2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Control of Linear Systems with Guarantee of Outputs in Given Sets at Any Time*

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***Abstract*— A new method for designing the control law for linear plants with a guarantee of finding outputs in given sets under conditions of unknown bounded disturbances is proposed. The problem is solved in two stages. In the first stage, a coordinate transformation is used to reduce the original constrained problem to the problem of studying the input-to-state stability of a new extended system without constraints. In the second stage, the control law for the transformed system is designed, where the adjustable parameters are selected from the solution of linear matrix inequalities (LMI). The simulations, which are performed in MATLAB, show the method's efficiency and confirm the theoretical results.**

I. Introduction

In practice, there are many control problems with a guarantee of finding outputs in given sets. For example, it is required to maintain the frequency and output voltage on electric generators within specified bounds in the electric power network [1]–[4] or pressure and flow rate at the wellhead in a given band when controlling the formation pressure stabilization process [5], [6].

The first approach for solving such problems is given in [7]. In this paper, the given sets are defined by a sequence of rectangles. The size of each rectangle corresponds to the desired maximum deviation of the output from the equilibrium point and to the desired time when the output belongs to the respective rectangle. However, these rectangular regions are rather rough, and this approach only applies to systems with scalar input and output (SISO). The paper [8] proposes the adaptive control method for multi-input multi-output systems (MIMO) with the guarantee of belonging the output vector to a given set. However, implementing this method requires knowledge of the sign and the set of initial conditions. Moreover, obtained upper and lower bounds for transients are rather rough because these bounds are determined by the same function with different signs. Additionally, the upper and lower bounds asymptotically converge to some constants.

The papers [9], [10] propose a novel change of coordinate that allows switching from the original constrained problem to a new control one without constraints. Differently from [7], [8], the method [9], [10] allows the developer to consider a more general class of restrictions, for example, non-symmetric and taking any value by sign. However, the application of this method is considered for a linear system only with scalar input and output signals, and the parameters in the algorithm controls are selected manually. Moreover, the approach to designing control law for the scalar system in [9], [10] cannot apply to MIMO systems.

We propose a new method to design the nonlinear control law based on results [9], [10]. Differently from [9], [10], the contribution of the present paper is as follows:

- a novel method is designed for control of MIMO systems;
- the calculation of the control law parameters is based on the use of LMI;
- recommendations on the choice of parameters in the control law to reduce the influence of disturbances are proposed.

The paper is organized as follows. Section II gives a crucial lemma. Section III describes the problem of linear plant control with a guarantee that the output signals belong to given sets. Section IV proposes a design of control law for linear plants based on linear matrix inequalities. Section V illustrates the results obtained by simulation using MATLAB, and demonstrates the theoretical conclusions.

The present paper uses the following notation: $\mathbb{R}^n$ is the Euclidean space of dimension n with euclidean norm $|\cdot|$; $\mathbb{R}^{n\times m}$ is the set of all real $n \times m$ matrices with euclidean norm $\|\cdot\|$; $I, 0, diag\{\cdot\}$ denotes the identity, zero, and diagonal matrix (of the corresponding dimension); $\mathbf{1_m}$ denotes the all-one vector with m values 1; $col\{\cdot\} \in \mathbb{R}^m$ denotes a column vector in $\mathbb{R}^m$; for square matrices $A \in \mathbb{R}^{n\times n}$, $A \succ 0$ ($A \prec 0$) means that A is a positive-definite matrix (negative-definite matrix). $A \succeq 0$ ($A \preceq 0$) means that A is a non-negative definite matrix (non-positive definite matrix); the symmetric entries of a symmetric matrix will be indicated by $\star$.

II. Auxiliary Results

Let us consider a key lemma in the present paper. According to [11]–[13], S-procedure can be formulated as follows

Lemma 1 (S-Procedure): Let the quadratic forms

$$f_i(x) = x^T A_i x, i = 0, 1, ..., m,$$

where $x \in \mathbb{R}^n$, $A_i = A_i^T \in \mathbb{R}^{n\times n}$, and the numbers $\alpha_0, \alpha_1, ..., \alpha_m$. If there are numbers $\tau_i \geq 0, i = 1, ..., m$, such that

$$A_0 \preceq \sum_{i=1}^{m} \tau_i A_i, \quad \alpha_0 \geq \sum_{i=1}^{m} \tau_i \alpha_i,$$

*The work is carried out at the IPME RAS with the partial supported by goszadanie no. 121112500298-6 (EGISU NIOKTR) in Sections III, V and the Ministry of Science and Higher Education of the Russian Federation (Project No. 075-15-2021-573) in Section IV.

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then the inequalities

$$f_i(x) \le \alpha_i, i = 1, ..., m,$$

imply the inequality

$$f_0(x) \le \alpha_0,$$

for all $x \ne 0$.

III. Problem Statement

Consider the linear dynamical system with m inputs and m outputs

$$\begin{array}{c}\dot{x}(t) = Ax(t) + Bu(t) + Df(t),\ x(0) \in \mathcal{S}_0 \\ y(t) = Lx(t),\end{array} \tag{1}$$

where $t \ge 0$; $x(t) \in \mathbb{R}^n$ is the state vector; $x(0)$ is unknown initial condition, and $\mathcal{S}_0$ is the known compact set of all initial values $x(0)$; $u(t) \in \mathbb{R}^m$ is control signal; $y(t) = col\{y_1(t), ..., y_m(t)\} \in \mathbb{R}^m$ is the output signal; $f(t) \in \mathbb{R}^l$ is an unknown bounded disturbance with $|f(t)| \le \bar{f}$; the matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, L \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{n\times l}$ are known.

The goal of the paper is to design a control law that guarantees the output signal $y(t)$ of the system (1) stays in the following set

$$\mathcal{Y} = \{y(t) \in \mathbb{R}^m : \underline{g}_i(t) < y_i(t) < \overline{g}_i(t), i = 1, ..., m\}. \tag{2}$$

The functions $\underline{g}_i(t)$ and $\overline{g}_i(t)$ are differential and bounded with their first-time derivatives. These functions can be selected by designers based on the requirements for system operation.

We accept the following assumptions for the system (1).

Assumption 1: Let us define the set $\mathcal{Y}_0 := \{y_0 :\ y_0 = Lx_0,\ x_0 \in \mathcal{S}_0\}$ as the set of all possible initial values $y(0)$ of the output $y(t)$. The initial condition $x(0)$ must be such that the initial value $y(0)$ belongs to the given set $\mathcal{Y}$. That is $\mathcal{Y}_0 \subset \mathcal{Y}$ or $\underline{g}_i(0) < y_i(0) < \overline{g}_i(0), i = 1, ..., m$. Assumption (1) guarantees the fulfillment of condition (2) at the initial time $t = 0$.

Assumption 2: The system (1) has a relative degree equal to $\mathbf{1_m}$. Assumption (2) guarantees that the matrix LB is invertible [14], [15].

Assumption 3: The system (1) is controllable. This assumption is typical for problems of control.

Remark 1: If for the initial condition $x(0)$, the corresponding initial value of the output $y(0)$ is not in the set $\mathcal{Y}$, then due to the choice of restriction functions $\overline{g}(t)$ and $\underline{g}(t)$, we can modify the set $\mathcal{Y}$ so that new restriction functions can cover this initial value $y(0)$. We will discuss this in more detail in the Example section.

IV. Main result

Following [10], let us introduce the transformation of coordinates

$$\varepsilon = \Phi(y, t). \tag{3}$$

Here, $\varepsilon = col\{\varepsilon_1, ..., \varepsilon_m\} \in \mathbb{R}^m$ is an auxiliary variable; $\Phi(y, t) = col\{\Phi_1(y_1, t), ..., \Phi_m(y_m, t)\}$ is a vector function with $\Phi_i(y_i, t),\ i = 1, ..., m$ are defined as follows

$$\Phi_i(y_i, t) = ln\left(\frac{y_i(t) - \underline{g}_i(t)}{\overline{g}_i(t) - y_i(t)}\right). \tag{4}$$

Remark 2: According to [10], the function $\Phi(y, t)$ satisfies the following properties

(a) $\forall t \ge 0$ there exists an inverse mapping with respect to $\varepsilon \in \mathbb{R}^m$

$$y = \Phi^{-1}(\varepsilon, t), \tag{5}$$

where $\Phi^{-1}(\varepsilon, t) = col\{\Phi_1^{-1}(\varepsilon_1, t), ..., \Phi_m^{-1}(\varepsilon_m, t)\}$

(b) $\underline{g}_i(t) < \Phi_i^{-1}(\varepsilon_i, t) < \overline{g}_i(t), i = 1, ..., m, \forall \varepsilon_i \in \mathbb{R}$ and $t \ge 0$;

(c) $\forall \varepsilon \in \mathbb{R}^m$ and $t \ge 0$ we have $0 \prec \frac{\partial \Phi^{-1}}{\partial \varepsilon} \preceq \overline{\sigma} I$ and $\left|\frac{\partial \Phi^{-1}}{\partial t}\right| < \gamma$, where $\overline{\sigma},\ \gamma > 0$ are determined by the transformation (3);

Let us check these properties for (3).

It is easy to see that $\forall t \ge 0$, there exists an inverse function Φ_i^{-1} w.r.t $\varepsilon_i \in \mathbb{R}$

$$y_i = \Phi_i^{-1}(\varepsilon_i, t) = \frac{\overline{g}_i(t)e^{\varepsilon_i} + \underline{g}_i(t)}{e^{\varepsilon_i} + 1}, \tag{6}$$

then, there exists a vector function $y = \Phi^{-1}(\varepsilon, t) = col\{\Phi_1^{-1}(\varepsilon_1, t), ..., \Phi_m^{-1}(\varepsilon_m, t)\}$.

Due to $\overline{g}_i(t) > \underline{g}_i(t),\ \forall t \ge 0$, then, from (6) we have $\forall \varepsilon_i \in \mathbb{R}$

$$\underline{g}_i(t) < \Phi_i^{-1}(\varepsilon_i, t) < \overline{g}_i(t).$$

Due to the $\Phi_i^{-1}(\varepsilon_i, t)$ is a differential function w.r.t ε and t. Then, we can define the partial derivatives of $\Phi^{-1}(\varepsilon, t)$ as follows

$$\frac{\partial \Phi^{-1}}{\partial \varepsilon} = diag\left\{\frac{e^{\varepsilon_1}(\overline{g}_1 - \underline{g}_1)}{(e^{\varepsilon_1} + 1)^2}, ..., \frac{e^{\varepsilon_m}(\overline{g}_m - \underline{g}_m)}{(e^{\varepsilon_m} + 1)^2}\right\},$$

$$\frac{\partial \Phi^{-1}}{\partial t} = col\left\{\frac{\dot{\overline{g}}_1 e^{\varepsilon_1} + \dot{\underline{g}}_1}{e^{\varepsilon_1} + 1}, ..., \frac{\dot{\overline{g}}_m e^{\varepsilon_m} + \dot{\underline{g}}_m}{e^{\varepsilon_m} + 1}\right\}.$$

From the last expressions, it can be seen that $0 \prec \frac{\partial \Phi^{-1}}{\partial \varepsilon} \preceq \overline{\sigma} I$, and $\left|\frac{\partial \Phi^{-1}}{\partial t}\right| \le \gamma,\ \forall \varepsilon \in \mathbb{R}^m$ and $t \ge 0$, where

$$\overline{\sigma} = \frac{1}{4}\max_i\left[\sup_{t\ge 0}(\overline{g}_i(t) - \underline{g}_i(t))\right], \tag{7}$$

and

$$\gamma = \sqrt{m}\max_i\{\sup_{t\ge 0}|\dot{\overline{g}}_i(t)|, \sup_{t\ge 0}|\dot{\underline{g}}_i(t)|\}. \tag{8}$$

To design a control law, the information about the dynamics of the variable $\varepsilon(t)$ is required. Calculate the time derivative of the function $y(t)$ on (5)

$$\dot{y} = \frac{\partial \Phi^{-1}}{\partial \varepsilon}\dot{\varepsilon} + \frac{\partial \Phi^{-1}}{\partial t}. \tag{9}$$

Taking into account (1) and (5), rewrite (9) in the form

$$\dot{\varepsilon} = \left(\frac{\partial \Phi^{-1}}{\partial \varepsilon}\right)^{-1}\left[LAx + LBu + LDf - \frac{\partial \Phi^{-1}}{\partial t}\right]. \tag{10}$$

In the expression (10) $LDf(t)$ and $\frac{\partial\Phi^{-1}(\varepsilon,t)}{\partial t}$ are bounded functions, so we make the substitution $\psi(t) = LDf(t) - \frac{\partial\Phi^{-1}(\varepsilon,t)}{\partial t}$. Then $|\psi(t)| \le \kappa$, where

$$\kappa = \|LD\|\bar{f} + \gamma. \quad (11)$$

Taking into account the last replacement, we rewrite the expression (10) as

$$\dot{\varepsilon} = \left(\frac{\partial\Phi^{-1}}{\partial\varepsilon}\right)^{-1}\Big[LAx + LBu + \psi\Big]. \quad (12)$$

The following theorem and its proof follow directly from Theorem 1 of [10].

Theorem 1: If there exists a control law $u(t)$ with (3) such that the solution (12) is bounded, then $y(t) \in \mathcal{Y}$.

This theorem allows one to transfer the control problem (1) with constraints (2) on the output $y(t)$ to the control problem on the auxiliary variable $\varepsilon(t)$ without constraint. Moreover, according to (4), if $\varepsilon_i(t)$ converges to the origin, then, $y_i(t)$ converges to the middle of the "tube", i.e. $y_i(t) \to \frac{\overline{g}_i(t)+\underline{g}_i(t)}{2}$. If $\varepsilon_i(t)$ goes to infinity, then $y_i(t)$ converges to the boundaries of the set $\mathcal{Y}$.

In order to find a control $u(t)$ that ensures the boundedness of $\varepsilon(t)$, we consider Lyapunov function of the form $V = \frac{1}{2}\varepsilon^T\varepsilon$. According to (12), we get

$$\dot{V} = \varepsilon^T\dot{\varepsilon} = \varepsilon^T\left(\frac{\partial\Phi^{-1}}{\partial\varepsilon}\right)^{-1}\Big[LAx + LBu + \psi\Big]. \quad (13)$$

Then, we introduce the control law for the system (1) in the form

$$u = -(LB)^{-1}[K\varepsilon + LAx], \quad (14)$$

where $K \in \mathbb{R}^{m\times m}$ is the design matrix.

Putting the control law (14) into the expression (12), we get

$$\dot{\varepsilon} = \left(\frac{\partial\Phi^{-1}}{\partial\varepsilon}\right)^{-1}\left[-K\varepsilon + \psi\right], \quad (15)$$

We calculate the time derivative of Lyapunov function V along the solutions (15)

$$\dot{V} = \varepsilon^T\dot{\varepsilon} = \varepsilon^T\left(\frac{\partial\Phi^{-1}}{\partial\varepsilon}\right)^{-1}\left[-K\varepsilon + \psi\right]. \quad (16)$$

Suppose that Σ_0 is the set of all possible initial values $\varepsilon(0)$. It is evident that Σ_0 is a compact set due to the compactness of the set $\mathcal{S}_0$. Using the concept of input-to-state stability [16] to determine the control gain K in (14) such that the solution $\varepsilon(t)$ of the system (15) starting from any initial point $\varepsilon(0)$ from Σ_0 is bounded for any time. In [10], the system (15) is SISO ($m = 1$), then the gain K can be determined such that the following expression holds for some $\alpha > 0$ and $\beta > 0$

$$\dot{V} + 2\alpha V\left(\frac{\partial\Phi^{-1}}{\partial\varepsilon}\right)^{-1} - \beta\psi^2\left(\frac{\partial\Phi^{-1}}{\partial\varepsilon}\right)^{-1} \le 0. \quad (17)$$

As we can see when analyzing the expression (17) with $m = 1$, $\left(\frac{\partial\Phi^{-1}}{\partial\varepsilon}\right)^{-1}$ is a positive scalar value that does not affect the sign of this expression and can be neglected. However, for the case with $m > 1$, $\left(\frac{\partial\Phi^{-1}}{\partial\varepsilon}\right)^{-1}$ is a positive definite matrix, which can not be neglected as in the case $m = 1$. Then, the approach for determining the gain K from [10] can not be applied to MIMO systems. To analyze the ISS stability of the system (15) with an arbitrary value of $m \ge 1$, we need to prove the following auxiliary lemma.

Lemma 2: Let us consider the following block matrices

$$M = \begin{bmatrix} Q & 0 \\ \star & Q \end{bmatrix} \succ 0,\ N = \begin{bmatrix} N_{11} & N_{12} \\ \star & N_{22} \end{bmatrix} \prec 0,$$

where $Q, N_{11}, N_{12}, N_{21}, N_{22} \in \mathbb{R}^{n\times n}$ are real diagonal matrices. Then the matrix multiplication

$$MN = \begin{bmatrix} QN_{11} & QN_{12} \\ \star & QN_{22} \end{bmatrix},$$

is a negative definite matrix.

Proof: It is easy to see that the matrix MN is symmetric. Let $\lambda_i, x_i, i = 1, ..., 2n$ be eigenvalues and eigenvectors of matrix MN respectively. Then we have the following relation

$$x_i^T NMNx_i = \lambda_i x_i^T Nx_i.$$

From the last equation we can express λ_i as

$$\lambda_i = \frac{x_i^T NMNx_i}{x_i^T Nx_i}.$$

Since $M \succ 0$ and $N = N^T \prec 0$, then $NMN \succ 0$, i.e. $x^T NMNx > 0\ \forall x \ne 0$. Taking into account $x^T Nx < 0\ \forall x \ne 0$, we obtain that $\lambda_i < 0, i = 1, ..., 2n$. The symmetric matrix MN has all negative eigenvalues, then MN is a negative definite. ■

The following theorem gives an approach to determine the gain matrix K in (14).

Theorem 2: For a given number $c > 0$, if there exists a diagonal matrix $K \in \mathbb{R}^{m\times m}$ and positive coefficients τ_1, τ_2 and τ_3, such that for any $\alpha > 0$ and $\beta > 0$ and any fixed numbers $\sigma \in [\underline{\sigma},\ \overline{\sigma}]$, the following linear inequalities are feasible

$$\begin{bmatrix} -K + [(\tau_1 - \tau_2)\sigma - \alpha + \beta]I & 0.5I \\ \star & -(\tau_3\sigma - \beta)I \end{bmatrix} \preceq 0,$$
$$-2c\tau_1 + \rho^2\tau_2 + \kappa^2\tau_3 \le 0, \quad (18)$$

where $\rho = \max_{\Sigma_0}|\varepsilon(0)|$, and $\underline{\sigma} = \min_i\left(\inf_{t\ge 0}\left[\frac{\partial\Phi_i^{-1}(\varepsilon_i,t)}{\partial\varepsilon_i}\Big|_{\varepsilon_i=\rho}\right]\right)$, $\overline{\sigma}$ and κ are defined in (7) and (8).

Then, for any initial condition of the system (1) from $\mathcal{S}_0$, the control law (14) with K satisfying (18) provides the goal (2).

Proof: Let us define the sets Ω and Π as follows

$$\Omega := \{\varepsilon \in \mathbb{R}^m : |\varepsilon| < \sqrt{2c},\ c > 0\}, \quad (19)$$

$$\Pi := \{\varepsilon \in \mathbb{R}^m : |\varepsilon| \le \rho,\ \rho > 0\}, \quad (20)$$

where $c < 0.5\rho^2$ is some positive number and $\rho = \max_{\Sigma_0}|\varepsilon(0)|$. It is evident that for fixed values c and ρ, these above sets are invariant. Moreover, as we obtain that $\Omega \subset \Pi$ and $\Sigma_0 \subseteq \Pi$.

In order to stabilize the trajectory of system (15) starting from any initial point of the set Σ_0 in the presence of the bounded disturbance $f(t)$ into the set Ω, we require the condition $\dot{V}(\varepsilon) \leq -\alpha\varepsilon^T \left(\frac{\partial\Phi^{-1}}{\partial\varepsilon}\right)^{-1} \varepsilon < 0$, for all ε such that $c \leq V(\varepsilon) \leq 0.5\rho^2$, where α is any positive number. We can rewrite the above conditions as

$$-\varepsilon^T \left(\frac{\partial\Phi^{-1}}{\partial\varepsilon}\right)^{-1} (K+\alpha I)\varepsilon + \varepsilon^T \left(\frac{\partial\Phi^{-1}}{\partial\varepsilon}\right)^{-1} \psi \leq 0,$$
$$\forall(\varepsilon,\psi) : 0.5\varepsilon^T\varepsilon \geq c,\ \varepsilon^T\varepsilon \leq \rho^2,\ \psi^T\psi \leq \kappa^2. \tag{21}$$

Denoting $z = [\varepsilon, \psi]^T$ and $\overline{K} = -(K+\alpha I)$, rewrite (21) in matrix form:

$$z^T \begin{bmatrix} \left(\frac{\partial\Phi^{-1}}{\partial\varepsilon}\right)^{-1}\overline{K} & 0.5\left(\frac{\partial\Phi^{-1}}{\partial\varepsilon}\right)^{-1} \\ \star & 0 \end{bmatrix} z \leq 0,$$
$$z^T \begin{bmatrix} -I & 0 \\ \star & 0 \end{bmatrix} z \leq -2c, z^T \begin{bmatrix} I & 0 \\ \star & 0 \end{bmatrix} z \leq \rho^2, z^T \begin{bmatrix} 0 & 0 \\ \star & I \end{bmatrix} z \leq \kappa^2. \tag{22}$$

According to S-procedure, the inequalities (22) are satisfied if the following inequalities hold

$$\begin{bmatrix} \left(\frac{\partial\Phi^{-1}}{\partial\varepsilon}\right)^{-1}\overline{K} + (\tau_1-\tau_2)I & 0.5\left(\frac{\partial\Phi^{-1}}{\partial\varepsilon}\right)^{-1} \\ \star & -\tau_3 I \end{bmatrix} \prec 0,$$
$$-2c\tau_1 + \rho^2\tau_2 + \kappa^2\tau_3 \leq 0. \tag{23}$$

The first inequality in (23) is equivalent to

$$\begin{bmatrix} \left(\frac{\partial\Phi^{-1}}{\partial\varepsilon}\right)^{-1} & 0 \\ \star & \left(\frac{\partial\Phi^{-1}}{\partial\varepsilon}\right)^{-1} \end{bmatrix} \times$$
$$\times \begin{bmatrix} \overline{K} + (\tau_1-\tau_2)\frac{\partial\Phi^{-1}}{\partial\varepsilon} & 0.5I \\ \star & -\tau_3\frac{\partial\Phi^{-1}}{\partial\varepsilon} \end{bmatrix} \prec 0. \tag{24}$$

Since $\left(\frac{\partial\Phi^{-1}(\varepsilon,t)}{\partial\varepsilon}\right)^{-1} \succ 0$ and according to Lemma 2 the inequality (24) holds if the following inequality holds

$$\Gamma := \begin{bmatrix} \overline{K} + (\tau_1-\tau_2)\frac{\partial\Phi^{-1}}{\partial\varepsilon} & 0.5I \\ \star & -\tau_3\frac{\partial\Phi^{-1}}{\partial\varepsilon} \end{bmatrix} \prec 0. \tag{25}$$

By homogeneity, we can write (25) in a non-strictly form $\Gamma \preceq -\beta I$, where β can be chosen as any small positive number, i.e.

$$\begin{bmatrix} \overline{K} + (\tau_1-\tau_2)\frac{\partial\Phi^{-1}}{\partial\varepsilon} + \beta I & 0.5I \\ \star & -\tau_3\frac{\partial\Phi^{-1}}{\partial\varepsilon} + \beta I \end{bmatrix} \preceq 0. \tag{26}$$

By condition (21), we care about the sign of the derivative of Lyapunov function only for ε from the set $\Pi \setminus \Omega = \{\varepsilon \in \mathbb{R}^m : \sqrt{2c} \leq |\varepsilon| \leq \rho\}$. Moreover, $\forall\varepsilon \in \Pi \setminus \Omega$, it is evident that $\underline{\sigma}I \preceq \frac{\partial\Phi^{-1}}{\partial\varepsilon} \preceq \overline{\sigma}I$, where $\underline{\sigma} = \min_i \left(\inf_{t\geq 0}\left[\frac{\partial\Phi_i^{-1}(\varepsilon_i,t)}{\partial\varepsilon_i}\Big|_{\varepsilon_i=\rho}\right]\right)$. Then, we obtain the following polytopic type time-invariant uncertainty

$$\Gamma \in \Xi = \{\Gamma \in \mathbb{R}^{2m\times 2m} : \Gamma = p\Gamma_1 + (1-p)\Gamma_2\}, \tag{27}$$

for some $p \in [0,\ 1]$.

We define the two vertices Γ_1 and Γ_2 of the polytope in (27) corresponding to $\underline{\sigma}I$ and $\overline{\sigma}I$ as follows

$$\Gamma_1 = \begin{bmatrix} \overline{K} + (\tau_1-\tau_2)\underline{\sigma}I + \beta I & 0.5I \\ \star & -\tau_3\underline{\sigma}I + \beta I \end{bmatrix},$$
$$\Gamma_2 = \begin{bmatrix} \overline{K} + (\tau_1-\tau_2)\overline{\sigma}I + \beta I & 0.5I \\ \star & -\tau_3\overline{\sigma}I + \beta I \end{bmatrix}. \tag{28}$$

According to [17], [18], if there exist a matrix K and numbers τ_1, τ_2 and τ_3 such that the inequality (26) and the second inequality in (23) hold at all vertices Γ_1 and Γ_2 in (28), then (18) have solutions. Therefore, the control law (14) with K satisfying (18) ensures the input-to-state stability in the system (15), then $\varepsilon(t)$ is bounded. According to Theorem 1, goal (2) is satisfied. Theorem 2 is proved. ■

Remark 3: The LMI technique and S-procedure allow us to analyze the input to state stability of the closed-loop MIMO system under the influence of unknown bounded disturbances. Moreover, the problem of finding the control gain matrix in (14) can be reduced to the problem of finding the solutions to the feasible problem (18), which can be easily solved using popular solvers for semidefinite programming (like SEDUMI [19], SDPT3 [20], CSDP [21] and others.)

Remark 4: It can be seen that the parameter c in (19) is associated with the radius of the open ball in which the trajectories of the system (15) are attracted (the radius of the ball is equal to $\sqrt{2c}$). If we decrease the value of c, then the radius of the ball will decrease and, in turn, the limiting value of $\varepsilon(t)$. Therefore, with a decrease in the limiting value of $\varepsilon(t)$, the oscillation of the variable $y(t)$ in the set $\mathcal{Y}$, which is caused by the influence of the external disturbance $f(t)$, also decreases.

V. Example

Let us demonstrate the performance of control for an unstable system (1) with two inputs and two outputs with the following parameters

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.1 & 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix},$$
$$D = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, L = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix},$$
$$f(t) = 0.1 + \sin(3t) + 0.5sat\{d(t)\},$$

where $sat\{\cdot\}$ is the saturation function, $d(t)$ is the signal modeled in MATLAB/SIMULINK using the "Band-Limited White Noise" block with a noise power and a sampling time of 0.1. Then $\bar{f} = 1.6$. The graph of the disturbance $f(t)$ is shown in Fig. 1.

Let $\Phi(y(t),t) = diag\{\Phi_1(y_1(t),t), \Phi_2(y_2(t),t)\}$, where $\Phi_i, i = 1,2$ are defined in (4).

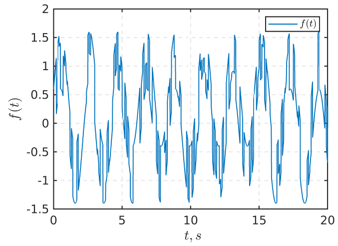


Fig. 1: The disturbance $f(t)$.

Define the parameters of the restriction functions $\underline{g}(t)$ and $\overline{g}(t)$ as

$$\begin{aligned} \overline{g}_1(t) &= 3.52e^{-0.5t} + 0.1, \\ \underline{g}_1(t) &= 1.62e^{-0.5t} - 0.1, \\ \overline{g}_2(t) &= 1.62\cos(0.5t) + 1.52, \\ \underline{g}_2(t) &= \cos(0.5t) + 0.8. \end{aligned} \tag{29}$$

According to (14) the control law is defined as

$$u = -(LB)^{-1}\left[K col\left\{ \ln\left(\frac{y_i - \underline{g}_i}{\overline{g}_i - y_i}\right)\right\} + LAx\right], i = 1, 2.$$

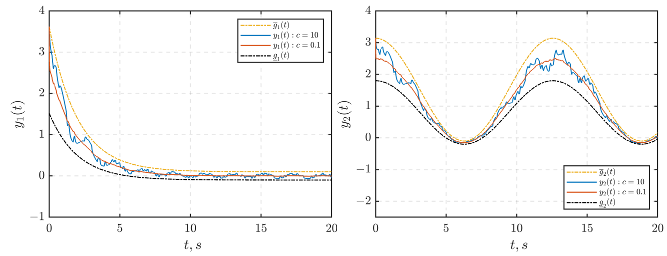
Let the initial condition of the system (1) be $x(0) = col\{1.48,\ 1,\ -0.34001\}$, then we obtain the initial value $\varepsilon(0) = col\{12.25,\ 11.81\}$. Moreover, we assume that the given above $\varepsilon(0)$ lies on the bound of the set Π. That is, the trajectory $\varepsilon(t)$ of the system (15) starts from the "farthest from the origin" point in the set of initial value Σ_0. Then, according to (29), we can calculate $\rho = \sqrt{12.25^2 + 11.81^2} = 17.02$ and $\underline{\sigma} = 4.1 \cdot 10^{-8}$. From (7), we obtain $\overline{\sigma} = 0.54$. Choose $\alpha = 0.01, \beta = 10^{-7}$, let us find solutions $(K, \tau_1, \tau_2, \tau_3)$ to the inequalities (18) at $\sigma = \underline{\sigma}$ and $\sigma = \overline{\sigma}$ for $c = 0.1$ and $c = 10$. The results are given in Table 1.

TABLE I: The solutions $(K, \tau_1, \tau_2, \tau_3)$ of the inequalities (18) for $c = 0.1$ and $c = 10$

c	σ	K	τ_1	τ_2	τ_3
0.1	$\overline{\sigma}$	$diag\{47.6,\ 47.6\}$	60.56	0.01	0.04
0.1	$\underline{\sigma}$	$diag\{32332,\ 32332\}$	17431	9.28	118
10	$\overline{\sigma}$	$diag\{7.41,\ 7.41\}$	8.66	0.05	1.1
10	$\underline{\sigma}$	$diag\{3145.2,\ 3145.2\}$	12506	363.68	1222

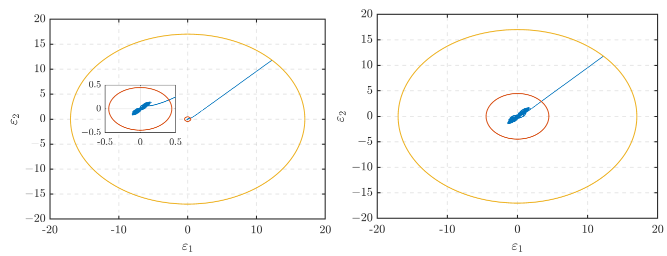
Let us select $\sigma = \overline{\sigma}$ to illustrate the simulation results. The transients in $y_1(t), y_2(t)$, the phase trajectories $\varepsilon(t)$ and inputs $u_1(t), u_2(t)$ are shown in Fig. 2, Fig. 3 and Fig. 4, respectively. Fig. 2 demonstrates that the output signals $y_1(t)$ and $y_2(t)$ never reach the boundaries of the provided sets. This observation is explained by the choice of the function $\varepsilon_i(t)$ in equation (4). Indeed, if the output signal $y_i(t)$ were to reach the boundaries $\underline{g}_i(t)$ or $\overline{g}_i(t)$ of the set $\mathcal{Y}$, then it follows from equation (4) that the corresponding signal $\varepsilon_i(t)$ would tend to infinity. However, as the phase trajectories illustrate in Fig. 3, the trajectory of $\varepsilon(t)$ always remains bounded within the set Π. This contradicts the aforementioned assumption. Furthermore, as shown in Fig. 3, the signal $\varepsilon(t)$ stabilizes within predefined invariant set. In particular, for $c = 10$ (Fig. 3a), $\varepsilon(t)$ stabilizes inside a ball with radius $r < 4.47$, while for $c = 0.1$ (Fig. 3b), it stabilizes inside a ball with radius $r < 0.45$. Comparing Fig. 2a and 2b, as well as Fig. 3a and 3b, it is crucial to emphasize that a decrease in the value of the parameter c results in a greater suppression of disturbances.

In Fig. 4, the oscillations of the control signal are explained by the presence of the disturbance $f(t)$ in the system.



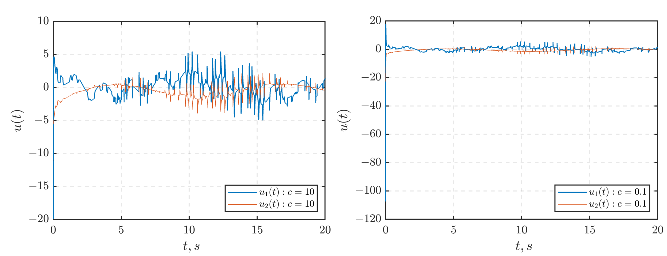
(a) (b)

Fig. 2: The transients of outputs $y_1(t)$ (a), $y_2(t)$ (b) in the closed-loop system under $c = 0.1$ and $c = 10$.



(a) (b)

Fig. 3: The phase trajectories of the closed-loop system under $c = 0.1$ (a) and $c = 10$ (b).



(a) (b)

Fig. 4: The inputs $u_1(t)$ (a), $u_2(t)$ (b) in the closed-loop system under $c = 0.1$ and $c = 10$.

Remark 5: In this example, we have considered that the initial values of the output signals belong to a given set. However, if the output signals start outside the given set, then the developed method does not work since according to transformation 3, the initial values of the output signals must be determined inside the given sets. As mentioned in Remark 1, this disadvantage can be eliminated by adding an additive fast, exponentially decaying function to the

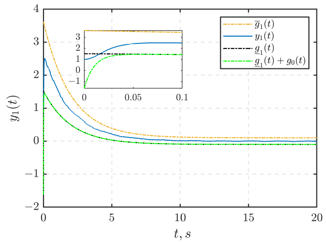


Fig. 5: The transients of output $y_1(t)$ for $x(0) = col\{-0.3233, 1.1467, 0.5\}$ with $c = 0.1$.

restriction functions so that the new restriction functions cover the new initial condition. Fig. 5 shows the transients in $y_1(t)$ at $x(0) = col\{-0.3233, 1.1467, 0.5\}$, i.e. $y_1(0) = 1$ does not belong to the orginal set $\mathcal{Y}$. Then we add a term $g_0(t) = -3.14e^{-100t}$ to the function $\underline{g}_1(t)$ so that the initial condition $y_1(0)$ is bounded from below by the new restriction function. As we can see in Fig. 5, the new below restriction function converges to the old restriction function $\underline{g}_1(t)$ for 0.03 second.

VI. Conclusion

The paper proposes a new method for designing the control law of linear systems with a guarantee of finding output signals in given set based on the basic method and technique of linear matrix inequalities. The proposed method is used to control a linear plant by state feedback under unknown bounded disturbance. The proposed method allows calculating the controller parameters using linear matrix inequalities, which extends the applicability of the obtained method in practice. The simulation results showed the effectiveness of the proposed method and confirmed the theoretical conclusions.

Acknowledgment

This paper is carried out at the IPME RAS with the partial supported by goszadanie no. 121112500298-6 (EGISU NIOKTR) in Sections III, V and the Ministry of Science and Higher Education of the Russian Federation (Project No. 075-15-2021-573) in Section IV.

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