \ldots, \bar{n}\}$, then it satisfies

$$\sum_{i=\bar{t}_0}^{t-1} |\Psi(i)| \le (n+m)^2, \tag{22}$$

and the signal

$$\hat{e}(i+1) := (1-\Psi(i)) \sum_{j=1}^{\bar{n}} g_j(i+1) e_{\hat{\theta}(i_j)}(i+1),$$

for $i = \bar{t}_0, ..., t-1$, satisfies

$$\begin{aligned} &\sqrt{\sum_{i=\bar{t}_0}^{t-1} \lambda^{t-i-1} |\hat{e}(i+1)|^2} \\ &\le (n+m)\left(1+\frac{2\|\hat{\mathscr{S}}\|}{\delta}\right)^{n+m} \sqrt{\sum_{i=\bar{t}_0}^{t-1} \lambda^{t-i-1} |\Delta w(i)|^2} \\ &\quad + \bar{k}(n+m)\left(1+\frac{2\|\hat{\mathscr{S}}\|}{\delta}\right)^{n+m} \sqrt{\lambda}^{t-\bar{t}_0} \|W(\bar{t}_0)^{\frac{1}{2}}\|. \end{aligned} \tag{23}$$

If we now define these new signals:

$$\begin{aligned} \tilde{c}(i) &:= \Psi(i-1) \sum_{j=1}^{\bar{n}} g_j(i) \tilde{\theta}(i_j), \\ \bar{g}(i) &:= \begin{bmatrix} 1 \\ \mathbf{0}_{n+m} \end{bmatrix} \left(1 + (\Psi(i-1)-1) \sum_{j=1}^{\bar{n}} g_j(i)\right), \\ &\quad i = \bar{t}_0+1, \ldots, t, \end{aligned}$$

then we can modify (21) to express it as

$$\begin{aligned} e_{\hat{\theta}(i+1)}(i+1) &= \hat{e}(i+1) - (\tilde{c}(i+1) + \bar{c}(i+1))^\top R^\top \varphi(i) \\ &+ \left(1 + (\Psi(i)-1) \sum_{j=1}^{\bar{n}} g_j(i+1)\right) \Delta w(i), \quad i = \bar{t}_0, ..., t-1. \end{aligned}$$

By substituting this into (12), we obtain:

$$\begin{aligned} \varphi(i+1) &= \left(A_{\hat{\theta}(i+1)\hat{\theta}(i+1)} - \begin{bmatrix} 1 \\ \mathbf{0}_{n+m} \end{bmatrix} (\tilde{c}(i+1) + \bar{c}(i+1))^\top R^\top \right) \varphi(i) \\ &+ \begin{bmatrix} 1 \\ \mathbf{0}_{n+m} \end{bmatrix} \hat{e}(i+1) + \bar{g}(i+1) \Delta w(i), \quad i = \bar{t}_0, ..., t-1, \end{aligned} \tag{24}$$

which looks like a perturbed system of the sort considered in Lemma 2. Using (20) and (22), it is clear that

$$\sum_{i=\bar{t}_0}^{t-1} \|\tilde{c}(i+1)\| \le 2\|\hat{\mathscr{S}}\|(n+m)^3 \left(1+\frac{2\|\hat{\mathscr{S}}\|}{\delta}\right)^{n+m},$$

which is independent of t. Also, we know that $\|\bar{c}(i)\| \le \delta$, $i = \bar{t}_0+1, \ldots, t$. Thus,

$$\begin{aligned} &\sum_{i=i_1}^{i_2-1} \left\| \begin{bmatrix} 1 \\ \mathbf{0}_{n+m} \end{bmatrix} (\tilde{c}(i+1) + \bar{c}(i+1))^\top R^\top \right\| \\ &\le 2\|\hat{\mathscr{S}}\|\|R\|(n+m)^3 \left(1+\frac{2\|\hat{\mathscr{S}}\|}{\delta}\right)^{n+m} + \delta\|R\|(i_2-i_1), \\ &\qquad \bar{t}_0 \le i_1 < i_2 \le t. \end{aligned}$$

Hence, this 'perturbation' is small on average. Since $A_{\hat{\theta}(t)\hat{\theta}(t)}$ is stable with margin $\tilde{\lambda} < \lambda$ (14), we can apply Lemma 2 and it follows that if we fix $\delta \in (0, \sigma]$ such that $\delta < \frac{\lambda - \tilde{\lambda}}{\|R\|\tilde{\gamma}}$, then (24) is a stable system with margin λ. Now define

$$\gamma_1 := 1 + (n+m)\left(1+\frac{2\|\hat{\mathscr{S}}\|}{\delta}\right)^{n+m},$$

and observe that $\bar{g}(i) \le \gamma_1$. From Lemma 2, we conclude that there exists a $\gamma_2 \ge 1$ so that for every $t \ge \bar{t}_0+1$ for which $\|\tilde{\theta}(t)\| \ge \delta$, we have

$$\begin{aligned} \|\varphi(t)\| &\le \gamma_2 \lambda^{t-\bar{t}_0} \|\varphi(\bar{t}_0)\| \\ &+ \gamma_2 \sum_{i=\bar{t}_0}^{t-1} \lambda^{t-i-1} (|\hat{e}(i+1)| + \gamma_1 |w(i)|). \end{aligned}$$

Using the Cauchy–Schwarz inequality and (23), we obtain [1]

$$\begin{aligned} &\sum_{i=\bar{t}_0}^{t-1} \lambda^{t-i-1} |\hat{e}(i+1)| \\ &\le \frac{n+m}{\sqrt{1-\lambda}} \left(1+\frac{2\|\hat{\mathscr{S}}\|}{\delta}\right)^{n+m} \sum_{i=\bar{t}_0}^{t-1} \sqrt{\lambda}^{t-i-1} |\Delta w(i)| \\ &\quad + \bar{k}\frac{n+m}{\sqrt{1-\lambda}} \left(1+\frac{2\|\hat{\mathscr{S}}\|}{\delta}\right)^{n+m} \sqrt{\lambda}^{t-\bar{t}_0} \|W(\bar{t}_0)\|^{0.5}. \end{aligned}$$

It follows that there exists a constant γ_3 so that for every $t \ge \bar{t}_0+1$ such that $\|\tilde{\theta}(t)\| \ge \delta$, we have

$$\begin{aligned} \|\varphi(t)\| &\le \gamma_3 \sqrt{\lambda}^{t-\bar{t}_0} \left\| \begin{bmatrix} \varphi(\bar{t}_0) \\ \mathrm{vec}\left(W(\bar{t}_0)^{\frac{1}{2}}\right) \end{bmatrix} \right\| \\ &+ \gamma_3 \sum_{i=\bar{t}_0}^{t-1} \sqrt{\lambda}^{t-i-1} |\Delta w(i)|. \end{aligned} \tag{25}$$

Part 3: A bound on $\|\varphi(t)\|$ on the whole interval. By combining (17) with (25), it follows that there exists a constant γ_4 such that

$$\begin{aligned} \|\varphi(t)\| &\le \gamma_4 \sqrt{\lambda}^{t-\bar{t}_0} \left\| \begin{bmatrix} \varphi(\bar{t}_0) \\ \mathrm{vec}\left(W(\bar{t}_0)^{\frac{1}{2}}\right) \end{bmatrix} \right\| \\ &+ \gamma_4 \sum_{i=\bar{t}_0}^{t-1} \sqrt{\lambda}^{t-i-1} |\Delta w(i)|, \quad t \ge \bar{t}_0. \end{aligned} \tag{26}$$

Part 4: A bound on $\|W(t)\|$. From (2), we see that $W(t)$ is a stable filter with a pole at λ and $\Delta\phi(t-1)$ and $\Delta y(t)$ as inputs. After some manipulation, it can be shown that $\|W(t)^{\frac{1}{2}}\|$ is bounded by a convolution involving φ and Δw. Combining this with (26) yields the result (1). The full derivation may be found in [18, Section 5.3]. □

[1] It is this step which appears to not be reproducible in the continuous-time version of Supervisory Control. See [18, Remark 3.5] for more details.

6. Robustness

The Supervisory Controller is nonlinear, so the linear-like convolution bound of Theorem 1 is somewhat surprising. It is also very useful: it can be leveraged to show that the system remains exponentially stable in the presence of time-varying parameters, nonlinearities, and unmodeled dynamics. To see this, consider a time-varying version of the plant subjected to an additive disturbance $w(t)$ and some unmodeled dynamics which enter the system via $d_\Delta(t)$:

$$y(t+1) = \phi(t)^\top \theta^*(t) + w(t) + d_\Delta(t). \qquad (27)$$

We adopt a common model of acceptable time-variations used in adaptive control: we let $s(\mathscr{S}, \zeta, \eta)$ denote the subset of $l_\infty(\mathbf{R}^{n+m})$ whose elements θ^* satisfy $\theta^*(t) \in \mathscr{S}$ for every $t \geq t_0$ as well as

$$\sum_{i=t_1}^{t_2-1} \|\theta^*(i+1) - \theta^*(i)\| \leq \zeta + \eta(t_2 - t_1), \quad t_2 > t_1 \geq t_0.$$

We also use a common model of unmodeled dynamics:

$$\begin{aligned} m(t+1) &= \beta m(t) + \beta\|\phi(t)\|, \\ |d_\Delta(t)| &\leq \mu m(t) + \mu\|\phi(t)\|, \quad t \geq t_0; \end{aligned} \qquad (28)$$

this model subsumes the classical additive uncertainty, multiplicative uncertainty, and uncertainty in a coprime factorization - see [12] for a more detailed discussion. Also, by letting $\beta = 0$, this can be viewed as an additive sector nonlinearity.

Before presenting this result, we must first define the new vector, which is the same as $\phi(t)$, except that it contains copies of y and u one additional step into the past:

$$\bar{\phi}(t) := \begin{bmatrix} y(t) & \dots & y(t-n) & u(t) & \dots & u(t-m) \end{bmatrix}^\top \in \mathbf{R}^{n+m+2};$$

this is necessary because of the estimator operating on the *difference* of the input-output data.

Theorem 2 *For every $\zeta \geq 0$, $\beta \in (0,1)$, $\lambda \in (0,1)$, $\tilde{\lambda} \in (0,\lambda)$ and $\hat{\lambda} \in (\max\{\beta, \sqrt{\lambda}\}, 1)$, there exists a $\gamma' \geq 1$, $\eta > 0$ and $\mu > 0$ so that for every $\theta^* \in s(\mathscr{S}, \zeta, \eta)$, $t_0 \in \mathbf{Z}$, $\bar{\phi}(t_0) \in \mathbf{R}^{n+m+2}$, $W(t_0) \in \mathbf{R}^{(n+m+1)\times(n+m+1)}$ positive semidefinite and symmetric, $m(t_0) \geq 0$, $\hat{\theta}(t_0) \in \hat{\mathscr{S}}$, $\tau(t_0) \in \{0, \dots, n+m\}$, $r \in R$, and $w \in l_\infty$, when the supervisory controller given by (2) - (5) and (10) is applied to the plant (27) with $d_\Delta(t)$ satisfying (28), the following bound holds:*

$$\left\| \begin{bmatrix} \bar{\phi}(t) \\ \text{vec}\left(W(t)^{\frac{1}{2}}\right) \\ m(t) \end{bmatrix} \right\| \leq \gamma' \hat{\lambda}^{t-\bar{t}_0} \left\| \begin{bmatrix} \bar{\phi}(\bar{t}_0) \\ \text{vec}\left(W(\bar{t}_0)^{\frac{1}{2}}\right) \\ m(\bar{t}_0) \end{bmatrix} \right\| + \gamma' \sum_{i=\bar{t}_0}^{t-1} \hat{\lambda}^{t-i-1} |w(i)| + \gamma' |r|, \quad t \geq \bar{t}_0 \geq t_0.$$

Proof: One can use Theorem 1 to convert a bound on φ to a bound on $\bar{\phi}$, and then apply Theorems 1 and 3 of [16] to prove robustness. The details can be found in [18, Section 5.4]. □

7. A Simulation Example

Here we provide a simulation demonstrating Supervisory Control of a time-varying plant with the goal being to track a constant reference $r = 1$ in the presence of a random disturbance with non-zero mean. Consider the first-order time-varying plant with relative degree one:

$$y(t+1) = a^*(t)y(t) + b^*(t)u(t) + w(t), \quad t \geq 0,$$

with initial conditions of $y(0) = u(0) = u(-1) = 0$, and with parameters constrained to the set

$$\mathscr{S} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a \in [-4,4], b \in [1,3] \cup [-3,-1] \right\};$$

this set admits a natural partitioning into $\hat{\mathscr{S}}_1$ and $\hat{\mathscr{S}}_2$ with the properties required in Lemma 1. The plant's time-varying parameters are chosen as

$$\begin{aligned} a^*(t) &= 3 - 0.03t, \quad t \in \{0, \dots, 199\}, \\ b^*(t) &= \begin{cases} -3 + 0.02t, & t \in \{0, \dots, 99\} \\ -1 + 0.02t, & t \in \{100, \dots, 199\}, \end{cases} \end{aligned}$$

and the disturbance $w(t)$ is a Gaussian random signal with standard deviation 0.05 and mean 1. The supervisory estimator uses a forgetting factor of $\lambda = 0.8$ and its dwell period is 3, and uses initial conditions of $\hat{\theta}(0) = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$, $W(0) = \mathbf{0}_{3\times 3}$, and $\tau(0) = 2$. We plot the results in Figure 2. There is a large initial transient, but from $t = 5$ onward good tracking ensues: the RMS of the tracking error is just 0.139, which is only a factor of 2.78 greater than the variance of the disturbance.

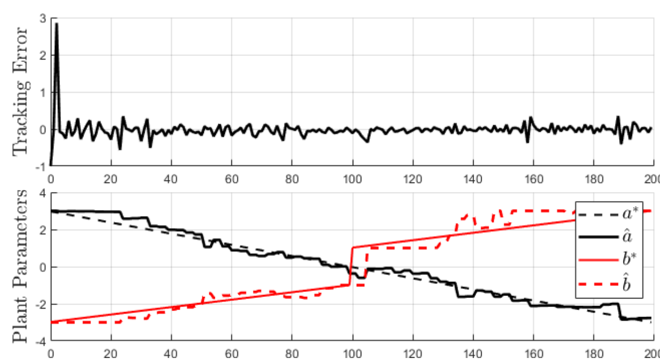

Figure 2: Plot of $\varepsilon(t)$ and estimated vs true parameters.

8. Summary and Conclusions

Here we show that if the Supervisory Control technique is applied to the discrete-time adaptive step tracking problem, then we are able to prove **linear-like convolution bounds on the closed-loop behavior**, which confers exponential stability and a bounded noise gain, and tolerance to a degree of unmodeled dynamics, nonlinearities, and plant parameter variation. Such linear-like bounds have never before been proven in the Supervisory Control paradigm.

We are working on several extensions of the approach. We would like to prove similar results when the plant model is nonlinearly dependent on the parameter vector. Also, we would like to determine if a dwell time of $n+m+1$ is necessary for stability, and hopefully relax that requirement.

References

[1] A. Feuer and A. S. Morse, "Adaptive control of single-input, single-output linear systems," in *IEEE Transactions on Automatic Control*, vol. 23(4), 1978, pp. 557–569.

[2] A. S. Morse, "Global stability of parameter-adaptive control systems," in *IEEE Transactions on Automatic Control*, vol. 25(3), 1980, pp. 433–439.

[3] G. C. Goodwin, P. J. Ramadge, and P. E. Caines, "Discrete-time multivariable adaptive control," in *IEEE Transactions on Automatic Control*, vol. 25(3), 1980, pp. 449–456.

[4] K. S. Narendra, Y. H. Lin, and L. S. Valavani, "Stable adaptive controller design, part ii: Proof of stability," in *IEEE Transactions on Automatic Control*, vol. 25(3), 1980, pp. 440–448.

[5] K. S. Narendra and Y. H. Lin, "Stable discrete adaptive control," in *IEEE Transactions on Automatic Control*, vol. 25(3), 1980, pp. 456–461.

[6] C. E. Rohrs, L. Valavani, M. Athans, and G. Stein, "Robustness of continuous-time adaptive control algorithms in the presence of unmodelled dynamics," in *IEEE Transactions on Automatic Control*, vol. 30(9), 1985, pp. 881–889.

[7] A. S. Morse, "Supervisory control of families of linear set-point controillers - part 1: Exact matching," in *IEEE Transactions on Automatic Control*, vol. 41(10), 1996, pp. 1413–1431.

[8] A. S. Morse, "Supervisory control of families of linear set-point controillers - part 2: Robustness," in *IEEE Transactions on Automatic Control*, vol. 42(11), 1997, pp. 1500–1515.

[9] L. Vu and D. Liberzon, "Supervisory control of uncertain linear time-varying systems," in *IEEE Transactions on Automatic Control*, vol. 56(1), 2011, pp. 27–42.

[10] D. Miller, "A parameter adaptive controller which provides exponential stability: The first order case," in *Systems and Control Letters*, vol. 103, 2017, pp. 23–31.

[11] D. Miller, "Classical discrete-time adaptive control revisited: Exponential stabilization," in *2017 IEEE Conference on Control Technology and Applications (CCTA)*, 2017, pp. 1975–1980.

[12] D. E. Miller and M. T. Shahab, "Classical pole placement adaptive control revisited: Linear-like convolution bounds and exponential stability," in *Math Control Signals Syst*, vol. 30(4):19, 2018.

[13] D. E. Miller and M. T. Shahab, "Adaptive tracking with exponential stability and convolution bounds using vigilant estimation," in *Math. Control Signals Syst*, vol. 32, 2020, pp. 241–291.

[14] D. E. Miller and M. T. Shahab, "Linear-like properties arise naturally in the adaptive control setting," in *International Journal of Adaptive Control and Signal Processing*, vol. 35(6), 2021, pp. 965–990.

[15] M. T. Shahab and D. E. Miller, "Asymptotic tracking and linear-like behavior using multi-model adaptive control," in *IEEE Transactions on Automatic Control*, vol. 67(1), 2022, pp. 203–219.

[16] M. T. Shahab and D. E. Miller, *A convolution bound implies tolerance to time-variations and unmodelled dynamics*, 2022. DOI: `10.48550/ARXIV.1910.02112v2`.

[17] D. Miller and M. Shahab, "The inherent robustness of a new approach to adaptive control," in *2020 IEEE Conference on Control Technology and Applications (CCTA)*, 2020, pp. 510–515.

[18] C. J. Lalumiere, "Supervisory adaptive control revisited: Linear-like convolution bounds," M.S. thesis, University of Waterloo, 2022.

[19] C. J. Lalumiere and D. E. Miller, "Supervisory adaptive control revisited: Linear-like convolution bounds," in *Systems and Control Letters*, Under review, 2022.

[20] K. J. Åström and B. Wittenmark, "Adaptive control," in Second. New York: Dover Publications, 2013, ch. 11. Practical Issues and Implementation, pp. 448–498.

[21] M. T. Shahab and D. E. Miller, "Adaptive set-point regulation using multiple estimators," in *2019 IEEE 58th Conference on Decision and Control (CDC)*, 2019, pp. 84–89.

[22] G. Kreisselmeier, "Adaptive control of a class of slowly time-varying plants," in *Systems and Control Letters*, vol. 8, 1986, pp. 97–103.